Mathematics

On the Riemann-Hilbert-Poincaré Problem and I. Vekua's Representation of Holomorphic Functions

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ABSTRACT. Generalizing the well-known I. Vekua's integral representation of holomorphic functions, we solve the Riemann-Hilbert-Poincaré problem in the classes of functions whose *m*th derivative is representable by a Cauchy type integral with a density from a weighted Lebesgue space with a variable exponent in simply connected domains with piecewise-smooth boundaries.

***Key words**: Lebesgue space with a variable exponent, integral representation of a holomorphic function, Cauchy type integral, Riemann-Hilbert-Poincaré problem, domains with piecewise-smooth boundaries.*

The Riemann-Hilbert-Poincaré problem (see, e.g., [1, p. 239])

$$\mathrm{Re}\left[\sum_{k=0}^{m}\left(a_k(t)\Phi^{(k)}(t)+\int_{\Gamma}h_k(t,\tau)\Phi^{(k)}(\tau)ds\right)\right]=f(t) \qquad (1)$$

embraces many boundary value problems of function theory. I. Vekua was the first mathematician who investigated this problem [2, 3]. For its investigation he used new original representations of holomorphic functions which N. Muskhelishvili subsequently called I. Vekua's representations. These representations were later generalized in [4, 5].

At the present time various boundary value problems are intensively studied in formulations which take into account the local properties of the integrability of given functions. These conditions are described by variable Lebesgue or variable Sobolev spaces [6-12] and other works. The study of these spaces was stimulated by various problems of nonlinear elasticity, fluid mechanics, calculus of variations and differential equations with nonstandard growth conditions.

In [11-12], the particular cases of problem [1] are investigated in the framework of Lebesgue spaces with a variable exponent. In the present paper we propose a generalization of the results of [2-5] in two directions. Firstly, the set of domains in which problem (1) is investigated has been essentially widened. Secondly, as a class of sought functions we consider the $K_D^{p(\cdot)}(\Gamma;\omega)$-set of functions whose *m*th derivative is representable by a Cauchy type integral with a density from the Lebesgue space $L^{p(\cdot)}(\Gamma;\omega)$ with a variable exponent (this set is essentially wider than the classes of functions considered in [2-5]). For this purpose, in the first place we establish I. Vekua's representations in a general situation, namely, for functions whose *m*th derivative is a Cauchy type integral with a density from the space $L^{p(\cdot)}(\Gamma;\omega)$ in domains with piecewise smooth boundaries.

Following [2] (see also [1: 115-116] we come to the definite Riemann-Hilbert problem in the class $K_D^{p(\cdot)}(\Gamma;\omega)$, having an arbitrary integer index (and not a non-negative one as in [2-5]). We have succeeded in showing that in the case of a negative index the conditions of solvability are fulfilled. This has enabled us, using the results from [9] and [12], to justify I. Vekua's representations under the considered assumptions. Having these representations, it becomes possible to formulate statements concerning problem (1) that are analogous to the results in [1]-[4]. It should be emphasized that the analogy concerns only the outward wording of the formulations. Relations between the considered values are similar to the previous ones, but they essentially depend on the geometry of a boundary, on the discontinuity points of a function $a_m(t)$ and on the values of a function $p(t)$ at these points.

1^0. Let $t=t(s)$, $0 \le s \le l$, be the equation of a simple rectifiable curve Γ with respect to the arc coordinate. Furthermore, let p be a positive measurable function on Γ, and ω be an almost everywhere non-negative measurable function. We denote by $L^{p(\cdot)}(\Gamma;\omega)$ the set of measurable functions f on Γ for which $|f(t(s))\omega(t(s))|^{p(t(s))}$ is summable.

Definition 1. We say that a function p belongs to the class $\widetilde{P}(\Gamma)$ if there exist positive constants A and ε such that for every $t_1,\ t_2 \in \Gamma$,

$$|p(t_1)-p(t_2)| < \frac{A}{|\ln|t_1-t_2||^{1+\varepsilon}} \tag{2}$$

and $p_- = \min_{t\in\Gamma} p(t) > 1$.

The set of functions p for which $p_->1$ and which satisfy (2) for $\varepsilon = 0$ is denoted by $P(\Gamma)$.

We denote by $C_D^1(A_1,\ldots,A_i;\ \upsilon_1,\ldots,\upsilon_i)$ the set of simple closed piecewise smooth curves Γ having angular points $A_1,\ldots,A_i$ whose angle values with respect to the domain D with boundary Γ are equal to $\pi\upsilon_k$, $k=\overline{1,i}$, $0 \le \upsilon_k \le 2$.

The set of piecewise Lyapunov curves Γ contained in that class is denoted by $C_D^{1,L}(A_1,\ldots,A_i;\ \upsilon_1,\ldots,\upsilon_i)$.

Definition 2. Let $m \ge 0$ be an integer and let $K_{D,m}^{p(\cdot)}(\Gamma;\omega)$ be the set of holomorphic functions Φ in D for which $\Phi^{(m)}(z)$ $\left(\Phi^{(0)}(z) \equiv \Phi(z)\right)$ is representable in the form

$$\Phi^{(m)}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-z}, \qquad z \in D, \quad \varphi \in L^{p(\cdot)}(\Gamma;\omega).$$

2^0. **Theorem 1.** *Let*:

(i) $\Gamma \in C_{D+}^1(A_1,\ldots,A_i;\ \upsilon_1,\ldots,\upsilon_i)$, $0<\upsilon_k\le 2$ *and* $p \in \widetilde{P}(\Gamma)$ or $\Gamma \in C_{D+}^{1,L}(A_1,\ldots,A_i;\ \upsilon_1,\ldots,\upsilon_i)$, $0<\upsilon_k\le 2$, *and* $p\in P(\Gamma)$;

(ii) $\omega(t) = \prod_{k=1}^{\upsilon}|t-t_k|^{\alpha_k}$, $\quad t_k \in \Gamma$, $\quad -\frac{1}{p(t_k)} < \alpha_k < \frac{1}{p'(t_k)}$, $\quad p'(t) = \frac{p(t)}{p(t)-1}$;

(iii) *$z=z(w)$ be a conformal mapping of* $V=\{w:\ |w|<1\}$ *onto* D_+, *and $w=w(z)$ be the inverse transformation*; $a_k = w(A_k)$, $k=\overline{1,i}$, $\tau_k = w(t_k)$, $k=\overline{1,\upsilon}$, *and put* $w_1 = a_1 = \tau_1,\ldots,w_\mu = a_\mu = \tau_\mu$; $w_{\mu+1} = a_{\mu+1},\ldots,w_{\mu+p} = a_{\mu+p}$; $w_{\mu+p+1} = \tau_{\mu+1},\ldots,w_{\mu+p+M} = \tau_{\mu+M}$;

(iv) $$\delta_k = \begin{cases} \alpha_k\upsilon_k + \dfrac{\upsilon_k-1}{l(w_k)} + \upsilon_k - 1, & k=\overline{1,\mu}, \quad l(\tau) = p(z(\tau)), \\ \dfrac{\upsilon_k-1}{l(w_k)} + \upsilon_k - 1, & k=\overline{\mu+1,\mu+p}, \\ \alpha_{k-p}, & k=\overline{\mu+p+1,\mu+p+M}; \end{cases}$$

(v) $\Phi \in K_{D+,m}^{p(\cdot)}(\Gamma;\omega)$.

Then if

$$\{\delta_k\} \ne \frac{1}{l'(w_k)}, \qquad l'(\tau) = \frac{l(\tau)}{l(\tau)-1} \tag{3}$$

(it is assumed that $\{a\}$ denotes the fractional part of the number a) and the origin lies in D_+, *then there exist a real function* $\mu \in L^{p(\cdot)}(\Gamma;\omega)$ *and a real constant d such that*

$$\Phi(z)=\int_{\Gamma}\frac{\mu(t)ds}{1-\frac{z}{t}}+id \quad \text{for } m=0 \tag{4}$$

and

$$\Phi(z)=\int_{\Gamma}\mu(t)\left(1-\frac{z}{t}\right)^{m-1}\ln\left(1-\frac{z}{t}\right)ds+\int_{\Gamma}\mu(t)ds+id \quad \text{for } m\geq 1. \tag{5}$$

This theorem is a generalization of I. Vekua's theorem proved by him in the case where Γ is a Lyapunov curve and $\Phi^{(m)}$ belongs to the Hölder class in $\overline{D_+}$ [2]. When $\Phi^{(m)}\in K^{p}_{D_+}(\Gamma;\omega)$, $p=const>1$, and Γ is a Lyapunov curve, the theorem was established by B. Khvedelidze. When $\omega=1$, K. Aptsiauri showed the validity of representations (4), (5) for the definite subclass of smooth curves [5].

3^0. Let: a) $\Gamma\in C^1_{D_+}(A_1,\ldots,A_i;\ \upsilon_1,\ldots,\upsilon_i)$, $0<\upsilon_k\leq 2$; b) $a_k(t)\in L^{p(\cdot)}(\Gamma;\omega)$, $k=\overline{0,m-1}$; c) $a_m(t)$ be piecewise-continuous; d) the operators

$$H_k\varphi=\int_{\Gamma}h_k(t,\tau)\varphi(\tau)d\tau, \quad t\in\Gamma,$$

be compact in $L^{p(\cdot)}(\Gamma;\omega)$.

Let us consider the problem: find a function $\Phi\in K^{p(\cdot)}_{D_{+,m}}(\Gamma;\omega)$ which satisfies condition (1) almost everywhere on Γ.

By virtue of representations (4)-(5) the investigation of this problem reduces to the investigation, in the class $L^{p(\cdot)}(\Gamma;\omega)$, of the equation

$$N\mu\equiv A_0(t_0)\mu(t_0)+\frac{A_1(t_0)}{\pi i}\int_{\Gamma}\frac{\mu(t)dt}{t-t_0}+(T_\mu)(t_0)=\tilde{f}(t_0),$$

where

$$A_k(t_0)=\frac{1}{2}(-1)^m(m-1)!\pi i\left[t_0^{1-m}\bar{t}_0\ a_m(t_0)+(-1)^{1+k}\,\bar{t}_0^{1-m}t_0'\,\overline{a_m(t_0)}\right], \qquad k=0,1,$$

$$\tilde{f}(t_0)=f(t_0)-d\sigma(t_0), \qquad \sigma(t_0)=\operatorname{Re}\left[ia_0(t_0)+i\int_{\Gamma}h(t_0,t)dt\right].$$

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled. Then for problem* (1) *to be solvable in the class* $K^{p(\cdot)}_{D+,m}(\Gamma;\omega)$ *it is necessary that for some real d the function* $\tilde{f}(t_0)$ *should satisfy the conditions*

$$\int_{\Gamma}\tilde{f}(t_0)g_k(s_0)ds_0=0, \qquad k=\overline{1,n'},$$

where $g_1,\ldots,g_{n'}$ *are linearly independent solutions from the class* $L^{p'(\cdot)}(\Gamma;\omega^{-1})$ *of the equation* $N'g=0$, *where* N' *is the adjoint operator to the operator N* ([1: 164]).

In order that the problem be solvable for any right-hand part of f it is necessary and sufficient that $n'=0$ *or* $n'=1$, *and in the latter case the solution g of the equation* $N'g=0$ *must satisfy the condition*

$$(g,\sigma)=\int_{\Gamma}g(t_0)\sigma(t_0)ds_0\neq 0.$$

In both cases the homogeneous problem has æ $+1$ *linearly independent solutions (where* æ ≥ -1).

Here the integer number æ is defined by means of Γ, p, ω and the jumps of the function a_m (see [9]; [12, Subsect.7]).

If the conditions of Theorem 2 are not observed (for example, when æ <-1), in that case: if for any $k=\overline{1,n'}$ we have $(\sigma,g_k)=0$, then the homogeneous problem has æ $+n'$ linearly independent solutions, and if among the numbers (σ,g_k) there is at least one number different from zero, then it has æ $+n'+1$ solutions.

4^0. Let

$$\Psi(z)=\int_\Gamma g(t)\Omega^*(t,z)ds\,,\qquad z\in D_-\,,$$

where

$$\Omega^*(t_0,z)=\sum_{k=0}^{m}\left\{a_k(t_0)N_k(t_0,z)+\int_\Gamma h_k(t_0,t)N_k(t,z)ds\right\},$$

$$N_l(t_0,z)=(-1)^l\,\frac{(m-1)\cdots(m-l)}{z^l}\left(1-\frac{t_0}{z}\right)\times$$

$$\times\left(\ln\left(1-\frac{t_0}{z}\right)+\frac{1}{m-1}+\cdots+\frac{1}{m-l}\right),\qquad\qquad l=\overline{1,m-1}\,,$$

$$N_0(t_0,z)=\left(1-\frac{t_0}{z}\right)^{m-1}\ln\left(1-\frac{t_0}{z}\right)+1\,,\qquad N_m(t_0,z)=\frac{(-1)^m(m-1)!}{z^{m+1}(z-t_0)}\,.$$

Then the equation $N'g=0$ is equivalent to the problem $\operatorname{Re}\psi^-=0$ in the class $K^{p'(\cdot)}_{D-,m}(\Gamma;\omega)$. This problem has æ $+1$ linearly independent solutions $\psi_0(z),\ldots,\psi_{æ}(z)$, where

$$æ=N\left\{A_k\notin\cup\{t_j\}:\,p'(A_k)<\upsilon_k\right\}+$$

$$+\,N\left\{A_k\in\cup\{t_j\}:\frac{p'(A_k)}{1-\alpha_k\,p'(A_k)}<\upsilon_k<\frac{2\,p'(A_k)}{1-\alpha_k\,p'(A_k)}\right\}$$

(see [12], Subsect. 7). Here $N(E)$ is the number of elements of the set E.

Let us assume

$$\tilde\Omega(t_0,z)=\Omega^*(t_0,z)-\Psi_0(z)-\cdots-\Psi_{æ}(z),$$

$$\Omega(t_0,z)=\tilde\Omega(t_0,z)-i\operatorname{Im}\tilde\Omega(t_0,\infty)\,.$$

We prove that under the considered assumptions the function $\Omega(t_0,z)$ plays the role of the kernel for problem (1) (for details about the kernel see [2] and [1: 241-243]). Under the classical assumptions it is the function $\Omega^*(t_0,z)-i\operatorname{Im}\Omega^*(t_0,\infty)$ that is the kernel ([1: 242]).

5^0. In particular cases we obtain more complete information, for example, for the well-known problems of Poincaré and Neumann [1: 243, 247] considered in domains with piecewise-smooth boundaries.

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ნაშრომში განზოგადებულია ანალიზურ ფუნქციათა ი. ვეკუასეული წარმოდგენა და ამოხსნილია რიმან-ჰილბერტ-პუანკარეს სასაზღვრო ამოცანა უბან-უბან გლუვი წირებით შემოსაზღვრული ცალადბმული არეების შემთხვევაში იმ ფუნქციათა კლასში, რომელთა m რიგის წარმოებული წარმოიდგინება კოშის ტიპის ინტეგრალით სიმკვრივით ცვლადმაჩვენებლიანი ლებეგის სივრციდან.

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