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THE RIEMANN-HILBERT PROBLEM IN DOMAINS WITH PIECEWISE-SMOOTH BOUNDARIES IN CLASSES OF CAUCHY TYPE INTEGRALS WITH DENSITY FROM $L^{p(\cdot)}(\Gamma)$

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In the present report we present the solution of the Riemann-Hilbert problem

$$\operatorname{Re}\left[(a(t)+i\,b(t))\Phi^{+}(t)\right]=c(t), \tag{1}$$

which is formulated as follows: In the bounded domain D, find an analytic function $\Phi(z)$, representable in the form

$$\Phi(z)=\frac{1}{2\pi i\,\omega(z)}\int\limits_{\Gamma}\frac{\varphi(t)}{t-z}\,dt, \tag{2}$$

where

$$\omega(z)=\prod_{k=1}^{\nu}(z-t_k)^{\alpha_k},\quad t_k\in\Gamma,\quad \alpha_k\in\mathbb{R}, \tag{3}$$

the boundary Γ of the domain D is a piecewise-smooth curve free from external cuspidal points, and density φ is from the Lebesgue space with a variable exponent. We prove the conditions of solvability and construct solutions explicitly in all cases in which they exist. The conditions of solvability and solutions depend essentially on coefficients a, b and c, on exponents of the weight function ω, values of the function $p(t)$ at the angular points and on the angle sizes at these points.

A vast literature is devoted to the study of the Riemann-Hilbert problem in various assumptions regarding the given and unknown functions (see, for e.g., [1], pp. 145 and 156). This problem was also investigated in classes of functions, representable by the Cauchy type integral with density from the Lebesgue class L^p, $p>1$ ([2]).

Recently, the theory of Lebesgue spaces $L^{p(\cdot)}(\Gamma;\omega)$ with a variable exponent is intensively being elaborated. This is a variety of those measurable on a rectifiable curve Γ function f for which

$$\int\limits_{0}^{\ell}\left|f(t(s))\,\omega(t(s))\right|^{p(t(s))}ds<\infty.$$

Here $t=t(s)$, $0\le s\le\ell$ is the equation of Γ with respect to the coordinate, $\omega(s)$ is the weight function, and $p(t)$ is the positive measurable function. These spaces and operators

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in them when $p \in Q(\Gamma)$, i.e., $p_- = \inf\limits_{t\in\Gamma} p(t) > 1$ and there exists a constant A, such that

$$\forall\, t_1, t_2 \in \Gamma, \quad |t_1 - t_2| < \frac{1}{2}\,, \quad \big|p(t_1) - p(t_2)\big| < \frac{A}{|\ln|t_1 - t_2|\,|}\,,$$

are well-studied (see [3–4]).

There naturally arises the question regarding the possibility of studying the boundary value problems of the function, including the problem (1), in classes of holomorphic functions, representable by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma;\omega)$.

In [5–12] are considered some boundary value problems in which the boundary values of unknown solutions are required to be integrable with a variable exponent. In [12], the problem (1) for $p \in \mathcal{P}(\Gamma)$ and piecewise-Hölder $a(t)$ and $b(t)$ has been investigated in a simply connected domain D bounded by a tame piecewise-Ljapunov curve Γ with nonzero angles in the class $K^{p(\cdot)}(D;\omega)$, that is, in the class of functions Φ, representable in the form (2), where $z \in D$, and $\varphi \in L^{p(\cdot)}(\Gamma)$. If $\omega(t) = \omega^+(t) \in W^{p(\cdot)}(\Gamma)$, i.e., the operator $Tf = \omega\, S_\Gamma\, \omega^{-1} f$ (where S_Γ is the singular Cauchy operator) is bounded in $L^{p(\cdot)}(\Gamma)$, and Γ is the Carleson curve, then the class $K^{p(\cdot)}(D;\omega)$ coincides with the class $K^{p(\cdot)}(\Gamma;\omega)$, i.e., with a set of functions

$$\Phi(z) = \frac{1}{\pi\, i} \int\limits_\Gamma \frac{\varphi(t)\, dt}{t - z}\,, \quad z \in D, \quad \varphi \in L^{p(\cdot)}(\Gamma;\omega)$$

(see [12], Theorem 1).

Now we consider the problem (1) in the above-mentioned classes of functions, but under more general assumptions regarding p and coefficients a, b. Thus we assume that $\Gamma \in C^1(A_1 \dots, A_i; \nu_1, \dots, \nu_i)$, i.e., Γ is a tame piecewise-smooth curve with angular points A_k at which the angle sizes are equal to $\pi\, \nu_k$, $0 < \nu_k \le 2$, $k = \overline{1, i}$; the coefficients a and b are piecewise-continuous. However, it should be noted that the function p is taken in somewhat more narrow than $\mathcal{P}(\Gamma)$ class $\widetilde{\mathcal{P}}(\Gamma)$.

The measurable on Γ function p belongs to the class $\widetilde{\mathcal{P}}(\Gamma)$, if $p_- > 1$ and there exist positive constants A and ε, such that

$$\forall\, t_1, t_2 \in \Gamma, \quad |t_1 - t_2| < \frac{1}{2}\,, \quad \big|p(t_1) - p(t_2)\big| < \frac{A}{|\ln|t_1 - t_2|\,|^{1+\varepsilon}}\,.$$

In Section 1^0 we present auxiliary statements; the outline of our investigation of the problem (1) under adopted assumptions can be found in Section 2^0; in Section 3^0 we give the conditions of solvability of the problem (1) and write out its solutions.

1^0. Let $z = z(w)$ be the conformal mapping of the unit circle $U = \{w : |w| < 1\}$ with the boundary $\gamma = \{\tau : |\tau| = 1\}$ onto the domain D, and let $w = w(z)$ be an inverse mapping.

Lemma 1. *If* $\Gamma \in C^1(A_1, A_2, \dots, A_i; \nu_1, \nu_2, \dots, \nu_i)$, $0 < \nu_k \le 2$, $k = \overline{1, i}$ *and* $p \in \widetilde{\mathcal{P}}(\Gamma)$, *then the function* $\ell(\tau) = p(z(\tau))$ *belongs to* $\widetilde{\mathcal{P}}(\gamma)$.

We put $p'(\tau) = p(t)[p(t) - 1]^{-1}$.

Lemma 2. *Let* $p \in \mathcal{P}(\Gamma)$, $\ell(\tau) = p(z(\tau))$, $\Gamma \in C^1(A_1, \dots, A_i; \nu_1, \dots, \nu_i)$, $0 < \nu_k < [\ell'(a_k)]^{-1}$, $a_k = w(A_k)$, *then the function* $\rho(\tau) = \prod\limits_{k=1}^{i} (z(\tau) - z(a_k))$ *belongs to the class* $W^{\ell(\cdot)}(\gamma)$.

Lemma 3. *Let* $\Gamma \in C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0<\nu_k<[p'(A_k)]^{-1}$. *If* $p \in \widetilde{\mathcal{P}}(\Gamma)$, $a_k = w(A_k)$, *and* φ *is a continuous on* γ *real function* ($\varphi \in C(\gamma)$), *then the function*

$$\rho(\tau)=\prod_{k=1}^{i}(\tau-a_k)^{\nu_k}\exp\left\{\frac{1}{\ell(\tau)}\int_\gamma \frac{\varphi(\zeta)\,d\zeta}{\zeta-c}\right\}$$

belongs to $W^{\ell(\cdot)}(\gamma)$.

Lemma 4. *Let* $p \in \widetilde{\mathcal{P}}(\Gamma)$. *If* $\tau_k \in \gamma$, $\alpha_k \in \big(-[\ell(a_k)]^{-1},[\ell'(a_k)]^{-1}\big)$, $k=\overline{1,m}$, $\varphi \in C(\gamma)$, $\operatorname{Im}\varphi = 0$, *then the function*

$$\rho(\tau)=\prod_{k=1}^{m}(\tau-\tau_k)\exp\left\{\frac{1}{\ell(\tau)}\int_\gamma \frac{\varphi(\zeta)\,d\zeta}{\zeta-\tau}\right\}$$

belongs to $W^{\ell(\cdot)}(\gamma)$.

Lemma 5. *Let* $p \in \widetilde{\mathcal{P}}(\Gamma)$. *If*

$$X(w)=\exp\left\{\frac{1}{2\pi\, i}\int_\gamma \frac{\ln G(\zeta)\,d\zeta}{\zeta-w}\right\},$$

where $G\in C(\gamma)$, $\min|G|>0$ *and* $\operatorname{ind}G=\frac{1}{2\pi}[\arg G]_\Gamma = 0$, *then*

$$\rho\, X^+ \in W^{\ell(\cdot)}(\gamma).$$

2^0. Let $a(t)$ and $b(t)$ be piecewise-continuous on $\Gamma\in C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$ functions with the condition $\inf(a^2(t)+b^2(t))>0$, while B_k be the points of discontinuity of the function $[a(t)-i\,b(t)][a(t)+i\,b(t)]^{-1}$, and $b_k\in W(B_k)$. Next, let $\omega(z)$ be the function, given by equality (3), and

$$G(b_k-)\big[G(b_k+)\big]^{-1}=\exp(2\pi\, i\, u_k),\quad k=\overline{1,\lambda}.$$

Following [1] (§§ 39–43), we assume that

$$\Psi(w)=\Phi(z(w)),\quad |w|<1$$

and

$$\Omega(w)=\begin{cases}\Psi(w), & |w|<1,\\ \overline{\Psi\Big(\dfrac{1}{\overline{w}}\Big)}, & |w|>1.\end{cases}\tag{4}$$

If for any function F, analytic in the plane, cut along γ, we assume that

$$F_*(w)=\overline{F\Big(\frac{1}{\overline{w}}\Big)},\quad |w|\neq 1,$$

then the problem (1) is equivalent to the system of the following conditions:

$$\begin{cases}\Omega^+(\tau)=G(\tau)\,\Omega^-(\tau)+c_1(\tau), & \tau\in\gamma,\\ \Omega_*(w)=\Omega(w), \quad |w|\neq 1,\\ \Omega(\infty)=\text{const},\end{cases}\tag{5}$$

where $G(\tau)=-[a(z(\tau))-i\,b(z(\tau))][a(z(\tau))+i\,b(z(\tau))]^{-1}$,

$$c_1(\tau)=\frac{c(z(\tau))}{a(z(\tau))+i\,b(z(\tau))}.$$

Suppose

$$r_k(w) = \begin{cases} (w-b_k)^{u_k}, & |w|<1, \\ \overline{\left(\dfrac{1}{\overline{w}} - b_k\right)^{u_k}}, & |w|>1, \end{cases} \qquad r(w) = \prod_{k=1}^{\lambda} r_k(w),$$

$$X_1(w) = \begin{cases} C \exp\left\{ \dfrac{1}{2\pi i} \displaystyle\int_{\gamma} \dfrac{\ln G_1(\tau)\tau^{-\varkappa_1}\,d\tau}{\tau - w} \right\}, & |w|<1, \\ C w^{-\varkappa_1} \exp\left\{ \dfrac{1}{2\pi i} \displaystyle\int_{\gamma} \dfrac{\ln G_1(\tau)\tau^{-\varkappa_1}\,d\tau}{\tau - w} \right\}, & |w|>1, \end{cases} \qquad C = \text{const},$$

and let

$$X(w) = X_1(w) \prod_{k=1}^{\lambda} r_k(w). \tag{6}$$

Here $G_1(\tau) = G(\tau) \prod\limits_{k=1}^{\lambda} [r_k^{+}(\tau)][r_k^{-}(\tau)]^{-1}$ is the continuous on Γ function, and $\varkappa_1 = \operatorname{ind} G_1(\tau)$.

In order to find in what class we have to seek for a solution of the boundary value problem (5) in which the solutions of the problem (1) correspond in the class $K^{p(\cdot)}(D;\omega)$, we construct in a definite manner a sequence of numbers $\{\delta_k\}_{k=1}^{j}$ through the parameters of the problem (i.e., angle sizes of Γ, exponents of the weight function ω, values of the function p at the points A_k, t_k, B_k). Further, we introduce the rational function

$$Q(w) = \prod_{k=1}^{j} (w-w_k)^{\gamma_k}, \quad w_k \in \gamma, \tag{7}$$

where the exponents γ_k are defined through δ_k (for details, see [12]).

Lemma 6. *If* $p \in \widetilde{\mathcal{P}}(\Gamma)$, *then the function* $R(r) = Q(r)\Omega(\tau)[X^{+}(\tau)]^{-1}$ *belongs to* $L^{p(\cdot)}(\gamma;\rho)$, *where*

$$\rho(\tau) = \prod_{k=1}^{j} (\tau - w_k)^{\delta_k - \gamma_k},$$

and $\rho \in W^{p(\cdot)}(\gamma)$.

Lemma 7. *Let* $\Phi \in K^{p(\cdot)}(D;\omega)$, $\Psi(w) = \Phi(z(w))$; *the functions* Ω, X *and* Q *are defined by equalities* (4), (6), (7), *and* $F(w) = \Omega(w)\,Q(w)\,X^{-1}(w)$, $|w| \neq 1$. *Next, let* $\varkappa = \varkappa_0 + \varkappa_1$, *where* $\varkappa_0$ *is the order of the function* Q *at infinity, and* $(-\varkappa_1)$ *is the order of the function* X. *Then there exists the polynomial* $p_\varkappa(w)$ *(of order* $\varkappa$*), such that the function* $F(w) - P_\varkappa(w)$ *is representable by the Cauchy type integral with density from* $L^{p(\cdot)}(\gamma;\rho)$ $(\rho_\varkappa(w) \equiv 0$ *for* $\varkappa < 0)$.

3^0. Relying on Lemmas 1–7, we prove the following

Theorem. *Let:* 1) D *be the bounded simply connected domain with the boundary* $\Gamma \in C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0<\nu_k\le 2$; 2) $\omega(z)$ *be the function, given by equality* (3); 3) $a(t)$, $b(t)$ *be piecewise-continuous real functions with the condition* $\inf(a^2(t)+b^2(t))>0$, *such that the function* $G(t) = [a(t) - i\,b(t)][a(t)+i\,b(t)]^{-1}$ *has the discontinuity points* B_k, *and* $G(B_k-)[G(B_k+)]^{-1} = \exp(2\pi\, i\, u_k)$; 4) $c\,\omega \in L^{p(\cdot)}(\Gamma)$, $p \in \widetilde{Q}(\Gamma)$.

Let, further, X and Q be the functions, defined by equalities (6) and (7), $\varkappa_0$ and $(-\varkappa_1)$ be the order of these functions at infinity, and $\varkappa = \varkappa_0 + \varkappa_1$. Then:

(a) if $\varkappa < 0$, then for the solvability of the problem (1) in the class $K^{p(\cdot)}(D;\omega)$ it is necessary and sufficient that the conditions

$$\int\limits_{\Gamma} \frac{c(t)\,Q(w(t))\,w^k(t)}{X^+(w(t))(a(t)+i\,b(t))}\,w'(t)\,dt = 0, \quad k=\overline{0,|\varkappa|},$$

be fulfilled, and such being the case, we have a unique solution

$$\Phi_c(z) = \widetilde{\Omega}_c(w(z)) = \frac{1}{2}\big(\Omega_c(w(z)) + (\Omega_c)_*(w(z))\big),$$

where

$$\Omega_c(w) = \frac{1}{2\pi\,i}\int\limits_{\Gamma} \frac{c(t)\,Q(w(t))}{X^+(w(t))(a(t)+i\,b(t))}\,\frac{w'(t)\,dt}{w(t)-w(z)}\,;$$

(b) if $\varkappa \geq 0$, then the problem is, unconditionally, solvable, and all its solutions are given by the equality

$$\Phi(z) = \Phi_c(z) + X(w(z))\,Q^{-1}(w(z))\,P_\varkappa(w(z)),$$

where $P_\varkappa(w) = \prod\limits_{k=0}^{\varkappa} h_k\,w^k$ is an arbitrary polynomial whose coefficients h_k satisfy the condition

$$\widetilde{h}_k = A\,\widetilde{h}_{\varkappa-k}, \quad k=\overline{0,\varkappa}, \quad A=(-1)^{\varkappa_0}\prod_{k=1}^{j} w_k^{-\gamma_k}.$$

Remark 1. All the statements in [12] concerning particular cases, remain valid.

Remark 2. If Γ is the piecewise-Ljapunov curve, and $a(t)$, $b(t)$ are the piecewise-Hölder functions, then the condition $p\in\widetilde{\mathcal{P}}(\Gamma)$ can be replaced by the condition $p\in\mathcal{P}(\Gamma)$.

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