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THE RIEMANN BOUNDARY VALUE PROBLEM IN VARIABLE EXPONENT SMIRNOV CLASS OF GENERALIZED ANALYTIC FUNCTIONS

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Abstract. The present paper studies the Riemann boundary value problem for generalized analytic in I. Vekua sense functions. The problem is formulated as follows: on the plane, cut along a simple, closed, rectifiable curve Γ, find the generalized analytic function $W(z)$ which in the domains G^+ and G^-, bounded by the curve Γ, belongs to the Smirnov classes with a variable exponent and $W^{\pm}(t)$ its boundary values almost for all $t \in \Gamma$ satisfy the condition

$$W^+(t) = a(t)W^-(t) + b(t),$$

where $a(t)$ and $b(t)$ are the given on Γ functions.

Various conditions of solvability are revealed and solutions (if any) are constructed.

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$$W^+(t) = a(t)W^-(t) + b(t),$$

სადაც $a(t)$ და $b(t)$-ზე მოცემული ფუნქციებია.

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1. Introduction

In the boundary value problems appearing in various fields of mathematics it is frequently required of the solution that unknown functions would belong to a certain Lebesque class (see, e.g., [2], [6], [7], [26], etc.)

Recently, the problems of pseudo-differential equations are being intensively studied in nonstandard Banach functional spaces, in particular, in the framework of variable exponent Lebesgue spaces. Such a statement of the problem is motivated by the fact that the classes of functions in definition of which the integration exponent is, generally speaking, a function, more precisely take into account local singularities of the given functions. Such spaces are natural ones in which we seek for solutions.

There is a vast literature devoted to the investigation of variable exponent Lebesgue spaces. It suffices to mention monographs [1], [3], [12] and references therein.

The works [3], [9], [11], [12], [22], [24], [25], etc. dealing with the boundary value problems for analytic and harmonic functions and related singular integral equations have been studied in the framework of variable exponent Lebesgue spaces.

In these problems regarding $p(t)$ it is more frequently assumed that $p(t) \in P(\Gamma)$, i.e., the conditions:

(a) there exists the number M such that for any t_1, $t_2 \in \Gamma$ we have

$$|p(t_1) - p(t_2)| < M|\ln|t_1 - t_2||^{-1}; \tag{1}$$

(b)

$$\min_{t\in\Gamma} p(t) = \underline{p} > 1, \tag{2}$$

are fulfilled.

Further, the generalized Cauchy type integral and the generalized singular integral have been investigated in [13], and Smirnov classes with a variable exponent $p(t)$ for generalized analytic functions have been introduced and studied in [21]. The results obtained in these works give every reason to investigate boundary value problems for generalized analytic functions when boundary values of unknown functions and those prescribed in the boundary conditions belong to $L^{p(t)}(\Gamma)$.

The Riemann problem for continuous statement has been considered in [17]. Smirnov classes for a constant p are studied in [18]. A number of problems in these classes have been investigeted in [14], [5], [7], [8], [15], [16], [18], [19].

In the present paper we investigate the Riemann problem which is formulated as follows.

Let Γ be a simple, closed, rectifiable curve dividing the plane $\mathbb{C}$ into two domains G^+ and G^-. Find such a generalized analytic in I. Vekua sense function W which

1) is a regular solution of the class $U^{s,2}(A,B,\mathbb{C})$ $s>2$ of equation;

$$LW=\partial_{\overline{z}}W+A(z)W(z)+B(z)\overline{W(z)}=0; \tag{3}$$

2) belongs to the class $K^{p(t)}(A;B;\Gamma)$, i.e., is representable by the generalized Cauchy type integral

$$W(z)=(\widetilde{K}_{\Gamma}\varphi)(z)=$$
$$=\frac{1}{2\pi}\int\limits_{\Gamma}\Omega_1(z,\tau)\varphi(\tau)d\tau-\Omega_2(z,\tau)\overline{\varphi}(\tau)d\overline{\tau},\quad \varphi\in L^{p(\cdot)}(\Gamma),\quad z\overline{\in}\Gamma; \tag{4}$$

3) the boundary functions $W^+(t)$ and $W^-(t)$ almost everywhere on Γ satisfy the condition

$$W^+(t)=a(t)W^-(t)+b(t), \tag{5}$$

where $b(t)\in L^{p(t)}(\Gamma)$.

Regarding Γ, $p(t)$ and $a(t)$, it is assumed that

(a) Γ is a curve of the class I^* containing, in particular, piece-wise smooth and Radon's curves without external peaks;

(b) $p(t)$ is the function of the class $\mathcal{P}(\Gamma)$;

(c) $a(t)$ belongs to the $A(p(t),\Gamma)$ class of measurable functions on Γ which is a natural generalization of I. Simonenko's class (see [26]).

Under the adopted assumptions, the generalized Cauchy type integrals (2) on the domains G^+ and G^- belong to the Smirnov classes $E^{p(\cdot)}(A;B;G^+)$ and $E^{p(\cdot)}(A;B;G^-)$, respectively [21].

A set of generalized analytic functions in the plane, cut along the closed curve Γ such that in the domains G^+ and G^- bounded by Γ they belong to the classes $E^{p(\cdot)}(A;B;G^{\pm})$, we denote by $PE^{p(\cdot)}(A;B;\Gamma)$. Such functions in the conditions (1), (a) and (b) are representable by the generalized Cauchy type integral in the domains G^+ and G^-, and therefore are representable by the Cauchy integral with density from $L^{p(\cdot)}(\Gamma)$ [21]. By virtue of the abovesaid, a picture of solvability of the Riemann problem in classes $K^{p(\cdot)}(A;B;\Gamma)$ and $PE^{p(\cdot)}(A;B;\Gamma)$ is the same.

2. Preliminaries

2.1. **The function of the class $L^{s,\nu}(G)$.** Let G be the domain in the plane $\mathbb{C}$, and $f(z)$ be the function of the class $L^s(G)$, $s>0$. We continue it on $\mathbb{C}\backslash G$ by zero and for the obtained function we preserve the notation $f(z)$. Assume $f_\nu(z)=z^\nu f\left(\frac{1}{z}\right)$.

A set of functions f for which

$$f\in L^s(\mathbb{C}),\quad f_\nu(z)\in L^s(U),\quad U=\{z:|z|<1\}. \tag{6}$$

we denote by $L^{s,\nu}(\mathbb{C})$ [27, p. 29].

2.2. **Regular solutions of equation (3).** We say that the function $W = W(z)$ is a regular solution in G of equation (3), if every point $z_0 \in G$ possesses a neighbourhood G_0 in which W has a generalized in Sobolev sense derivative $\partial_{\overline{z}} W = \frac{1}{2}\Big(\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y}\Big)$, $(z = x+iy)$ and almost everywhere in $G_0 - LW = 0$.

A set of regular solutions of equation (3), when $A, B \in L^{s,2}(G)$, we denote by $U^{s,2}(A;B;G)$.

For $s > 2$, every function $W \in U^{s,2}(A;B;G)$ is representable in the form

$$W = \Phi_W \exp \omega_W, \quad \omega_W(z) = \iint\limits_G \Big(A + B\frac{\overline{W}}{W}\Big)\frac{d\zeta d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta, \tag{7}$$

where Φ_W is holomorphic in G, ω_W belongs to the Hölder class $H_{\frac{s-2}{s}}(\mathbb{C})$, and $\omega_W(\infty) = 0$ [27, pp. 160–162].

The function Φ_W is called an analytic divisor and ω_W is a logarithmic difference of the generalized analytic function $W(z)$.

2.3. **The principal kernels.** Let $A, B \in L^{s,2}(G)$, $s > 2$, Φ be an analytic function in G and t be a fixed point from $\mathbb{C}$. It is proved in [27] (p. 175-7) that there exists a regular solution $W(z;t)$ of equation (3) such that: 1) $W_0 = \frac{W(z,t)}{\Phi(z)}$ is continuous in G and continuously extendable on $\mathbb{C}$; 2) $W_0(z) \neq 0$; 3) $W(t) = 1$; 4) $W_0(z)$ is holomorphic outside of G.

The operator which assigns to each pair Φ and t the function $W(z;t)$ we denote by $R_t^{A;B}(\Phi(z))$.

If $\Phi_1(z) = \frac{1}{2(t-z)}$, $\Phi_2(z) = \frac{1}{2i(t-z)}$ and $X_j(z,t) = R_t^{A,B}(\Phi_j(z))$, $j = 1,2$ are regular solutions of equation (3) in $\mathbb{C}\{t\}$, then the functions

$$\Omega_1(z,t) = X_1(z,t) + iX_2(z,t), \quad \Omega_2(z,t) = X_1(z,t) - iX_2(z,t)$$

are called the principal kernels of the class $U^{s,2}(A;B;G)$.

2.4. **Generalized polynomials.** A generalized polynomial of order n of the class $U^{s,2}(A;B;\mathbb{C})$ is called that regular solution of equation (3) whose analytic divisor is a classical polynomial of order n [27, p. 167].

Suppose

$$\nu'_{2k} = R_\infty^{-A,-\overline{B}}(z^k), \quad \nu'_{2k+1}(z) = R_\infty^{-A,-\overline{B}}(iz^k).$$

2.5. **The space $L^{p(\cdot)}(\Gamma)$.** For the measurable on Γ function $f(t)$ we put

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \int\limits_0^b \Big|\frac{f(t(\sigma))}{\lambda}\Big|^{p(t(\sigma))} d\sigma \leq 1\},$$

where $t = t(\sigma)$, $0 \le \sigma \le l$ is the equation of the curve Γ with respect to the arc abscissa σ. And let

$$L^{p(\cdot)}(\Gamma) = \{f : \|f\|_{p(\cdot)} < \infty\}.$$

For $p \in P(\Gamma)$, the set $L^{p(\sigma)}(\Gamma)$ with the norm $\|\cdot\|_{p(\cdot)}$ is the Banach space.

3. The Variable Exponent Smirnov Class

3.1. **Definition.** We say that the generalized analytic function W belongs to the class $E^{p(\cdot)}(A;B;G)$, if $W \in U^{s,2}(A;B;G)$, $s > 2$ and

$$\sup_{0<\rho<1} \int_0^{2\pi} |W(z(\rho e^{i\theta}))|^{p(\theta)} |z'(\rho e^{i\theta})| d\theta < \infty, \quad p(\theta) \equiv p(z(e^{i\theta})), \tag{8}$$

where $z = z(\rho e^{i\theta})$ is the comformal mapping of U onto G (for details on those classes, see [21]).

If $W \in E^{p(\cdot)}(A;B;G)$, $p \in P(\Gamma)$ then almost for all $t \in \Gamma$, there exists an angular boundary value $W^+(t)$, and the function $t \to W^+(t)$ belongs to $L^{p(\cdot)}(\Gamma)$.

It follows from the representation (7) that the belonging of W to the class $E^{p(\cdot)}(A;B;G)$ is equivalent to the fact that the function Φ_W belongs to the class $E^{p(\cdot)}(G)$, i.e.,

$$\sup \int_0^{2\pi} |\Phi_W(z(\rho e^{i\theta}))|^{p(\theta)} |z'(\rho e^{i\theta})| d\theta < \infty.$$

If G is an unbounded domain and there is the polynomial $Q(z)$ such that $[\Phi(z) - Q(z)] \in E^{p(\cdot)}(G)$, we write $\Phi \in \widetilde{E}^{p(t)}(G)$.

3.2. **Classes of functions representable by the generalized Cauchy type integral.** Let Γ be a simple rectifiable curve bounding the domains G^+ and G^-, $\Omega_1(z,t)$ and let $\Omega_2(z,t)$ be the kernels of the class $U^{s,2}(A;B;\mathbb{C})$, $f \in L(\Gamma)$. The function

$$(\widetilde{K}_\Gamma f)(z) = \int_\Gamma \Omega_1(z,t) f(t) dt - \Omega_2(z,t) \overline{f}(t) d\bar{t}, \quad z \overline{\in} \Gamma$$

is called the generalized Cauchy type integral [27, p. 198].

This function is a regular solution of (3) of the class $U^{s,2}(A;B;\mathbb{C})$.

Assume

$$K^{p(t)}(A;B;\Gamma) = \{W : W(z) = (\widetilde{K}_\Gamma f)(z), \ \ f \in L^{p(\cdot)}(\Gamma)\};$$

$$K^{p(t)}(\Gamma) = K^{p(t)}(0;0;\Gamma).$$

$\widetilde{K}^{p(t)}(A;B;\Gamma) = \{W : \exists$ polynomial $p_{_W} : W(z) = W_0(z) + p_{_W}(z),\ W_0 \in K^{p(t)}(A;B;\Gamma)\}$.

4. Classes of Curves

4.1. **Lavrent'ev's curves (of the class Λ).** The curve Γ belongs to the class Λ, if $\sup_{t_1,t_2\in\Gamma} s(t_1,t_2)[|t_1-t_2|^{-1}]<\infty$, where $s(t_1,t_2)$ is the length of the least of two arcs lying on Γ and joining t_1 and t_2.

4.2. **The class I_0.** I_0 is a set of curves Γ with the equation $t=t(\sigma)$, $0\le\sigma\le l$ (with respect to the arc abscissa) for which there exists a smooth curve with the equation $\mu=\mu(\sigma)$, $0\le\sigma\le l$ such that

$$\operatorname*{ess\,sup}_{0\le\sigma_0\le l}\int_0^l\Big|\frac{t'(\sigma)}{t(\sigma)-t(\sigma_0)}-\frac{\mu'(\sigma)}{\mu(\sigma)-\mu(\sigma_0)}\Big|d\sigma<\infty.$$

4.3. **The class I^*.** The simple curve Γ belongs to the class I^*, if $\Gamma\in\Lambda$ and it can be represented as a finite union of arcs of the class I_0, having tangents at the ends.

4.4. **Examples.** I^* contains piecewis-smooth and piecewise-Radonean curves without cusps (see [4], pp. 23-30, [1], pp. 146-7).

5. The class of functions $A(p(t),\Gamma)$.

A measurable function $a(t)$ belongs to the class $A(p(t),\Gamma)$, if

1) $0<m=\operatorname*{ess\,inf}_{t\in\Gamma}|a(t)|\le\operatorname*{ess\,sup}_{t\in\Gamma}|a(t)|=M<\infty$;

2) for every point $\tau\in\Gamma$, there exists the arc $\Gamma_\tau\subset\Gamma$ containing τ on which almost all values $a(t)$ lie inside of the angle with vertex at the origin, of size less than

$$\alpha_\tau=2\pi\Big[\sup_{t\in\Gamma_\tau}\max(p(t),q(t))\Big]^{-1},\quad q(\tau)=\frac{p(\tau)}{p(\tau)-1}.$$

For the function $a(t)$ from $A(p(t),\Gamma)$, following [26], we define a branch of the function $\arg a(t)$. We select a finite covering of Γ by the arcs $\Gamma_k=\Gamma_{\tau_k}$.

Let c be the point on Γ_{τ_1} at which there exists the tangent and the point $a(\sigma)$ lies inside of the angle of size α_{r_1}. We fix $(\arg a(c))^-\in[0,2\pi)$. Moving along γ, we define the value $\arg a(t)$ so as for t_1,t_2, lying on one of the arcs Γ_{τ_k}, to have $|\arg a(t_1)-\arg a(t_2)|<\alpha_{\tau_k}$. Going around Γ, the point c falls into Γ_{τ_1} with a new value $(\arg(c))^+$.

The number

$$\varkappa=\frac{1}{2\pi}\Big[(\arg a(c))^+-(\arg a(c))^-\Big] \tag{9}$$

is the integer, independent of the covering of Γ by the arcs Γ_k, and the choice of c. We call this number an index of the function $a(t)$ and write $\varkappa=\operatorname{ind}a(t)$.

For $p=\text{const}$, the class $A(p,\Gamma)$ coincides with the known I. Simonenko's class [26].

6. Statement of the Riemann problem.

When Γ is the Carleson curve bounding the domains G^+ and G^-, and

$$A, B \in L^{s,2}(G^+), \quad s > 2, \quad p \in P(\Gamma), \quad \overline{p} = \sup_{t \in \Gamma} p(t),$$

$$\overline{p}' = \frac{\overline{p}}{\overline{p} - 1}, \quad \frac{s}{2} > \overline{p}' \tag{10}$$

then as is proved in [21], the equality

$$K^{p(\cdot)}(A; B; G^+) = E^{p(\cdot)}(A; B; G^+) \tag{11}$$

holds.

In particular, inclusion (11) holds if

$$A, B \in L^\infty(G^+), \quad p \in P(\Gamma). \tag{12}$$

When

$$\Gamma \in I^*, \quad p \in P(\Gamma), \quad a \in A(p(t), \Gamma), \quad b \in L^{p(t)}(\Gamma) \tag{13}$$

problem (5) in the class $K^{p(t)}(\Gamma)$ has been investigated in [22].

Since when solving problem (5) in the class $K^{p(t)}(A; B; \Gamma)$, of importance for us is equality (11) and knowledge of a picture of its solvability in $K^{p(t)}(A; B; \Gamma)$, we will assume that the condition

$$\Gamma \in I^*, \quad A, B \in L^{s,2}(G), \quad s > 2, \quad p \in P(\Gamma), \quad \frac{s}{2} > \overline{p}', \tag{14}$$

or

$$\Gamma \in I^*, \quad A, B \in L^\infty(G), \quad p \in P(\Gamma), \tag{15}$$

is fulfilled.

In the first case, the choice for A, B is wide, but the set of admissible $p(t)$ is bounded by the condition $\frac{s}{2} > \overline{p}'$. In the second case, the set of A and B contracts, but now $p(t)$ is arbitrary from $P(\Gamma)$.

Thus, let condition (14) or (15) be fulfilled and we are required to find a generalized analytic function W which is a regular solution of equation (3), representable by the generalized Cauchy type integral with density $L^{p(\cdot)}(\Gamma)$ and almost everywhere on Γ equality (5) is valid.

When we say that W is a regular solution of problem (5), we regard that all the conditions adopted in this section for W are satisfied.

7. Solution of the Problem

7.1. One necessary condition of solvability. If

$$\Gamma \in I^*, \quad a \in A(p(t), \Gamma) \quad p \in P(\Gamma). \tag{16}$$

then the function

$$X(z) = \begin{cases} \exp h(z), & z \in G^+, \\ (z - z_0)^{-\varkappa} \exp h(z), & z_0 \in G^+, \quad z \in G^- \end{cases} \tag{17}$$

satisfies the following conditions: there exists $\delta > 0$ such that

$$X(z) \in \widetilde{E}^{p(t)+\delta}(G^{\pm}), \tag{18}$$

$$[X(z)]^{-1} \in \widetilde{E}^{q(t)+\delta}(G^{\pm}) \tag{19}$$

$$a(t) = X^{+}(t)[X^{-}(t)]^{-1} \tag{20}$$

(see [22]).

We write condition (5) in the form

$$W^{+}(X^{+})^{-1} - W^{-}(X^{-})^{-1} = b(X^{+})^{-1}, \tag{21}$$

and assume

$$V = W(X)^{-1}. \tag{22}$$

Lemma 1. *Let*

$$LW = \partial_{\overline{z}} W + AW + B\overline{W}, \quad L_1 V = \partial_{\overline{z}} V + AV + B\frac{\overline{X}}{X}\overline{V}.$$

If $LW = 0$, then $L_1 V = 0$, where V is given by equality (22). *Conversely, if $L_1 V = 0$ and $W = VX$, then $LW = 0$.*

Proof. Since $X(z)$ and $(X(z))^{-1}$ are the functions, analytic in G, it can be easily verified that

$$L_1 V = L_1 \frac{W}{X} = \frac{1}{X}(\partial_{\overline{z}} W + AW + B\overline{W}) = \frac{1}{X} LW.$$

From the above equality follow two statements of the lemma. □

Corollary 1. *If $W \in U^{s,2}(A; B; G)$, $s > 2$ then $V \in U^{s,2}(A; B\frac{\overline{X}}{X}; G)$.*

Corollary 2. *If V is the function given by equality* (22), *then*

$$V = \Phi_V \exp \omega_V, \tag{23}$$

where

$$\Phi_V = \frac{\Phi_W}{X}, \quad \omega_V = \omega_W. \tag{24}$$

Proof. We have

$$V = \frac{W}{X} = \frac{\Phi_W}{X} \exp \omega_W,$$

$$\omega_W = \iint\limits_G \Big(A + B\frac{\overline{W}}{W}\Big)\frac{d\xi d\eta}{\zeta - z} = \iint\limits_G \Big(A + B\frac{\overline{XV}}{XV}\Big)\frac{d\xi d\eta}{\zeta - z} =$$

$$= \iint\limits_G \Big(A + B\frac{\overline{X}}{X}\frac{\overline{V}}{V}\Big)\frac{d\xi d\eta}{\zeta - z} = \omega_V; \tag{25}$$

(We have used here the equality $W = VX$ and Corollary 1).

Thus,

$$V = \Phi_V \exp \omega_{_V} = \frac{\Phi_W}{X} \exp \omega_{_V}$$

and hence,

$$\Phi_V = \frac{\Phi_W}{X}. \tag{26}$$

Equalities (25) and (26) are just the provable by us equalities (24). □

Since $\Phi_W \in E^{p(\cdot)}(G^+)$ and $\frac{1}{X} \in \widetilde{E}^{q(\cdot)+\delta}(G^\pm)$ (see (19)), it follows from (26) that $\Phi_V \in E^{1+\varepsilon}(G^+)$, $\varepsilon > 0$ and hence $V \in E^{1+\varepsilon}\Big(A; B\frac{\overline{X}}{X}; G^+\Big)$. Behavior of the function $\phi_{_V}$ in the domain G^- depends on $\frac{1}{X}$.

If $\varkappa = \operatorname{ind} a \geq 0$, then it is easily seen from (17) that $\lim_{z\to\infty} V(z) = 0$ for $\varkappa = 0$ and $\lim_{z\to\infty} V(z) = \text{const}$ for $\varkappa = 1$.

For $\varkappa > 1$, the function V at the point $z = \infty$ admits the pole of order $\varkappa - 1$. Therefore there exist $\widetilde{\Phi} \in E^{1+\varepsilon}(G^-)$, $\varepsilon > 0$ and the polynomial $Q_{\varkappa-1}$ of order $\varkappa - 1$ such that

$$\Phi_V = \widetilde{\Phi} + Q_{\varkappa-1}.$$

By virtue of (21)–(22), we have

$$V^+ - V^- = \frac{b}{X^+}.$$

Since Φ_V and $\widetilde{\Phi}_V$ belong to $E^{1+\varepsilon}(G^\pm)$, then W belongs to the class $E^{1+\varepsilon}\Big(A; B\frac{\overline{X}}{X}; G^\pm\Big)$.

Let $\Omega_{1,1}(z,t)$ and $\Omega_{2,1}(z,t)$ be the principal kernels of the class $U^{s,2}\Big(A; B\frac{\overline{X}}{X}; G^\pm\Big)$ $s > 2$. Then

$$\frac{W(z)}{X(z)} = V(z) = \widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big) + V_{\varkappa-1}(z),$$

where

$$\widehat{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big) = \int\limits_\Gamma \Omega_{1,1}(z,t)\frac{b(t)}{X^+(t)} - \Omega_{2,1}(z,t)\Big(\overline{\frac{b(t)}{X^+(t)}}\Big)dt$$

where $\widehat{V}_{\varkappa-1}(z)$ is the generalized polynomial of order $\varkappa - 1$. This implies that one possible solution of problem (5) will be

$$W(z) = X(z)W_b(z) + X(z)\widehat{V}_{\chi-1}(z), \tag{27}$$

where

$$W_b(z) = \widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big).$$

Since

$$\frac{b}{X^+} \in L^{1+\eta}(\Gamma), \quad \eta > 0$$

we have

$$W = X\Big(\widetilde{K}_{\Gamma,1}\frac{b}{X^+}\Big) = X\Phi_{W_b}\exp\omega_{W_b} \in E^\eta\Big(A; B\frac{\overline{X}}{X}; G^\pm\Big). \tag{28}$$

Next,

$$W_b^+ = \frac{1}{2}\Big(b + X^+\widetilde{S}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)\Big), \quad W_b^- = \frac{1}{2a}\Big(-b + X^+\widetilde{S}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)\Big). \tag{29}$$

This implies that for the inclusion $W_b \in K^{p(\cdot)}(A;B;\Gamma)$ it is necessary that

$$(Tb)(t) = X^+(t)\widetilde{S}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)(t) \in L^{p(t)}(\Gamma). \tag{30}$$

Conversely, if (30) holds, then $W_b \in E^\eta\Big(A; B\frac{\overline{X}}{X}; G^\pm\Big)$ and $(W_b)^+ \in L^{p(t)}(\Gamma)$. According to the generalized Smirnov's theorem (see [17]), we will have $W_b \in E^{\widetilde{p}(t)}(A;B;G)$, where $\widetilde{p}(t) = \max(p(t),\eta) = p(t)$, i.e.,

$$W_b(z) \in E^{p(t)}(A;B;G^\pm). \tag{31}$$

Thus the following lemma is valid.

Lemma 2. *For problem* (5) *to be solvable for $\varkappa \geq 0$, it is necerssary and sufficient that inclusions* (30) *be fulfilled.*

Let condition (30) be fulfilled. Find out under what additional conditions W_b is a particular solution of problem (5) and construct its general solution. We consider separately the cases $\varkappa \geq 0$ and $\varkappa < 0$.

7.2. **The case $\varkappa \geq 0$.** By virtue of (31),

$$W_b(z) = \widetilde{K}_{\Gamma,1}(W_b^+ - W_b^-) = (\widetilde{K}_{\Gamma,1}f)(z), \quad f(t) = (W_b^+(t) - W_b^-(t)) \in L^{p(t)}(\Gamma).$$

Since $\Gamma \in I^*$; $p \in H(\Gamma)$ and $W_b(\infty) = 0$, we have $W_b(z) \in K^{p(\cdot)}\Big(A; B\frac{\overline{X}}{X}; \Gamma\Big)$ (see Corollary 1).

Therefore $(\widetilde{K}_{\Gamma,1}f)(z) \in E^{p(\cdot)}\Big(A; B\frac{\overline{X}}{X}; G^\pm\Big)$, and hence, W_b is the solution of problem (5). Now, to find its general solution, we have to solve the problem

$$V^+ - V^- = 0 \tag{32}$$

in the class of functions whose analytic divisor admits the representation $\Phi_v = \widetilde{\Phi}_v + Q_{\varkappa-1}$, $\widetilde{\Phi} \in E^{p(t)}(G^\pm)$.

It follows from (32) that $\widetilde{\Phi}_v^+ - \widetilde{\Phi}_v = 0$, and since $W_b \in K^{p(\cdot)}\Big(A, \frac{B\overline{X}}{X}, \Gamma\Big)$, therefore $\widetilde{\Phi}_v = 0$. Consequently, the solutions of (32) are the functions V for which analytic divisor is the polynomial $Q_{\varkappa-1}$.

We denote such a function by $\widehat{V}_{\varkappa-1}$. Then if condition (30) is fulfilled, a general solution of the problem is

$$W(z) = X(z)\widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)(z) + X(z)\widetilde{V}_{\varkappa-1}(z). \tag{33}$$

7.3. **The case $\varkappa < 0$.** In this case the only one possible solution of the problem may be only the function $W_b(z)$; however, for this function to be of the class $E^{p(\cdot)}(A;B;G^-)$, it is necessary and sufficient that the function $\widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)(z)$ at the point $z=\infty$ have zero of order $|\varkappa|$. For this to be so, it is necessary and sufficient that

$$\operatorname{Im}\int\limits_{\Gamma} u_k(t)b(t)dt = 0, \quad k = 0,1,\dots,2(1+|\varkappa|)-3, \tag{34}$$

where u_k are linearly independent solutions of the homogeneous problem

$$u^+(t) = \frac{1}{a(t)}u^-(t) \tag{35}$$

(see [4], p. 53).

Let us show that u_k belongs to $E^{q(t)}(G^\pm)$.

Since $\frac{1}{a(t)} \in A(q(t),\Gamma)$ and $\operatorname{ind}\frac{1}{a(t)} = -\varkappa > 0$, according to the result obtained in item 7.2, the solutions of problem (35) are given by the equality

$$u(z) = \frac{1}{X(z)}\widetilde{u}_{|\varkappa|-1},$$

where $\widehat{u}_{|\varkappa|-1}$ is the generalized polynomial of order $|\varkappa|-1$.

Consequently, the analytic divisor of the generalized analytic function $u(z)$ is $\frac{Q_{|\varkappa|-1}(z)}{X(z)}$.

By virtue of the fact that we have inclusion (19) and $\Phi_u(\infty)=0$, we can conclude that $\Phi_W(z) \in E^{q(t)}(G^\pm)$, and hence, $u(z) \in E^{q(\cdot)}(A;B;G^\pm)$. This implies that the function W_b under conditions (30) and (34) is the solution of problem (5).

7.4. **The main theorem.** From the results obtained in items 6.2 and 6.3 it follows that if condition (14) or (15) with respect to Γ, $p(t)$, $a(t)$, $b(t)$ are fulfilled, then for the Riemann problem considered in the class $K^{p(t)}(A;B;\Gamma)$ (or in $PE^{p(t)}(A;B;\Gamma)$), the theorem, analogous to that appearing in the classical assumptions and in the class $K^{p(t)}(\Gamma)$, is valid.

Theorem. *Let Γ be the simple closed curve bounding the domains G^+ and G^- and let the condition* (14) *or* (15) *be fulfilled. If, moreover, $a(t) \in \Lambda(p(t),\Gamma)$, $b(t) \in L^{p(\cdot)}(\Gamma)$ and $\varkappa = \operatorname{ind} a(t)$, then for problem* (5) *to be solvable in the class $K^{p(t)}(A;B;\Gamma)$, it is necessary and sufficient that the condition*

$$(\widetilde{T}b)(t) = X^+(t)\Big(\widetilde{S}_{\Gamma,1}\frac{b}{X^+}\Big)(t) \in L^{p(t)}(\Gamma)$$

be fulfilled, where $\widetilde{S}_{\Gamma,1}$ is the generalized Cauchy singular integral with principal kernels $\Omega_{1,1}$ and $\Omega_{1,2}$ of the class $U^{s,2}(A;B\frac{\overline{X}}{X};G^\pm)$, and $X(z)$ is the function given by equality (17).

If this condition is fulfilled, then:

(i) when $\varkappa = \operatorname{ind} a \geq 0$, the problem is solvable and its general solution is given by the equality

$$W(z) = X(z)\widetilde{K}\Big(\frac{b}{X^+}\Big)(z) + X(z)\widehat{V}_{\varkappa-1}(z),$$

where $\widetilde{V}_{\varkappa-1}(z)$ is an arbitrary generalized polynomial of order $(V_{\varkappa-1}(z) = 0)$;

(ii) when $\varkappa < 0$, then for the solvability of the problem it is necessary and sufficient that the condition $\widetilde{T}b \in L^{p(t)}(\Gamma)$ and

$$\operatorname{Im} \int\limits_{\Gamma} u_k(t)b(t)dt = 0, \quad k = 0, 1, \dots, 2(1 + |x|) - 3$$

be fulfilled, where u_k are linearly independent solutions of the class $K^{p(t)}(-A; -B\frac{\overline{X}}{X}; \Gamma)$ of the problem

$$u^+(t) = \frac{1}{a(t)}u^-(t).$$

Remark. If $b \in L^{p(t)+\delta}(\delta)$, $\delta > 0$ then $\widetilde{T}_b \in L^{p(t)}(\Gamma)$.

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