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THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS FROM VARIABLE EXPONENT SMIRNOV CLASSES IN DOMAINS WITH PIECEWISE SMOOTH BOUNDARY

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Dedicated to the memory of Academician Niko Muskhelishvili on the occasion of his 120th birthday

The Dirichlet problem is solved for harmonic functions from variable exponent Smirnov classes in domains with piecewise smooth boundaries. The solvability conditions are established. Depending on the boundary geometry and value of the space exponent at angular points, the Dirichlet problem may turn out to be unsolvable, solvable uniquely and non-uniquely. In the unsolvable case, for boundary functions the necessary and sufficient conditions are found, which govern the solvability. In all solvability cases, solutions are constructed in explicit form. Bibliography: 19 *titles.*

1 Introduction

The Dirichlet problem for harmonic functions of two variables was studied under various assumptions as to the sought for functions, boundary data, and domains in which they are considered (cf., for example, [1]–[5] and other references). In recent years, boundary value problems were considered in Lebesgue spaces with a variable exponent (cf., for example, [6]– [8] and other references). Representing the sought for functions by the Cauchy type integral with a density from variable exponent Lebesgue spaces, we investigated the following boundary value problems of the theory of functions of a complex variable: the Riemann, Riemann–Hilbert, and

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Riemann–Hilbert–Poincaré boundary value problems and also the Dirichlet problem in the class of harmonic functions which are the real parts of above–mentioned Cauchy type integrals (cf., for example, [9]– [12]).

We managed to obtain a complete picture of solvability when the Dirichlet problem is considered in a bounded simply connected domain with an arbitrary piecewise smooth boundary and the variable exponent is a function satisfying the Log-Hölder condition. In that case, the considered problem reduces to the Dirichlet problem for a circle. The Muskhelishvili method played an important role in reducing this problem to the Riemann problem [2, Sections 40 and 41].

For $p(t) = \text{const} > 1$ the Cauchy type integral with a density from $L^p(\Gamma)$, where Γ is a simple closed Carleson curve bounding the domain D, belongs to the Smirnov class $E^p(D)$ ([13, p. 29]) and thus we have additional useful information on the problem solution.

In [14]– [15], the variable exponent Hardy and Smirnov classes were introduced and studied in simply connected domains, whereas, in [16], these classes were studied in both simply and doubly connected domains. It turned out, in particular, that if Γ is a simple closed piecewise Lyapunov curve bounding the domain D, then the Cauchy type integral with a density from $L^{p(t)}(\Gamma)$ belongs to the class $E^{p(\cdot)}(D)$. The Dirichlet problem as it is formulated above is thereby solved in the class $\operatorname{Re} E^{p(\cdot)}(D)$ too. However, the important and interesting case of domains with arbitrary piecewise smooth boundaries has so far been remaining uninvestigated.

In the present paper, for such domains we study the Dirichlet problem for harmonic functions from variable exponent Smirnov classes. Owing to the conformal mapping, this problem again reduces to a problem for the circle, but this time in the weight class $h^{p(\cdot)}(\omega) = \operatorname{Re} E^{p(\cdot)}(\omega)$, where the weight ω is more general than the power weight usually considered in such problems. After solving the problem in the circle for harmonic functions of class $h^{p(\cdot)}(\omega)$, we proceed to our main goal which is to study the case of "bad domains." Depending on the boundary geometry and the values of exponent $p(t)$ at angular points, the Dirichlet problem may turn out to be unsolvable, solvable uniquely or non-uniquely. In the unsolvable case, a necessary and sufficient condition is found for the boundary function, which provides the problem solvability. In all solvability cases, solutions are constructed in an explicit form.

2 Notation, Definitions, and Auxiliary Statements

2.1 Variable exponent Lebesgue spaces

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \leqslant s \leqslant \ell\}$ be a simple rectifiable curve whose equation is given with respect to the arc abscissa s, and $p(t) = p(t(s))$ be a positive measurable function on Γ. If ω is a measurable, a.e. nonzero function on Γ, then for measurable functions f on Γ it is assumed that

$$\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} = \inf\left\{\lambda > 0 : \int_0^\ell \left|\frac{f(t(s))\omega(t(s))}{\lambda}\right|^{p(t(s))} ds \leqslant 1\right\}.$$

We denote by $L^{p(\cdot)}(\Gamma;\omega)$ the set of all functions f for which

$$\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} = \|f\omega\|_{L^{p(\cdot)}(\Gamma)} < \infty.$$

2.2 Curves

If Γ is a piecewise smooth simple curve with a finite number of angular points A_k, $k = \overline{1,n}$, where the values of angles with respect to the domain D bounded with Γ are equal to $\pi\nu_k$, $0 \leqslant \nu_k \leqslant 2$, then we write $\Gamma \in C^1_D(A_1, \dots, A_n; \nu_1, \dots, \nu_n)$. The set of such piecewise Lyapunov curves is denoted by $C^{1,L}_D(A_1, \dots, A_n; \nu_1, \dots, \nu_n)$.

2.3 Classes of exponents

Let Γ be a simple rectifiable curve. Denote by $\mathscr{P}_{1+\varepsilon}(\Gamma)$, $\varepsilon \geqslant 0$, the set of positive measurable functions p on Γ, for which the following conditions are fulfilled:

(a) there exists a constant B (depending on p and ε) such that for any $t_1, t_2 \in \Gamma$

$$|p(t_1) - p(t_2)| < B(|\ln|t_1 - t_2||)^{-(1+\varepsilon)};$$

(b) $\inf\limits_{t\in\Gamma} p(t) = \underline{p} > 1$.

Let us expand $\mathscr{P}(\Gamma) = \mathscr{P}_1(\Gamma)$ and $\widetilde{\mathscr{P}}(\Gamma) = \bigcup\limits_{\varepsilon>0} \mathscr{P}_{1+\varepsilon}(\Gamma)$.

Proposition 2.1 (cf. [16, Theorem 7.2])**.** *Let D be a domain bounded by a simple closed rectifiable curve Γ, $z = z(w)$ the conformal mapping of the circle $U = \{w : |w| < 1\}$ onto D, and $p \in \mathscr{P}(\Gamma)$. If $z' \in \bigcup\limits_{\delta>1} H^\delta$, where H^δ is the Hardy class of analytic functions in U, then the function $\ell(\tau) = p(z(\tau))$ belongs to $\mathscr{P}(\gamma)$, $\gamma = \{\tau : |\tau| = 1\}$.*

For the Hardy classes cf., for example, [17].

Proposition 2.2 (cf. [12, Lemma 2])**.** *If $\Gamma \in C^1_D(A_1, \dots, A_n; \nu_1, \dots, \nu_n)$, $0 < \nu_k \leqslant 2$, $p \in \widetilde{\mathscr{P}}(\Gamma)$, and $\ell(\tau) = p(z(\tau))$, then $\ell \in \widetilde{\mathscr{P}}(\gamma)$.*

Remark 2.1. If we follow the proof of Theorems 7.1 and 7.2 in [16], then it can be easily verified that for $z' \in \bigcup\limits_{\delta>1} H^\delta$ and $p \in \widetilde{\mathscr{P}}(\Gamma)$ we have $\ell \in \widetilde{\mathscr{P}}(\gamma)$.

Definition 2.1. Let Γ be a simple closed curve bounding the domain D, and let $z = z(w)$ be the conformal mapping of the circle U with boundary γ onto D. We say that a function p on Γ belongs to the class $Q(\Gamma)$ if $p \in \widetilde{\mathscr{P}}(\Gamma)$ and $\ell \in \widetilde{\mathscr{P}}(\gamma)$, where $\ell(\tau) = p(z(\tau))$, $\tau \in \gamma$.

As follows from Proposition 2.2, if Γ is a piecewise smooth curve without internal zero angles, then $Q(\Gamma)$ coincides with $\widetilde{\mathscr{P}}(\Gamma)$. Note that if Γ is an arbitrary piecewise smooth curve and p is a function from $\widetilde{\mathscr{P}}(\Gamma)$ such that p is constant in a neighborhood of the points with zero internal angles, then $p \in Q(\Gamma)$.

2.4 Set of weight functions $W^{p(\cdot)}(\Gamma)$

Let p be a function given on Γ, $\underline{p} > 1$. We say that an almost everywhere different from zero function ω on Γ belongs to the class $W^{p(\cdot)}(\Gamma)$ if the Cauchy singular operator

$$S_\Gamma : f \to S_\Gamma f, \quad (S_\Gamma f)(t) = \frac{\omega(t)}{\pi i} \int\limits_\Gamma \frac{1}{\omega(\tau)} \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma,$$

is continuous in $L^{p(\cdot)}(\Gamma)$.

2.5 Classes of functions $E^{p(\cdot)}(D;\omega)$, $H^{p(\cdot)}(\omega)$, $e^{p(\cdot)}(D;\omega)$, $h^{p(\cdot)}(\omega)$

Suppose that D is the internal domain bounded by a rectifiable closed curve Γ, p is a measurable positive function on Γ, ω is a measurable, a.e. different from zero function in D, and $z = z(w)$ is the conformal mapping of the unit circle U onto D. Denote by $E^{p(\cdot)}(D;\omega)$ the set of analytic functions Φ in D for which

$$\sup_{0<r<1} \int_0^{2\pi} |\Phi(z(re^{i\vartheta}))\omega(re^{i\vartheta})|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\, d\vartheta < \infty.$$

Assume that

$$H^{p(\cdot)}(\omega) = E^{p(\cdot)}(U;\omega),$$
$$e^{p(\cdot)}(D;\omega) = \{u : \operatorname{Re}\Phi,\ \Phi \in E^{p(\cdot)}(D;\omega)\},$$
$$h^{p(\cdot)}(\omega) = e^{p(\cdot)}(U;\omega), \quad h^{p(\cdot)} = h^{p(\cdot)}(1).$$

When p is a positive constant, $\omega \equiv 1$, and the classes $E^{p(\cdot)}(D;\omega)$ and $H^{p(\cdot)}(\omega)$ coincide with the well-known Smirnov $E^p(D)$ and Hardy H^p classes.

If $\omega \in \bigcup\limits_{\delta>0} H^\delta$, then for all $t \in \Gamma$ every function Φ of class $E^{p(\cdot)}(D;\omega)$ has the angular boundary value $\Phi^+(t)$ and the function Φ^+ belongs to the class $L^{p(\cdot)}(\Gamma;\omega^+)$ ($\omega^+(t)$ is a boundary function ω; it exists since $\omega \in \bigcup\limits_{\delta>0} H^\delta$).

2.6 Class of weight functions $W^{p(\cdot)}(U)$

Assume that

$$W^{p(\cdot)}(U) = \Big\{\omega :\ [\omega(re^{i\vartheta})]^{\pm 1} \in \bigcup_{\delta>0} H^\delta,\ \ \omega^+ \in W^{p(\cdot)}(\gamma)\Big\}.$$

Examples. 1. Let $p \in \mathscr{P}(\gamma)$, $a_k \in \gamma$, $k = \overline{1,n}$. Let us draw cuts ℓ_k connecting the points a_k with $z = \infty$ and lying outside U. Fix an arbitrary branch of an analytic function $(w-a_k)^{\alpha_k}$, $\alpha_k \in \mathbb{R}$, in the plane cut along ℓ_k. The function

$$\omega(w) = \prod_{k=1}^n (w-a_k)^{\alpha_k}, \tag{2.1}$$

where

$$-\frac{1}{p(a_k)} < \alpha_k < \frac{1}{p'(a_k)}\,, \quad p'(t) = \frac{p(t)}{p(t)-1}\,, \tag{2.2}$$

belongs to $W^{p(\cdot)}(U)$ because $\omega^+ \in W^{p(\cdot)}(\gamma)$ under the condition (2.2) (cf. [18]).

2. Suppose that φ is a continuous real-valued function on γ, $a_k \in \gamma$, $\alpha_k \in \mathbb{R}$, and the conditions (2.2) are fulfilled. Then the function

$$\omega(w)=\prod_{k=1}^{n}(w-a_k)^{\alpha_k}\exp\int_\gamma\frac{\varphi(t)\,dt}{t-w}\,,\quad w\in U, \tag{2.3}$$

belongs to $W(U)$ (cf. [9, p. 57] and Theorem A below).

3. Let $z=z(w)$ be the conformal mapping of the circle U onto the domain D bounded by a simple closed curve $\Gamma\in C_D^1(A_1,\dots,A_n;\nu_1,\dots,\nu_k)$, $0\leqslant\nu_k\leqslant 2$, $k=\overline{1,n}$. Then

$$z'(w)\sim\prod_{k=1}^{n}(w-a_k)^{\alpha_k}\exp\int_\gamma\frac{\psi(\tau)\,d\tau}{\tau-w}\,,\quad z(a_k)=A_k,\quad w\in U, \tag{2.4}$$

where ψ is a continuous real-valued function on γ (cf. [13, p. 144]) and $f\sim g$ means that

$$0<\inf\left|\frac{f}{g}\right|\leqslant\sup\left|\frac{f}{g}\right|<\infty.$$

If $p\in\mathscr{P}(\gamma)$ and $0<\nu_k<p(A_k)$, then the following function belongs to $W^{p(\cdot)}(\gamma)$:

$$m(t)=\prod(t-a_k)^{\frac{\nu_k-1}{p(A_k)}}\exp\frac{1}{p(t)}\int_\gamma\frac{\psi(\tau)\,d\tau}{\tau-t}.$$

2.7 Some results to be used in the sequel

Theorem A [9, Theorem 6.1]. *If $\omega\in W^{p(\cdot)}(\gamma)$ and $\frac{1}{\omega}\in L^{p'(\cdot)+\varepsilon}(\gamma)$ for some $\varepsilon>0$, and φ is an arbitrary real-valued function on γ, then the function*

$$m(t)=\omega(t)\exp\int_\gamma\frac{\varphi(\tau)\,d\tau}{\tau-t}\quad t\in\gamma \tag{2.5}$$

belongs to the class $W^{p(\cdot)}(\gamma)$.

Theorem B [15, Theorem 6]. *If D is a domain bounded by a curve Γ from*

$$C_D^{1,L}(A_1,\dots,A_n;\nu_1,\dots,\nu_n),\quad 0<\nu_k\leqslant 2,$$

$p\in\mathscr{P}(\Gamma)$, and $f\in L^{p(\cdot)}(\Gamma)$, then a Cauchy type integral $(K_\Gamma f)(z)$ belongs to $E^{p(\cdot)}(D)$.

3 The Dirichlet Problem in $H^{p(\cdot)}(\omega)$

3.1 Statement of the problem

Assume that (i) $p\in\mathscr{P}(\gamma)$ and (ii) ω is an analytic function in U and $\omega^{\pm1}\in\bigcup\limits_{\delta>1}H^\delta$. Consider the Dirichlet problem formulated as follows: Find a function u if

$$\begin{cases} u\in h^{p(\cdot)}(\omega), \\ u^+(t)=f(t),\quad t\in\gamma,\quad g\in L^{p(\cdot)}(\gamma;\omega^+). \end{cases} \tag{3.1}$$

3.2 Reducing the problem (3.1) to the Riemann problem

If u is a solution of the problem (3.1), then $u = \operatorname{Re}\Phi$, where $\Phi \in H^{p(\cdot)}(\omega)$. Therefore, the boundary condition (3.1) can be written in the form

$$\operatorname{Re}[\Phi^+(t)] = f(t), \quad \Phi \in H^{p(\cdot)}(\omega). \tag{3.2}$$

Assume that

$$\Psi(w) = \Phi(w)\omega(w). \tag{3.3}$$

Then (3.2) takes the form

$$\operatorname{Re}\left[\frac{\Psi^+(t)}{\omega^+(t)}\right] = f(t), \quad \Psi \in H^{p(\cdot)},$$

or, which is the same,

$$\frac{\Psi^+(t)}{\omega^+(t)} + \overline{\left[\frac{\Psi^+(t)}{\omega^+(t)}\right]} = 2f(t).$$

Hence

$$\Psi^+(t) = -\frac{\omega^+(t)}{\overline{\omega^+(t)}}\,\overline{\Psi^+(t)} + g(t), \quad g(t) = 2f(t)\omega^+(t), \quad g \in L^{p(\cdot)}(\gamma). \tag{3.4}$$

Consider the function

$$\Omega(w) = \begin{cases} \Psi(w), & |w| < 1, \\ \overline{\Psi\left(\frac{1}{\overline{w}}\right)}, & |w| > 1. \end{cases} \tag{3.5}$$

Then the condition (3.4) takes the form

$$\Omega^+(t) = -\frac{\omega^+(t)}{\overline{\omega^+(t)}}\,\Omega^-(t) + g(t), \quad g(t) = 2f(t)\omega^+(t). \tag{3.6}$$

Note that any function of the form (3.5) has the property

$$\Omega(w) = \Omega_*(w), \tag{3.7}$$

where

$$\Omega_*(w) = \overline{\Omega(1/\overline{w})}, \quad |w| \neq 1. \tag{3.8}$$

Thus, we have to seek for a solution Ω of the problem (3.6) that satisfies the condition (3.7), where Ω_* is defined by the equality (3.8).

Since $p \in \mathscr{P}(\gamma)$, we have $\underline{p} > 1$ and therefore $H^{p(\cdot)} \subset H^1$. By the well-known property of functions from H^1 (cf., for example, [17, p. 39]), we have

$$\Psi(w) = \frac{1}{2\pi i}\int_\gamma \frac{\Psi^+(t)\,dt}{t-\gamma}, \quad w \in U, \quad \Psi^+ \in L^{p(\cdot)}(\gamma).$$

Lemma 3.1. *A function $\Omega_1(w) = \Omega(w) - \Omega(\infty) = \Omega(w) - \overline{\Psi(0)}$, $|w| \neq 1$, is representable by the Cauchy type integral with a density from $L^{p(\cdot)}(\gamma)$.*

Proof. Since $p \in \mathscr{P}(\gamma)$ and $\Omega_1(\infty) = 0$, the restrictions of the function Ω_1 on U and $C\overline{U}$ belong to $H^{p(\cdot)} \subset H^1$ and $H^{p(\cdot)}(C\overline{U}) \subset H^1(C\overline{U})$ respectively. Hence they are representable by Cauchy integrals in these domains and therefore

$$\Omega_1(w) = K_\gamma(\Omega_1^+ - \Omega_1^-)(w), \quad (\Omega_1^+ - \Omega_1^-) \in L^{p(\cdot)}(\gamma). \qquad \square$$

Definition 3.1. Denote by $\widetilde{K}^{p(\cdot)}(\gamma)$ the set of analytic functions Φ in the plane cut along γ that are representable in the form

$$\Phi(w) = \frac{1}{2\pi i} \int\limits_\gamma \frac{\varphi(t)\,dt}{t-w} + \text{const}, \quad |w| \neq 1, \quad \varphi \in L^{p(\cdot)}(\gamma). \tag{3.9}$$

By Lemma 3.1 we conclude that $\Omega \in \widetilde{K}^{p(\cdot)}(\gamma)$.

We have established that any solution Φ of the problem (3.2) obtained by means of the equalities (3.3) and (3.5) generates a solution of the problem (3.6) belonging to $\widetilde{K}^{p(\cdot)}(\gamma)$. Though, by Theorem B, the restriction of the solution $\Omega \in \widetilde{K}^{p(\cdot)}(\gamma)$ on U belongs to $H^{p(\cdot)}$. This fact, speaking in general, does not give the desired solution of the problem (3.4). For this it is necessary that the condition (3.7) be fulfilled. It is not difficult to verify that the restriction Ψ of any solution of such a kind on U gives the desired solution of the problem (3.4) and, in that case, the function $u(w) = \operatorname{Re}[\Psi(w)(\omega(w))^{-1}]$ is a solution of the problem (3.1).

If Ω is a solution of the problem (3.6) of class $\widetilde{K}^{p(\cdot)}(\gamma)$, then the function $\Omega_*(w)$, too, will be such a solution. But then the function $\Omega^+(w) = \frac{1}{2}(\Omega(w) + \Omega_*(w))$, too, which satisfies the condition (3.7), will be a solution of the problem (3.6).

Thus, the following theorem is valid.

Theorem 3.1. *If $p \in \mathscr{P}(\gamma)$ and $[\omega(w)]^{\pm 1} \in \bigcup\limits_{\delta>0} H^\delta$, then any solution of the problem* (3.1) *by means of the equalities* (3.3) *and* (3.5) *generates a solution Ω of the problem* (3.6) *that belongs to $\widetilde{K}^{p(\cdot)}(\gamma)$ and satisfies the condition* (3.7).

Conversely, if $\Omega \in \widetilde{K}^{p(\cdot)}(\gamma)$ is a solution of the problem (3.6) *that satisfies the condition* (3.7) *and Ψ is the restriction of Ω on U, then $u(w) = [\Psi(w)(\omega(w))^{-1}]$, $w \in U$, is a solution of the problem* (3.1).

All solutions of the problem (3.1) *are given by the equalities*

$$u(w) = \frac{1}{2} \operatorname{Re} \frac{\Omega(w) + \Omega_*(w)}{\omega(w)}, \quad u \in U, \tag{3.10}$$

where Ω is an arbitrary solution of the problem (3.6) *of class $\widetilde{K}^{p(\cdot)}(\gamma)$.*

Note that in this subsection we have followed the method used in [2, Sections 40 and 41].

3.3 Solution of the Dirichlet problem in the class $h^{p(\cdot)}(\omega)$ when $\omega(w) = (w-a)^\alpha \exp\left(\int\limits_\gamma \frac{\psi(\tau)\,d\tau}{\tau - w}\right)$, $a \in \gamma$, $\alpha \in \mathbb{R}$, $\operatorname{Im}\psi = 0$, $\psi \in C(\gamma)$

Keeping in mind further applications, we investigate the problem (3.6) under the following assumptions as to α:

I. $-\dfrac{1}{p(a)} < \alpha < \dfrac{1}{p'(a)}$,

II. $-\dfrac{1}{p'(a)} < \alpha < \dfrac{1}{p'(a)} + 1$,

III. $\alpha = \dfrac{1}{p'(a)}$,

IV. $\alpha = -\dfrac{1}{p(a)}$.

I. Let $-\dfrac{1}{p(a)} < \alpha < \dfrac{1}{p'(a)}$. Assume that

$$X(w) = \begin{cases} -\omega(w), & |w| < 1, \\ \overline{\omega\left(\frac{1}{\overline{w}}\right)}, & |w| > 1. \end{cases} \tag{3.11}$$

It is easy to verify that

$$[X(w)]^{-1} \in \widetilde{K}^{p'(\cdot)}(\gamma) \subset \widetilde{K}^{1}(\gamma).$$

Therefore, only the functions

$$\Omega(w) = \frac{X(w)}{2\pi i} \int\limits_{\gamma} \frac{g(t)}{X^{+}(t)} \frac{dt}{t-w} + CX(w), \tag{3.12}$$

where C is an arbitrary (complex) constant, can be a solution of the problem (3.6). Since $\omega^{+} \in W^{p(\cdot)}$ by assumption I (cf. Subsection 2.6), we have $\Omega \in \widetilde{K}^{p(\cdot)}(\gamma)$. Thus, the function

$$\begin{aligned} \Omega(w) + \Omega_{*}(w) = & \frac{X(w)}{2\pi i} \int\limits_{\gamma} \frac{g(t)}{X^{+}(t)} \frac{dt}{t-w} + \overline{\left(\frac{X\left(\frac{1}{\overline{w}}\right)}{2\pi i}\right)} \overline{\int\limits_{\gamma} \frac{g(t)}{X^{+}(t)} \frac{dt}{t-\frac{1}{\overline{w}}}} \\ & + [CX(w) + (CX(w))_{*}] \end{aligned} \tag{3.13}$$

is a solution satisfying the condition (3.7).

The condition $[CX(w)]_{*} = CX(w)$ implies $\operatorname{Re} C = 0$. From (3.10) and (3.13) we obtain

$$u(w) = \operatorname{Re} \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(t)}{t} \frac{t+w}{t-w}\, dt. \tag{3.14}$$

II. Let $\dfrac{1}{p'(a)} < \alpha < \dfrac{1}{p'(a)} + 1$. If

$$X_1(w) = X(w)(w-a)^{-1}, \tag{3.15}$$

where $X(w)$ is given by the equality (3.11), then the function $\Omega(w)[X_1(w)]^{-1}$ may have a pole of first order at infinity. Therefore, all solutions of the problem (3.6) lie in the set of functions given by the equality

$$\Omega(w) = \frac{X_1(w)}{2\pi i} \int\limits_{\gamma} \frac{g(t)}{X_1^+(t)} \frac{dt}{t-w} + X_1(w)P(w), \quad g(t) = 2f(t)\omega^+(t), \tag{3.16}$$

where $P(w)$ is an arbitrary polynomial of first order. Since $(\alpha - 1) \in \left(-\frac{1}{p(a)}, \frac{1}{p'(a)}\right)$ by assumption II, we have $X_1^+ \in W^{p(\cdot)}(\gamma)$. Therefore, the functions Ω given by the equality (3.16) are solutions of the problem (3.6) of class $\widetilde{K}^{p(\cdot)}(\gamma)$.

Solutions of the homogeneous problem (3.6) (i.e., the problem (3.6) with $f = 0$) are given by the function

$$\Omega_0(w) = X_1(w)(\delta w + \beta), \tag{3.17}$$

where δ and β are arbitrary complex numbers. Using the condition (3.7), we establish that the number β in (3.17) can be taken arbitrarily and $\delta \in \overline{\beta} a$ (cf. [13, p. 159]). Then

$$u_0(w) = M \operatorname{Re}\left[\frac{1}{\omega(w)}\left(\omega(w)\,\frac{w\overline{\beta}a + \beta}{w-a}\right)\right] = M \operatorname{Re} \frac{w\overline{\beta}a + \beta}{w-a},$$

where M is an arbitrary real constant. Hence (cf. [13, pp. 159–160])

$$u_0(w) = M \operatorname{Re} \frac{a+w}{a-w}\,. \tag{3.18}$$

According to (3.13), the function

$$u_f(w) = \operatorname{Re}\left[\frac{1}{w-a}\left(\frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(t)(t-a)}{t-w}\,dt - \frac{aw^2}{2\pi i}\int\limits_{\gamma} \frac{f(t)}{t}\,\frac{\overline{t}-\overline{a}}{t-w}\,dt\right)\right] \tag{3.19}$$

is a particular solution of the inhomogeneous problem (3.1).

III. Let $\alpha = \frac{1}{p'(a)}$. First we consider the homogeneous problem. If X is defined by the equality (3.11), then the function $F(w) = \Omega(w)[X(w)]^{-1}$ in the circle U belongs to the class H^η with $\eta > 0$, whereas, in $C\overline{U}$, we have $[F(w) - F(\infty)] \in H^\eta(C\overline{U})$. Moreover, $F^+(t) = F^-(t)$, $t \in \gamma$. By the same reasoning as in [13, pp. 161–162], F is an analytic function of the form

$$F(w) = \delta + \beta(w-a)^{-1},$$

where δ and β are arbitrary constants. But then all solutions of the homogeneous problem (3.6) lie in the set of functions

$$\Omega_0(w) = \delta X(w) + \beta(w-a)^{-1}X(w) = \delta X(w) + \beta X_1(w).$$

Let us consider two cases: (a) $X_1 \notin H^{p(\cdot)}$ and (b) $X_1 \in H^{p(\cdot)}$.

(a) In this case, Ω_0 may belong to $\widetilde{K}^{p(\cdot)}(\gamma)$ only when $\beta = 0$. Furthermore, the condition (3.7) yields the equality $\overline{\delta}X(w) = -\delta X(w)$, i.e., $\operatorname{Re}\delta = 0$. Thus, under condition (a), a general solution of the homogeneous problem (3.6) satisfying the condition (3.7) is given by the equality $\Omega_0(w) = \delta X(w)$, $\operatorname{Re}\delta = 0$. Therefore, (3.10) implies $u_0 = 0$.

(b) In this case, $\Omega_0(w) = X_1(w)P(w)$, where $P(w)$ is an arbitrary polynomial of first degree, and a solution of the homogeneous Dirichlet problem is given by the equality (3.18).

Let us return to the inhomogeneous problem.

Since $X_1^{-1} \in \widetilde{K}^{p'(\cdot)}(\gamma)$, all possible solutions lie in the set of functions given by the equality (3.13), where $X(w)$ is replaced by the function $X_1(w)$. Using the reasoning from [13, pp. 162–163], we see that the desired solution can be constructed by formula (3.13) only for those g for which the function

$$(Tg)(\zeta_0) = \frac{X_1^+(\zeta_0)}{2\pi i} \int\limits_\gamma \frac{g(\zeta)}{X_1^+(\zeta)} \frac{d\zeta}{\zeta - \zeta_0}, \quad \zeta_0 \in \gamma,$$

belongs to $L^{p(\cdot)}(\gamma)$. Since

$$X_1^+(\zeta) = (\zeta - a)^{-1}\omega^+(\zeta),$$

we therefore proved the following assertion.

Proposition 3.1. *If*

$$\omega(w) = (w - a)^{\frac{1}{p'(a)}} \exp\left(\int\limits_\gamma \frac{\psi(\tau)\, d\tau}{\tau - w}\right)$$

and a function f is such that

$$\frac{(\zeta_0 - a)^{-1}\omega^+(\zeta_0)}{\pi i} \int\limits_\gamma \frac{g(\zeta)}{\omega^+(\zeta)(\zeta - a)^{-1}} \frac{d\zeta}{\zeta - \zeta_0} \in L^{p(\cdot)}(\gamma), \quad g = 2f\omega^+, \tag{3.20}$$

then the problem (3.1) *is solvable and its solution is given by the equality*

$$u(w) = \operatorname{Re} \frac{1}{2\pi i} \int\limits_\gamma \frac{f(t)}{t} \frac{t + w}{t - w}\, dt + u_0(w),$$

where

$$u_0(w) = M(p) \operatorname{Re} \frac{a + w}{a - w}$$

and

$$M(p) = \begin{cases} 0 & \text{if } X_1(w) = (w - a)^{-\frac{1}{p(a)}} \exp\left(\displaystyle\int\limits_\gamma \frac{\psi(\tau)\, d\tau}{\tau - w}\right) \notin H^{p(\cdot)}, \\ \text{an arbitrary real constant } M & \text{if } X_1 \in H^{p(\cdot)}. \end{cases}$$

IV. Let $\alpha = -\dfrac{1}{p(a)}$. Then $[X(w)]^{-1} \in K^{p'(\cdot)}(\gamma)$ and thus we find that $\Omega_0 = \delta X$ and $u_0 = 0$. The inhomogeneous problem is solvable if and only if

$$\frac{\omega^+(\zeta_0)}{\pi i} \int\limits_\gamma \frac{g(\zeta)}{\omega^+(\zeta)} \frac{d\zeta}{\zeta - \zeta_0} \in L^{p(\cdot)}(\gamma), \quad g = 2f\omega^+. \tag{3.21}$$

Remark 3.1. The conditions (3.20) and (3.21) are equivalent to the condition

$$\frac{(\zeta_0 - a)^{-\frac{1}{p(a)}}}{\pi i} \int_\gamma \frac{g(\zeta)}{(\zeta - a)^{-\frac{1}{p(a)}}} \frac{d\zeta}{\zeta - \zeta_0} \in L^{p(\cdot)}(\gamma).$$

Let us summarize the obtained results.

Theorem 3.2. *Let $p \in \mathscr{P}(\gamma)$, and let*

$$\omega(w) = (w - a)^\alpha \exp\left(\int_\gamma \frac{\psi(\tau)\, d\tau}{\tau - w} \right), \quad a \in \gamma, \quad \operatorname{Im} \psi = 0, \quad \psi \in C(\gamma).$$

Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & u \in h^{p(\cdot)}(\omega), \\ u^+(\zeta) = f(\zeta), & f \in L^{p(\cdot)}(\gamma; \omega^+). \end{cases}$$

Then the following assertions hold.

I. *For $-1/p(a) < \alpha < 1/p'(a)$ the problem is uniquely solvable and its solution is given by the equality* (3.14)*.*

II. *For $1/p'(a) < \alpha < 1/p'(a) + 1$ the problem is solvable and its general solution contains an arbitrary real constant M,*

$$u(w) = u_f(w) + M \operatorname{Re} \frac{a + w}{a - w},$$

where u_f is the function defined by the equality (3.19)*.*

III. *For $\alpha = 1/p'(a)$ the problem is solvable only if the condition* (3.20) *is fulfilled and, in that case, the solution is defined by the equality*

$$u(w) = u_f(w) + M(p) \operatorname{Re} \frac{a + w}{a - w},$$

where $u_f(w)$ is given by formula (3.14) *and*

$$M(p) = \begin{cases} 0 & \text{if } X_1(w) = (w - a)^{-\frac{1}{p(a)}} \exp\left(\int_\gamma \frac{\varphi(\tau)\, d\tau}{\tau - w} \right) \notin H^{p(\cdot)}, \\ \text{an arbitrary real constant} & \text{if } X_1 \in H^{p(\cdot)}. \end{cases}$$

IV. *For $\alpha = -1/p(a)$ the problem is solvable if the condition* (3.21) *is fulfilled and, in that case, a solution is unique. The solution is given by the equality* (3.14)*.*

3.4 The Dirichlet problem in $h^{p(\cdot)}(\omega)$ for weight functions ω of the form (2.3)

Using the results of Subsection 3.3, we obtain the following assertion.

Theorem 3.3. *Let $p \in \mathscr{P}(\gamma)$ and ω be given by the equality (2.3), where*

$$-\frac{1}{p(a_k)} < \alpha_k < \frac{1}{p'(a_k)}\,, \quad k = \overline{1,m},$$

$$\frac{1}{p'(a_k)} < \alpha_k < \frac{1}{p'(a_k)} + 1\,, \quad k = \overline{m+1, m+j},$$

$$\alpha_k = \frac{1}{p'(a_k)}\,, \quad k = \overline{m+j+1, m+j+s},$$

$$\alpha_k = -\frac{1}{p(a_k)}\,, \quad k = \overline{m+j+s+1, n}.$$

Then for the Dirichlet problem to be solvable in the class $h^{p(\cdot)}(\omega)$, it is necessary and sufficient that the condition

$$\frac{\omega_1^+(\zeta)}{\pi i} \int\limits_\gamma \frac{g(\tau)}{\omega_1^+(\zeta)}\, \frac{d\tau}{\tau - \zeta} \in L^{p(\cdot)}(\gamma), \quad g = 2f\omega^+, \tag{3.22}$$

be fulfilled, where

$$\omega_1(w) = \rho(w)\omega(w), \quad \rho(w) = \begin{cases} \prod\limits_{k=m+1}^{m+j+s} (w - a_k)^{-1} & \text{if } j + s \geqslant 1, \\ 1 & \text{if } j + s = 0. \end{cases}$$

If (3.22) holds, then a general solution is given by the equality

$$u(w) = u_0(w) + u_f(w), \tag{3.23}$$

where

$$u_0(w) = \sum_{k=m+1}^{m+j+s} M_k(p) \operatorname{Re} \frac{a_k + w}{a_k - w}\,, \tag{3.24}$$

$$M_k(p) = \begin{cases} 0 & \text{if } k > m + j \text{ and } X_k \notin H^{p(\cdot)}, \\ & X_k(w) = (w - a_k)^{-\frac{1}{p(a_k)}} \exp \int\limits_\gamma \frac{\psi(\tau)\, d\tau}{\tau - w}\,, \\ \text{an arbitrary real constant } M_k & \text{if } m + 1 \leqslant k \leqslant m + j, \\ & m + j < k \leqslant m + j + s, \text{ and } X_k \in H^{p(\cdot)}, \end{cases}$$

$$u_f(w) = \operatorname{Re}\left[\frac{1}{\rho(w)}\left(\frac{1}{2\pi i} \int\limits_\gamma \frac{f(\zeta)\rho(\zeta)}{\zeta - w}\, d\zeta - \frac{(-1)^{j+1} w^{j+1}}{2\pi i} \prod_{k=m+1}^{m+j} a_k \int\limits_\gamma \frac{f(\zeta)}{\zeta}\, \frac{\overline{\rho(\zeta)}\, d\zeta}{\zeta - w}\right)\right], \tag{3.25}$$

where

$$\prod_{k=m+1}^{m+j} a_k = 1 \quad \text{if } j = 0.$$

4 The Dirichlet Problem in $h^{p(\cdot)}(C\overline{U};\widetilde{\omega})$

4.1 Class $H^{p(\cdot)}(C\overline{U};\widetilde{\omega})$

Let $C\overline{U} = \mathbb{C} \setminus \overline{U}$, and let

$$\widetilde{\omega}(w) = \left(\frac{w-a}{w-w_0}\right)^{\alpha} \exp \int\limits_{\gamma} \frac{\psi(\tau)\,d\tau}{\tau - w}\,, \qquad w \in C\overline{U}, \quad w_0 \in U, \quad \operatorname{Im}\psi = 0, \quad \psi \in C(\gamma). \tag{4.1}$$

We say that an analytic function Φ in $C\overline{U}$ belongs to the class $H^{p(\cdot)}(C\overline{U};\widetilde{\omega})$ if

$$\sup_{r>1} \int\limits_0^{2\pi} |\Phi(re^{i\vartheta})\widetilde{\omega}(re^{i\vartheta})|^{p(\vartheta)} r\, d\vartheta = \sup_{r>1} \int\limits_{|\zeta|=r} |\Phi(\zeta)\widetilde{\omega}(\zeta)|^{p(\zeta)}|d\zeta| < \infty. \tag{4.2}$$

Lemma 4.1. *Let $\Phi \in H^{p(\cdot)}(C\overline{U};\widetilde{\omega})$ and $\varphi(w) = \overline{\Phi\left(\frac{1}{\overline{w}}\right)}$, $|w| < 1$. Then $\varphi \in H^{p(\cdot)}\left(\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)\right)$ and $\varphi(0) = 0$. Conversely, if $\varphi \in H^{p(\cdot)}\left(\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)\right)$ and $\varphi(0) = 0$, then $\Phi(w) = \overline{\varphi\left(\frac{1}{\overline{w}}\right)}$, $|w| > 1$, belongs to the class $H^{p(\cdot)}(C\overline{U};\widetilde{\omega})$.*

Proof. Since $\widetilde{\omega}(\infty) \neq 0$, from (4.2) it follows that $\Phi(\infty) = 0$ and therefore $\varphi(0) = 0$. Consider the function

$$\varphi(w)\overline{\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)} = \overline{\Phi\left(\frac{1}{\overline{w}}\right)\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)}.$$

This function is analytic in U and

$$\begin{aligned}
\sup_{0<r<1} \int\limits_0^{2\pi} \left|\varphi(w)\overline{\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)}\right|^{p(\vartheta)} d\vartheta &= \sup_{0<r<1} \int\limits_0^{2\pi} \left|\Phi\left(\frac{1}{\overline{w}}\right)\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)\right|^{p(\vartheta)} d\vartheta \\
&= \sup_{0<r<1} \int\limits_0^{2\pi} \left|\Phi\left(\frac{e^{i\vartheta}}{r}\right)\widetilde{\omega}\left(\frac{e^{i\vartheta}}{r}\right)\right|^{p(\vartheta)} d\vartheta \\
&= \sup_{|\zeta|=\frac{1}{r}} \int\limits_0^{2\pi} |\Phi(\zeta)\widetilde{\omega}(\zeta)|^{p(\vartheta)} \frac{|d\zeta|}{|\zeta|} < \infty.
\end{aligned} \tag{4.3}$$

Therefore, $\varphi \in H^{p(\cdot)}\left(\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)\right)$.

Conversely, assume that $\varphi \in H^{p(\cdot)}\left(C\overline{U};\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)\right)$ and $\varphi(0) = 0$. Since $\underline{p} > 1$, from (4.3) it follows that

$$\Phi(w) = \overline{\varphi\left(\frac{1}{\overline{w}}\right)} \in H^{p(\cdot)}(C\overline{U};\widetilde{\omega}(w)).$$

□

4.2 The Dirichlet problem in $H^{p(\cdot)}(C\overline{U};\widetilde{\omega})$

Consider the following problem: Find a harmonic function V in $C\overline{U}$ satisfying the conditions

$$\begin{cases} \Delta V = 0, \\ V \in h^{p(\cdot)}(C\overline{U};\widetilde{\omega}) = \{V : V = \operatorname{Re}\Phi,\ \Phi \in H^{p(\cdot)}(C\overline{U};\widetilde{\omega})\}, \\ V\big|_{\gamma} = f, \quad f \in L^{p(\cdot)}(\gamma;\widetilde{\omega}^{+}), \end{cases} \tag{4.4}$$

where

$$\widetilde{\omega}(w) = \prod_{k=1}^{n} \left(\frac{w-a_k}{w-w_0}\right)^{\alpha_k} \exp\left(\int_{\gamma} \frac{\psi(\tau)\,d\tau}{\tau - w}\right), \quad \operatorname{Im}\psi = 0, \quad \psi \in C(\gamma).$$

We have

$$V = \operatorname{Re}\Phi, \quad \Phi \in H^{p(\cdot)}(C\overline{U};\widetilde{\omega}).$$

Hence $u(w) = V\left(\frac{1}{\overline{w}}\right)$ is a harmonic function in U and $u = \operatorname{Re}\overline{\Phi\left(\frac{1}{\overline{w}}\right)}$. According to Lemma 4.1, we have

$$u \in h^{p(\cdot)}\left(\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)\right).$$

It is easy to verify that

$$h^{p(\cdot)}\left(\widetilde{\omega}\left(\frac{1}{\overline{w}}\right)\right) = h^{p(\cdot)}(w),$$

where

$$\omega(w) = \prod_{k=1}^{n} (w-a_k)^{\alpha_k} \exp\left(\int_{\gamma} \frac{\psi(\tau)\,d\tau}{\tau - w}\right).$$

Thus, the function $u(w) = V\left(\frac{1}{\overline{w}}\right)$ satisfies the conditions

$$\begin{cases} \Delta u = 0, \quad u \in h^{p(\cdot)}(\omega(w)), \quad u(0) = 0, \\ u\big|_{\gamma} = f\left(\frac{1}{\overline{\tau}}\right) = f(\tau), \quad \tau \in \gamma. \end{cases} \tag{4.5}$$

If u is representable by a Poisson integral, then

$$u(0) = \frac{1}{2\pi}\int_{0}^{2\pi} u^{+}\,d\vartheta = \frac{1}{2\pi}\int_{0}^{2\pi} f(e^{i\vartheta})\,d\vartheta = 0.$$

Therefore, in this case, to solve the problem (4.5) and thereby the problem (4.4), it is necessary that the condition

$$\int_{0}^{2\pi} f(e^{i\vartheta})\,d\vartheta = 0 \tag{4.6}$$

be fulfilled.

If this condition is fulfilled, then, under the assumptions of Theorem 3.3 as to the weight function ω, a solution u of the problem (4.5) is given by (3.23)–(3.25) provided that (3.22) is fulfilled, whereas the solution of the problem (4.4) will be the function $V(w) = u\left(\frac{1}{\overline{w}}\right)$. In particular,

$$V_0(w) = u_0\left(\frac{1}{\overline{w}}\right) = \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{a_k + \frac{1}{\overline{w}}}{a_k - \frac{1}{\overline{w}}} = \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{a_k\overline{w}+1}{a_k\overline{w}-1}$$

$$= \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{a_k(\overline{w}_k+\overline{a}_k)}{a_k(\overline{w}_k-\overline{a}_k)} = \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{w+a_k}{w-a_k},$$

where

$$\sum_{k=1}^{n} M_k(p) = 0, \quad M_k(p) = 0 \quad \text{if } k \in \overline{0,m} \text{ or } k > m+j+s,$$

and $M_k(p)$ is defined according to Theorem 3.3 if $k = \overline{m+1, m+j+s}$.

We write solutions of the homogeneous problem (4.4) in the class $h^{p(\cdot)}(C\overline{U}_\rho; \widetilde{\omega})$, where $U_\rho = \{w : |w| > \rho\}$. Then $V_0(w) = u\left(\frac{\rho}{\overline{w}}\right)$ and therefore for $a_k = \rho e^{i\vartheta_k}$ we obtain

$$V_0(w) = \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{\frac{\rho}{\rho e^{-i\vartheta_k}} + \frac{\rho}{\overline{w}}}{\frac{\rho}{\rho e^{-i\vartheta_k}} - \frac{\rho}{\overline{w}}} = \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{e^{i\vartheta_k} + \frac{\rho}{\overline{w}}}{e^{i\vartheta_k} - \frac{\rho}{\overline{w}}}$$

$$= \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{\overline{w}+\rho e^{-i\vartheta_k}}{\overline{w}-\rho e^{-i\vartheta_k}} = \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{\overline{w}+\overline{a}_k}{\overline{w}-\overline{a}_k}$$

$$= \sum_{k=1}^{n} M_k(p)\operatorname{Re}\frac{w+a_k}{w-a_k},$$

$$\sum_{k=1}^{n} M_k(p) = 0.$$

5 The Dirichlet Problem in $e^{p(\cdot)}(D)$ in Simply Connected Domains with Piecewise Smooth Boundaries

Let $\Gamma \in C^1_D(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, $0 \leqslant \nu_k \leqslant 2$, and let $p \in \widetilde{\mathscr{P}}(\Gamma)$. Consider the following problem: Find a harmonic function $U(z)$ in D satisfying the conditions

$$\begin{cases} U \in e^{p(\cdot)}(D), \\ U^+(t) = f(t), \quad t \in \Gamma, \quad f \in L^{p(\cdot)}(\Gamma). \end{cases} \tag{5.1}$$

From the definition of $E^{p(\cdot)}(D)$ and $H^{p(\cdot)}(\omega)$ it follows that if $U(z) \in e^{p(\cdot)}(D)$, then the function $u(w) = U(z(w))$ belongs to the class $h^{p(z(\tau))}(\mathfrak{m})$, where

$$\mathfrak{m}(w) = |z'(re^{i\vartheta})|^{\frac{1}{p(z(\tau))}}, \quad w = re^{i\vartheta}, \quad \tau = e^{i\vartheta}, \tag{5.2}$$

and, conversely, if $u \in h^{p(z(\tau))}(\mathfrak{m})$, where $\mathfrak{m}$ is defined by (5.2), then $U(z) = u(w(z)) \in e^{p(t)}(D)$. Therefore, the problem (5.1) is equivalent to the following problem: Find a function if

$$\begin{cases} u \in h^{\ell(\cdot)}(\mathfrak{m}), \quad \ell(\tau) = p(z(\tau)), \\ u^+(\tau) = g(\tau), \quad g(\tau) = f(z(\tau)) \in L^{\ell(\cdot)}(\gamma; \mathfrak{m}^+). \end{cases} \tag{5.3}$$

In the sequel, to study the problem (5.3), we need that ℓ would belong to $\widetilde{\mathscr{P}}(\gamma)$. So, we assume that $p \in Q(\Gamma)$.

Thus, we study the problem (5.3) under the assumptions $\ell \in \widetilde{\mathscr{P}}(\gamma)$ and $g \in L^{\ell(\cdot)}(\gamma; \mathfrak{m}^+)$.

Let $t_k = z(a_k)$. Then for z' we have (cf. (2.4))

$$z'(w) \sim \prod_{k=1}^{n} (w - a_k)^{\nu_k - 1} \exp \int\limits_{\gamma} \frac{\psi(\tau)\, d\tau}{\tau - w}, \quad w \in U, \quad \operatorname{Im} \psi = 0, \quad \psi \in C(\gamma).$$

Together with the weight $\mathfrak{m}$ given by the equality (5.2), we consider the weight

$$\mathfrak{m}_0(w) = \prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{\ell(a_k)}} \exp \left(\frac{1}{\ell(\tau)} \int\limits_{\gamma} \frac{\psi(\xi)\, d\xi}{\xi - w} \right), \quad w = re^{i\vartheta}, \quad \tau = e^{i\vartheta}. \tag{5.4}$$

Lemma 5.1. *If* $\mathfrak{m}$ *and* $\mathfrak{m}_0$ *are given by* (5.2) *and* (5.4) *respectively, then* $\mathfrak{m} \sim \mathfrak{m}_0$ *and* $\mathfrak{m}_0^+ \in L^{\ell(\cdot)}(\gamma)$.

Proof. We have

$$F(w) = \frac{\mathfrak{m}(w)}{\mathfrak{m}_0(w)} = \prod_{k=1}^{n} (w - a_k)^{\left[\frac{1}{\ell(\tau)} - \frac{1}{\ell(a_k)}\right](\nu_k - 1)}.$$

In the proof of Proposition 2 in [15], it is shown that for $\ell \in \widetilde{\mathscr{P}}(\gamma)$, the functions $[F(w)]^{\pm 1}$ are bounded in U. Hence $\mathfrak{m} \sim \mathfrak{m}_0$. Let us show that $\mathfrak{m}_0^+ \in L^{\ell(\cdot)}(\gamma)$.

We have

$$I = |\mathfrak{m}_0^+(e^{i\vartheta})|^{\ell(e^{i\vartheta})} = |\mathfrak{m}_0^+(\tau)|^{\ell(\tau)}, \quad \tau = e^{i\vartheta}.$$

Therefore,

$$\begin{aligned} I &\leqslant \exp(2\pi \max \psi(\tau)) \left| \prod_{k=1}^{n} (\tau - a_k)^{\frac{\ell(\tau)(\nu_k - 1)}{\ell(a_k)}} \exp \int\limits_{\gamma} \frac{\psi(\zeta)\, d\zeta}{\zeta - \tau} \right| \\ &\leqslant M \left| \prod_{k=1}^{n} \exp \left(\frac{\ell(\tau)(\nu_k - 1)}{\ell(a_k)} \ln(\tau - a_k) \exp \int\limits_{\gamma} \frac{\psi(\zeta)\, d\zeta}{\zeta - \tau} \right) \right| \\ &= M \prod_{k=1}^{n} \left| \exp(\nu_k - 1) \ln(\tau - a_k) \exp \frac{\ell(\tau) - \ell(a_k)}{\ell(a_k)} (\nu_k - 1) \ln(\tau - a_k) \exp \int\limits_{\gamma} \frac{\psi(\zeta)\, d\zeta}{\zeta - \tau} \right) \right| \\ &= M \prod_{k=1}^{n} \left| (\tau - a_k)^{\nu_k - 1} \exp \int\limits_{\gamma} \frac{\psi(\zeta)\, d\zeta}{\zeta - \tau} \right| \exp \left| \frac{\ell(\tau) - \ell(a_k)}{\ell(a_k)} \ln(\tau - a_k) \right|. \end{aligned}$$

Since $\ell \in \widetilde{\mathscr{P}}(\gamma)$, we obtain

$$I \sim \left| \prod_{k=1}^{n} (\tau - a_k)^{\nu_k - 1} \exp \int\limits_{\gamma} \frac{\psi(\zeta)\, d\zeta}{\zeta - \tau} \right| \sim |z'(\tau)|.$$

But $z' \in H^1$. Therefore,

$$\int\limits_{0}^{2\pi} |\mathfrak{m}_0^+(e^{i\vartheta})|^{\ell(e^{i\vartheta})} d\vartheta < \infty$$

and thus $\mathfrak{m}_0^+ \in L^{\ell(\cdot)}(\gamma)$. The above arguments show that the lemma is valid. □

The above discussion leads to the following assertion.

Lemma 5.2. *If the domain D is bounded by a simple closed curve $\Gamma \in C_D^1(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$ and $p \in Q(\Gamma)$, then all solutions of the problem* (5.1) *are given by the equality $U(z) = u(w(z))$, where u is a solution of the problem*

$$\begin{cases} \Delta u = 0, \quad u \in h^{\ell(\cdot)}(\mathfrak{m}_0), \quad \ell(\tau) = p(z(\tau)), \\ u^+(\tau) = g(\tau), \quad g(\tau) = f(z(\tau)) \in L^{\ell(\cdot)}(\gamma; \mathfrak{m}_0^+), \end{cases} \tag{5.5}$$

where $\ell \in \widetilde{\mathscr{P}}(\gamma)$ and $\mathfrak{m}_0$ is defined by (5.4).

Lemma 5.3. *Let*

$$\omega_0(w) = \prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{\ell(a_k)}} \exp \int\limits_{\gamma} \frac{\varphi(\zeta)\, d\zeta}{\zeta - w}, \quad \varphi(\zeta) = \frac{\psi(\zeta)}{\ell(\zeta)}, \quad w \in U,$$

and let $\ell \in \widetilde{\mathscr{P}}(\gamma)$. Then $\mathfrak{m}_0(w) \sim \omega_0(w)$, $w \in U$.

Proof. We have

$$\exp\left(\frac{1}{\ell(\tau)} \exp \int\limits_{\gamma} \frac{\psi(\zeta)\, d\zeta}{\zeta - w} \right) = \exp \int\limits_{\gamma} \frac{\varphi(\zeta)}{\ell(\zeta)} \frac{d\zeta}{\zeta - w} \ \exp \int\limits_{\gamma} \frac{\ell(\zeta) - \ell(\tau)}{\ell(\zeta)\ell(\tau)} \frac{\varphi(\zeta)\, d\zeta}{\zeta - w}. \tag{5.6}$$

As is proved in [2, Section, pp. 50–52], if a function $\ell(\zeta)$ satisfies the Hölder condition, then the function

$$\int\limits_{\gamma} \frac{\ell(\zeta) - \ell(\tau)}{\zeta - w} d\zeta$$

uniformly tends to the limit

$$\int\limits_{\gamma} \frac{\ell(\zeta) - \ell(\tau)}{\zeta - \tau} d\tau$$

as $w \to \tau$. Owing to this proof, we easily establish that for $\ell \in \widetilde{\mathscr{P}}(\gamma)$

$$\int\limits_{\gamma} \frac{\ell(\zeta) - \ell(\tau)}{\zeta - w} \varphi(\zeta)\, d\zeta \to \int\limits_{\gamma} \frac{\ell(\zeta) - \ell(\tau)}{\zeta - \tau} \varphi(\zeta)\, d\zeta, \quad \varphi(\zeta) = \frac{\psi(\zeta)}{\ell(\zeta)},$$

uniformly with respect to $\tau \in \gamma$ as $w \to \tau$.

By virtue of this fact, from (5.6) we conclude that $(\mathfrak{m}_0 \omega_0^{-1})^{\pm 1}$ are bounded functions in U. Therefore, $\mathfrak{m}_0 \sim \omega_0$ in U. □

Since, by Lemma 5.3, in the problem (5.5) we can take ω_0 instead of $\mathfrak{m}_0$ and ω_0 has the form (2.3), Theorem 3.3 is applicable to the problem (5.5). If we assume that

$$\alpha_k = \frac{\nu_k - 1}{p(t_k)} \quad \left(= \frac{\nu_k - 1}{\ell(a_k)} \right),$$

then the following statement is valid.

Theorem 5.1. *Assume that*

(i) *D is a simply connected bounded domain with boundary $\Gamma \in C^1_D(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$,*

(ii) *$w = w(z)$ is the conformal mapping of the domain D onto the unit circle U with boundary γ,*

(iii) *$p \in Q(\Gamma)$,*

(iv) *ψ is a real-valued continuous function from the relation (2.4) for z' and*

$$\varphi(\tau) = \frac{\psi(\tau)}{\ell(\tau)}, \quad \ell(\tau) = p(z(\tau)), \quad \tau \in \gamma.$$

Assume also that

$$\begin{aligned}
&0 < \nu_1 < p(t_1), \dots, 0 < \nu_m < p(t_m), \\
&\nu_{m+1} > p(t_{m+1}), \dots, \nu_{m+j} > p(t_{m+j}), \\
&\nu_{m+j+1} = p(t_{m+j+1}), \dots, \nu_{m+j+s} = p(t_{m+j+s}), \\
&\nu_{m+j+s+1} = 0, \dots, \nu_n = 0.
\end{aligned}$$

Then the following assertions hold.

I. *If among the points t_k there are no points such that $\nu_k = p(t_k)$ or $\nu_k = 0$, then the problem (5.5) and therefore the problem*

$$\begin{cases} u(z) \in e^{p(\cdot)}(D), \\ u^+(z) = f(t), \quad t \in \Gamma, \quad f \in L^{p(\cdot)}(\Gamma), \end{cases}$$

is solvable and its solution contains j arbitrary real constants (i.e., as many as there are points at which $\nu_k > p(t_k)$). In that case, a solution of the problem is given by the equality

$$u(z) = u_f(z) + u_0(z), \tag{5.7}$$

where

$$u_0(z) = \sum_{k=m+1}^{m+j} M_k \operatorname{Re} \frac{w(t_k) + w(z)}{w(t_k) - w(z)}, \tag{5.8}$$

$$u_f(z) = \operatorname{Re}\left[\frac{1}{\rho(w(z))}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(z(\tau))\rho(\tau)}{\tau - w(z)}\,d\tau - \frac{(-1)^{j+1}}{2\pi i}\, w^{j+1}(z) \prod_{k=m+1}^{m+j} a_k \int_\gamma \frac{f(z(\tau))\overline{\rho(\tau)}}{\tau(\tau - w(z))}\,d\tau\right)\right], \tag{5.9}$$

where

$$\rho(w) = \prod_{k=m+1}^{m+j} (w - a_k)^{-1} \quad \textit{for } j \geqslant 1$$

if $\{k : \nu_k > p(t_k)\} = \varnothing$, *then* $\rho(w) = 1$, *and*

$$\prod_{k=m+1}^{m+j} a_k = 1 \quad \textit{for } j = 0,$$

M_k *are arbitrary real constants.*

In particular, if Γ *is a smooth curve or at all angular points* t_k *we have* $0 < \nu_k < p(t_k)$, $k = \overline{1,n}$, *then the problem is uniquely solvable and the solution is given by the equality*

$$u(z) = \operatorname{Re} \frac{1}{2\pi i} \int_\gamma \frac{f(z(\tau))}{\tau} \, \frac{\tau + w(z)}{\tau - w(z)}\,d\tau.$$

II. *If among the points* t_k *there are such points that* $\nu_k = p(t_k)$ *or* $\nu_k = 0$, *then, speaking in general, the problem is not solvable. For the problem to be solvable, it is necessary and sufficient that the function* $f(z(\tau))$ *satisfy the condition*

$$\frac{\omega^+(\zeta)}{2\pi i} \int_\gamma \frac{f(z(\tau))\,d\tau}{\tau - \zeta} \in L^{\ell(\cdot)}(\gamma), \tag{5.10}$$

where

$$\omega(w) = \prod_{k\in T} (w - w(t_k))^{-\frac{1}{p(t_k)}} \exp \int_\gamma \frac{\varphi(\tau)\,d\tau}{\tau - w},$$

$$T = \{k : \nu_k = p(t_k) \quad \textit{or} \quad \nu_k = 0\}.$$

If the condition (5.10) *is fulfilled, then the problem is solvable and its general solution is given by the equality* (5.7), *where* u_f *is defined by* (5.9), *whereas* $u_0(z)$ *is given by the equality*

$$u_0(z) = \sum_{k=m+1}^{m+j+s} M_k(p) \operatorname{Re} \frac{w(t_k) + w(z)}{w(t_k) - w(z)}, \tag{5.11}$$

where $M_{m+1}, \dots, M_{m+j}$ are arbitrary constants, and for $m+j < k \leqslant m+j+s$ we have

$$M_k(p) = \begin{cases} 0 & \text{if } X_k \notin H^{\ell(\cdot)} \text{ and} \\ & X_k(w) = (w-a_k)^{-\frac{1}{\ell(a_k)}} \exp \int\limits_{\gamma} \dfrac{\varphi(\tau)\,d\tau}{\tau - w}, \\ \text{an arbitrary constant } M_k & \text{if } X_k \in H^{\ell(\cdot)}. \end{cases} \tag{5.12}$$

III. *If, in addition to all the other assumptions of Theorem* 5.1*, it is assumed that* $\Gamma \in C_D^{1,L}(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$*, then for* $\nu_k = p(t_k)$ *we have* $X_k \notin H^{\ell(\cdot)}$ *and therefore, in that case,* $u_0(z)$ *is given by the equality* (5.8)*.*

In the formulation of assertion III, we took into account that $X_k \notin H^{\ell(\cdot)}$. This follows from Warschawski's theorem [19] by which

$$X_k(w) = (w-a)^{-\frac{1}{\ell(a_k)}} y(w),$$

where $[y(w)]^{\pm 1}$ are continuous functions.

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