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ON VARIABLE EXPONENT HARDY CLASSES OF ANALYTIC FUNCTIONS

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Abstract. The paper studies the Hardy type classes $H^{p(t)}$ and $h^{p(t)}$ of analytic and harmonic functions respectively when a variable exponent $p(t)$ satisfies the log-continuity condition and its least value equals to one. Generalizations of the Fichtenholz, Smirnov and Tumarkin's theorems known for the classical Hardy classes are given. The Dirichlet problem is solved in the framework of spaces $H^{p(t)}$ in two different statements.

რეზიუმე. ნაშრომში შესწავლილია ანალიზურ და ჰარმონიულ ფუნქციათა ჰარდის ტიპის $H^{p(t)}$ და $h^{p(t)}$ კლასები, როდესაც ცვლადი მაჩვენებელი $p(t)$ აკმაყოფილებს ლოგარითმული უწყვეტობის პირობას და მისი უმცირესი მნიშვნელობა ტოლია ერთის. განზოგადებულია ფიხტენგოლცის, სმირნოვის და ტუმარკინის თეორემები, რომლებიც კარგადაა ცნობილი ჰარდის კლასიკური სივრცეების შემთხვევაში. ამოხსნილია დირიხლეს ამოცანა ორი სხვადასხვა დასმით ჰარმონიულ ფუნქციათა ჰარდის ცვლადმაჩვენებლიან კლასებში.

The interest in new functional spaces including those which involve Lebesgue integration with a variable exponent $p(t)$ has appreciably increased in the last two decades, and these spaces have become the subject of study by many mathematicians. This was motivated by the fact that investigation of applied problems in such classes allows one to consider local singularities of the given and unknown functions in more detail (see, e.g., [1]–[7] et al.)

In studying boundary value problems of the theory of analytic functions and certain problems for harmonic functions, the more fruitful turned out to be the notion of variable exponent Hardy classes suggested in [8].

Here we introduce some definitions.

Let $U = \{w : |w| < 1\}$ be a circle with the boundary $\gamma = \{t : |t| = 1\}$ and $p(t) = p(e^{i\sigma}) \equiv p(\sigma)$, $0 \le \sigma \le 2\pi$ be the given on γ positive measurable function.

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We say that an analytic in U function $\Phi(w)$ belongs to the class $H^{p(\cdot)}$, if

$$\sup_{0<r<1}\int_0^{2\pi}\left|\Phi(re^{i\sigma})\right|^{p(\sigma)}d\sigma<\infty; \tag{1}$$

analogously, a harmonic function $u(w)$ belongs to the class $h^{p(\cdot)}$, if

$$\sup_{0<r<1}\int_0^{2\pi}\left|u(re^{i\sigma})\right|^{p(\sigma)}d\sigma<\infty. \tag{2}$$

Assume

$$\widetilde{h}^{p(\cdot)}=\left\{u:\exists\,\Phi\in H^{p(\cdot)}\ u(w)=\operatorname{Re}\Phi(w),\ w\in U\right\}.$$

In the most of the above-mentioned works it is assumed that $p(t)$ satisfies the following conditions:

(1) there exists the constant $C(p)$ such that for any $t_1,t_2\in\gamma$,

$$\left|p(t_1)-p(t_2)\right|<C(p)\left|\ln(t_1-t_2)\right|^{-1}; \tag{3}$$

(2) $\min\limits_{t\in\gamma}p(t)=\underline{p}>1$.

A set of such functions we denote by $\mathcal{P}(\gamma)$.

The class of functions $p(t)$ for which (3) holds and

(2′) $\min\limits_{t\in\gamma}p(t)=\underline{p}=1, \quad (4)$

we denote by $\mathcal{P}_1(\gamma)$.

The classes indicated in [8]–[11] have been investigated under the assumption that $p\in\mathcal{P}(\gamma)$. However, from the point of view of applications, it is desirable to maintain the case $\underline{p}=1$.

In the present paper we present some properties of functions from the classes $H^{p(\cdot)}$, $h^{p(\cdot)}$ and $\widetilde{h}^{p(\cdot)}$ for $p\in\mathcal{P}_1(\gamma)$. The classes Hardy are considered in the domain $U^-=\{w:|w|>1\}$, as well. It turns out that for $p\in\mathcal{P}(\gamma)$ the equality

$$h^{p(\cdot)}\approx\widetilde{h}^{p(\cdot)}$$

holds. For $p\in\mathcal{P}_1(\gamma)$, this is, generally speaking, false (the corresponding example can be found in item 4.2).

In the final part of the present work we consider the Dirichlet problem in two different statements:

I. Find a harmonic function $u(w)$ of the class $\widetilde{h}^{p(\cdot)}$ such that almost everywhere on γ we have

$$u^+(t)=b(t). \tag{5}$$

II. The certain new class of functions $V \subset L^{p(\cdot)}(\gamma)$ which is invariant with respect to the Cauchy singular operator

$$S : b \to Sb, \ \ (Sb)(t) = \frac{1}{\pi i} \int_{\gamma} \frac{b(\tau)d\tau}{\tau - t} \, ,$$

i.e.,

$$S(V) = V$$

has been introduced in [12].

We consider the problem: find the function u from the set

$$\widetilde{h}^{p}(\gamma; U) = \Big\{ u : \exists\, \Phi \in H^{p(\cdot)}, \quad \Phi(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{\Phi^{+}(\tau)d\tau}{\tau - w} \, , \ \Phi^{+} \in L^{p(\cdot)}(\gamma), \ u = \operatorname{Re}\Phi \Big\}$$

for which equality (5) holds.

We prove that for problem (5) to be solvable in the first statement, it is necessary and sufficient that

$$b(t) \in L^{p(\cdot)}(\gamma), \quad (Sb)(t) \in L^{p(\cdot)}(\gamma). \tag{6}$$

The problem in the second statement is solvable for any $b \in V$.

In both cases we have a unique solution.

2. Preliminaries

2.1. The Class $L^{p(\cdot)}(\gamma)$. Let $p(t)$ be a positive measurable function on γ. For the measurable on γ function $f(\tau) = f(e^{i\sigma})$, $0 \le \sigma \le 2\pi$ we put

$$\|f\|_{p(\cdot)} = \inf \Big\{ \lambda > 0 : \int_0^{2\pi} \Big| \frac{f(e^{i\sigma})}{\lambda} \Big|^{p(\sigma)} d\sigma \le 1, \, p(\sigma) = p(e^{i\sigma}) \Big\}.$$

Let

$$L^{p(\cdot)}(\gamma) = \big\{ f : \|f\|_{p(\cdot)} < \infty \big\}.$$

2.2. The Hardy Classes $H^{p(\cdot)}(U^{-})$.

Definition. We say that the function $\Phi(w)$, analytic in the domain $U^{-} = \{w : |w| > 1\}$, belongs to the class $H^{p(\cdot)}(U^{-})$, if

$$\sup_{R>1} \int_0^{2\pi} \big| \Phi(\mathrm{Re}^{i\sigma}) \big|^{p(\sigma)} R \, d\sigma < \infty.$$

For $p \equiv 1$, we write $H^{1}(U^{-})$.

2.3. **Classes of Functions Representable by the Cauchy Type Integral.** By $K^{p(\cdot)}(\gamma)$ we denote a set of functions $\Phi(w)$, analytic in the plane, cut along γ, and representable in the form of the integral

$$\Phi(w) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\varphi(\tau)}{\tau - w}\,d\tau = (K_\gamma\varphi)(w), \quad w\overline{\in}\gamma, \quad \varphi \in L^{p(t)}(\Gamma). \tag{7}$$

3. Some Properties of Hardy Class Functions

3.1. **The existence of boundary values.** Relying on the Fatou's lemma, it is not difficult to prove that functions of classes $h^{p(\cdot)}$ and $H^{p(\cdot)}$ for almost all points $t \in \gamma$ possess an angular boundary value, and the boundary functions belong to $L^{p(\cdot)}(\gamma)$.

3.2. **The condition for belonging of analytic function to the class $H^{p(\cdot)}$.**

Theorem 1. *Let $p \in \mathcal{P}_1(\gamma)$. If the analytic in U function $\Phi(w)$ is representable by one of the formulas*

$$\Phi(w) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\Phi^+(\tau)}{\tau - w}\,d\tau, \quad w \in U, \quad (\textit{the Cauchy formula}), \tag{8}$$

or

$$\Phi(w) = \Phi(r\,e^{i\vartheta}) = \frac{1}{2\pi}\int\limits_0^{2\pi} \Phi^+(e^{i\sigma})\frac{1-r^2}{1+r^2-2r\cos(\sigma-\vartheta)}\,d\sigma, \tag{9}$$

$$(\textit{the Poisson formula}),$$

where $\Phi^+ \in L^{p(\cdot)}(\gamma)$, then it is representable by another formula, as well.

A set of such functions coincides with the class $H^{p(\cdot)}$.

Proof. We make use of the following result from [6].

If f is 2π-periodic function from $L^{p(\cdot)}(T)$, $T = [0, 2\pi]$, $p \in \mathcal{P}_0(\gamma)$, then for the Poisson integral

$$u_f(r,\vartheta) = \frac{1}{2\pi i}\int\limits_0^{2\pi} f(e^{i\sigma})\frac{1-r^2}{1+r^2-2r\cos(\sigma-\vartheta)}\,d\sigma$$

the estimate

$$\big\|u_f(r,\vartheta)\big\|_{p(\cdot)} \le M\|f\|_{p(\cdot)} \tag{10}$$

is valid, where M does not depend on f.

This implies that for almost all $t \in \gamma$ there exists an angular limit $u^+(t)$ which is equal to $t(e^{i\vartheta})$, and

$$\lim_{r\to 1}\big\|u_f(re^{i\vartheta}) - f(e^{i\vartheta})\big\|_{p(\cdot)} = 0,$$

hence,

$$u_+(re^{i\vartheta}) \in h^{p(\cdot)}, \quad p \in \mathcal{P}_0(\gamma). \tag{11}$$

Let now (9) hold, where $\Phi = u + iv$ and $\Phi^+ = (u^+ + iv^+) \in L^{p(\cdot)}(\gamma)$, then $u(r,\vartheta) = u_{\operatorname{Re}\Phi^+}(r,\vartheta)$, $v(r,\vartheta) = u_{\operatorname{Im}\Phi^+}(r,\vartheta)$ and by virtue of equality (11), we have $u \in h^{p(\cdot)}$, $v \in h^{p(\cdot)}$. Thus $\Phi \in H^{p(\cdot)} \subset H^1$, and according to the Fichtenholz theorem, Φ is representable by formula (8), where $\Phi^+ \in L^{p(\cdot)}(\gamma)$.

If (8) is valid, then $\Phi \in H^1$ and $\Phi^+ \in L^{p(\cdot)}(\gamma)$. Again, by virtue of Fichtenholz theorem, formula (9), where $\Phi^+ \in L^{p(\cdot)}(\gamma)$, is valid according to the assumption, and $\Phi \in H^{p(\cdot)}$, by the above proven.

If $\Phi \in H^{p(\cdot)}$, then it belongs to H^1 and $\Phi^+ \in L^{p(\cdot)}(\gamma)$ (see item 3.1). This implies that both equalities (8) and (9) are valid. □

3.3. **On the functions of the class $H^1(U^-)$.** (a) If the analytic in U^- function $\Phi(w)$ belongs to $H^1(U^-)$, then the function $F(\zeta) = \Phi(\frac{1}{\zeta})$, $\zeta \in U$ belongs to H^1, and $F(0) = 0$.

Conversely, if $F(0) = 0$ and $F(\zeta) \in H^1$, then $\Phi(\zeta) = F(\frac{1}{\zeta})$, $\zeta \in U^-$ belongs to $H^1(U^-)$.

(b) For the analytic in U^- function $\Phi(w)$ to belong to $H^1(U^-)$, it is necessary and sufficient that it be representable by the Cauchy integral

$$\Phi(w) = -\frac{1}{2\pi i}\int\limits_\gamma \frac{\Phi^-(\tau)d\tau}{\tau - w}, \quad w \in U^-. \tag{12}$$

3.4. **On the representability of a pair of functions given on U and U^- by the Cauchy type integral.**

(a) If $\Phi_1 \in H^1$, $\Phi_2 \in H^1(U^-)$, then the function

$$F(w) = \frac{1}{2\pi i}\int\limits_\gamma \frac{\Phi_1^+(\tau) - \Phi_2^-(\tau)}{\tau - w}\, d\tau, \quad w \overline{\in} \gamma$$

coincides both with $\Phi_1(w)$ for $w \in U$ and with $\Phi_2(w)$ for $w \in U^-$. If, however, $\Phi_1 \in H^{p(\cdot)}$, $\Phi_2 \in H^{p(\cdot)}(U^-)$, then $F \in K^{p(\cdot)}(\gamma)$.

(b) If $\Phi_1 \in H^1$, $\Phi_2 \in H^1(U^-)$ and almost for all $t \in \gamma$ we have

$$\Phi_1^+(t) = \Phi_2^-(t),$$

then $\Phi_1(w) = 0$, $w \in U$, $\Phi_2(w) = 0$, $w \in U^-$.

3.5. **On the classes $h^{p(\cdot)}$ and $\widetilde{h}^p$.**

Theorem 2. *If $p \in \mathcal{P}(\gamma)$, then*

$$h^{p(\cdot)} = \widetilde{h}^{p(\cdot)}. \tag{13}$$

Proof. The fact that $\widetilde{h}^{p(\cdot)} \subset h^{p(\cdot)}$ is obvious. Let us prove that $h^{p(\cdot)} \subset \widetilde{h}^{p(\cdot)}$.

Let $u \in h^{p(\cdot)}$, then $u \in h^{\underline{p}}$, $\underline{p} > 1$; by the known Riesz theorem, the function v, harmonically conjugate to u, likewise belongs to $h^{\underline{p}}$, hence

$$\Phi(w) = [u(w) + iv(w)] \in H^{\underline{p}} \subset H^1.$$

By the Fichtenholz theorem,

$$\Phi(w) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{u(\tau) + iv(\tau)}{\tau - w}\, d\tau,$$

where $(u + iv) \in L^{p(\cdot)}(\gamma)$, $p \in \mathcal{P}(\gamma)$. But the Cauchy type integral $\frac{1}{2\pi i}\int_\gamma \frac{f(\tau)d\tau}{\tau - w}$, $w \in U$ for $f \in L^{p(\cdot)}(\gamma)$ belongs to $H^{p(\cdot)}$ (see [11], p. 76). Consequently, $\Phi \in H^{p(\cdot)}$, and $u = \operatorname{Re}\Phi$, i.e., $u \in \widetilde{h}^{p(\cdot)}$. □

3.6. **Generalization of one Smirnov's theorem.** The following Smirnov's theorem is well known [13].

Theorem. *If $\Phi \in H^p$, $\Phi^+ \in L^{p_1}(\gamma)$, $p_1 > p$, then $\Phi \in H^{p_1}$.*

For the variable p, the theorem below is valid.

Theorem 3. *If $\Phi \in H^{p(\cdot)}$, $\underline{p} > 0$ and $\Phi^+ \in L^{\mu(\cdot)}(\gamma)$, $\mu \in \mathcal{P}_1(\gamma)$, then $\Phi \in H^{\lambda(\cdot)}$, where $\lambda(t) = \max(p(t), \mu(t))$.*

In [11] (p. 76), this theorem has been proved under the assumptions $\underline{p} > 0$, $\mu \in \mathcal{P}(\gamma)$.

Proof. Let $\Phi(z) \in H^{p(\cdot)}$, $\underline{p} > 0$ and $\Phi \in L^{\mu(\cdot)}(\gamma)$. This implies that $\Phi^+ \in L^1(\gamma)$; consequently, $\Phi(z) \in H^1$. Then $\Phi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\Phi^+(\tau)d\tau}{\tau - z}$. Here $\Phi^+(t) \in L^{\mu(t)}(\gamma)$, where $\mu(t) \in \mathcal{P}_1(\gamma)$. By Theorem 1, we conclude that $\Phi \in H^{\mu(t)}$. Thus $\Phi \in H^{p(\cdot)}$ (by the assumption) and $\Phi \in H^{\mu(\cdot)}$ (by the above proven), hence $\Phi \in H^{\mu(t)}$. □

3.7. **On the convergence of a function sequence from $H^{p(\cdot)}$, $p \in \mathcal{P}_0(\gamma)$.**

Theorem 4. *Let $\{\Phi_n(\zeta)\}$ be a sequence of boundary values of functions $\Phi_n(z) \in H^{p(\cdot)}$, $p \in \mathcal{P}_1(\gamma)$ and*

$$\int\limits_{\gamma} \big|\Phi_n(\zeta)\big|^{p(\zeta)} |d\zeta| = \int\limits_0^{2\pi} \big|\Phi_n(\zeta^{i\vartheta})\big|^{p(\vartheta)} d\vartheta < C, \quad p(\vartheta) = p(e^{i\vartheta}),$$

where ζ is independent of n.

If $\{\Phi_n(\zeta)\}$ converges in measure on γ, then the sequence $\{\Phi_n(\zeta)\}$ converges uniformly in U to some function $\Phi(z)$ of the class $H^{p(\cdot)}$, and $\{\Phi_n(\zeta)\}$ converges in measure on γ to the function $\Phi^+(\zeta)$.

Proof. We have

$$\int_\gamma \left|\Phi_n(\zeta)\right||d\zeta| = \int_{\{\zeta:|\Phi_n(\zeta)|\le 1\}} \left|\Phi_n(\zeta)\right||d\zeta| + \int_{\{\zeta:|\Phi_n(\zeta)|>1\}} \left|\Phi_n(\zeta)\right||d\zeta| \le$$

$$\le 2\pi + \int_\gamma \left|\Phi_n(\zeta)\right|^{p(\zeta)}|d\zeta| \le 2\pi + C.$$

Using Tumarkin's theorem ([14], p. 263-9) (in which it is stated that the provable theorem is valid for $p = \text{const}$), we conclude that $\{\Phi_n(\zeta)\}$ converges in U to some function $\Phi \in H^1$. Let us show that $\Phi^+ \in L^{p(\cdot)}(\gamma)$.

From the converging in measure on γ sequence $\{\Phi_n(\zeta)\}$ we select the subsequence $\{\Phi_n(z)\}$, converging almost everywhere on γ. Then $|\Phi_{n_k}(e^{i\vartheta})|^{p(\vartheta)}$ converges almost everywhere on γ to the function $|\Phi(e^{i\vartheta})|^{p(\vartheta)}$. By the Fatou's lemma, we obtain

$$\int_\gamma \left|\Phi^+(\zeta)\right|^{p(\zeta)}|d\zeta| = \int_\gamma \lim_{k\to\infty} \left|\Phi_{n_k}(\zeta)\right|^{p(\zeta)}|d\zeta| \le \int_\gamma \left|\Phi_{n_k}(\zeta)\right|^{p(\zeta)}|d\zeta| < C.$$

Thus $\Phi \in H^1$, $\Phi^+ \in L^{p(\cdot)}(\gamma)$. Hence

$$\Phi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\Phi^+(t)dt}{t-z}, \quad \Phi^+ \in L^{p(\cdot)}(\gamma).$$

By Theorem 1, $\Phi(z) \in H^{p(\cdot)}$. □

Theorem 4 is a partial generalization of G. Tumarkin's theorem (see [14], p. 268–269).

4. The Dirichlet Problem in the Class $\widetilde{h}^{p(\cdot)}$

4.1. For $p \in \mathcal{P}(\gamma)$, the Dirichlet problem is solved in the class $h^{p(\cdot)}$ for $b \in L^{p(\cdot)}(\gamma)$ (see, e.g., [11], p. 219). The solution is unique and representable by the Poisson integral.

When $p \in \mathcal{P}_1(\gamma)$, situation changes in the main. Here we have the following

Theorem 5. *Let $p \in \mathcal{P}_1(\gamma)$; for the solvability of the Dirichlet problem in the class $\widetilde{h}^{p(\cdot)}$, that is, for the existence of the function $u(w)$ which is the real part of some function from $H^{p(\cdot)}$ and*

$$u^+(t) = b(t) \tag{14}$$

it is necessary and sufficient that the conditions

$$b(t) \in L^{p(\cdot)}(\gamma), \quad (Sb)(t) \in L^{p(t)}(\gamma). \tag{15}$$

be fulfilled. If these conditions are fulfilled, then the Dirichlet problem in the class $\widetilde{h}^{p(\cdot)}(\gamma)$ *is uniquely solvable and the solution* $u(w)$ *is given by the equality*

$$u(w) = \Re \frac{1}{2\pi} \int_{\gamma} b(\tau) \frac{\tau + w}{\tau - w} \frac{d\tau}{\tau}\,, \tag{16}$$

or what us the same,

$$u(w) = \frac{1}{2\pi} \int_{\gamma} b(e^{i\sigma}) \frac{1 - r^2}{1 + r^2 - 2r\cos(\sigma - \vartheta)}\, d\sigma, \quad w = re^{i\vartheta}. \tag{17}$$

Proof. The necessity. We use the following result.

If $\Phi(w) \in H^1$, then it is representable in the form

$$\Phi(w) = \frac{1}{2\pi} \int_{\gamma} \operatorname{Re} \Phi^+(\tau) \frac{\tau + w}{\tau - w} \frac{d\tau}{\tau} + i \operatorname{Im} \Phi(0). \tag{18}$$

(This statement is well-known for the functions Φ, analytic in U and continuous in $\overline{U}$. In the above formulation, this statement can be found in [15] (see also [11], p. 11)).

Thus, let $u(w) \in \widetilde{h}^{p(\cdot)}$ and satisfy the condition (14), then there exists the function $\Phi(w) \in H^{p(\cdot)} \subset H^1$ such that $u(w) = \operatorname{Re} \Phi(w)$.

By virtue of the statement from item 3.2, a solution $u(w)$ may be only the function given by equality (16). For this function to be a solution, it is necessary that the function

$$\Phi_b(w) = \frac{1}{2\pi} \int_{\gamma} b(\tau) \frac{\tau + w}{\tau - w} \frac{d\tau}{\tau}$$

belongs to H^1, i.e., the equality

$$\Phi_b(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{\Phi_b^+(\tau) d\tau}{\tau - w}$$

be valid.

Since

$$\Phi_b(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{2b(\tau) d\tau}{\tau - w} - \frac{1}{2\pi i} \int_{\gamma} \frac{b(\tau)}{\tau}\, d\tau, \tag{19}$$

by virtue of Sokhotskii-Plemelj formula we, find

$$\Phi_b^+(t) = b(t) + (Sb)(t) + \text{const}\,. \tag{20}$$

By the statement from item 3.1, we should have $[b(t) + (Sb)(t)] \in L^{p(\cdot)}(\gamma)$. Since $b(t) \in L^{p(\cdot)}(\gamma)$, we should have $(Sb)(t) \in L^{p(\cdot)}(\gamma)$. Hence conditions (15) are fulfilled.

The sufficiency. Let the conditions (15) be fulfilled. Let us prove that $\Phi_b(w) \in H^{p(\cdot)}$. It is seen from (19) that $\Phi_b(w)$, as the Cauchy type integral, belongs to $\bigcap\limits_{\delta<1} H^\delta$ (see [14]. p. 96). It follows from (15) that $\Phi_b(w) \in H^1$, and hence,

$$\Phi_b(w) = \frac{1}{2\pi i} \int\limits_\gamma \frac{\Phi_b^+(\tau)d\tau}{\tau - w}.$$

According to (20) and (15), we find that $\Phi_b(w)$ is representable by the Cauchy integral with density $\Phi_b^+ \in L^{p(\cdot)}(\gamma)$. By virtue of Theorem 1, we conclude that $\Phi_b \in H^{p(\cdot)}$. Consequently, $u = \operatorname{Re}\Phi_b$ is the solution of problem (14) of the class $\widetilde{h}^{p(t)}$. □

Remark. The fact that $\Phi_b(w)$ belongs to the class $H^{p(\cdot)}$ can be also proved as follows.

As is mentioned above, $\Phi_b \in \bigcap\limits_{\delta<1} H^\delta$; assume $\Phi_b \in H^{1/2}$. Next, owing to (15), the function $\Phi_b^+ \in L^{p(t)}(\gamma)$.

Using Theorem 3, we find that $\Phi_b \in H^{\lambda(t)}$, where $\lambda(t) = \max(\frac{1}{2}; p(t)) = p(t)$.

4.2. On the functions $b(t)$ for which problem (14) is unsolvable. If $p \in \mathcal{P}(\gamma)$, then for any function $b \in L^{p(\cdot)}(\gamma)$ we have $Sb \in L^{p(\cdot)}(\gamma)$ (see [15] and also [11], p. 44). But when $p \in \mathcal{P}_1(\gamma)$, then this is, generally speaking, impossible at least for such $p(t)$ which admit value 1 on some arc $\gamma_0 \subset \gamma$. Indeed, were Sb for any b from $L^{p(\cdot)}(\gamma)$ belong to $L^{p(\cdot)}(\gamma)$, the Cauchy operator $S : b \to Sb$ would be continuous in $L^{p(\cdot)}(\gamma)$ (see [16], and also [11], p. 101). But this is impossible, since there exist the functions $\widetilde{b} \in L^1(\gamma_0)$ for which $S\widetilde{b} \overline{\in} L^1(\gamma_0)$; taking as $b(t)$ the function b_1 from $L^{p(\cdot)}(\gamma)$ which equals $\widetilde{b}$ on γ_0, we have $b_1 \overline{\in} L^1(\gamma_0)$, and hence, $Sb_1 \overline{\in} L^{p(\cdot)}(\gamma)$.

Obviously, in the case under consideration there exist linearly independent functions $b_1, b_2, \dots, b_n, \dots$ for which problem (14) is unsolvable in the class $\widetilde{h}^{p(\cdot)}$.

4.3. Certain subsets of functions from $L^{p(\cdot)}(\gamma)$, $\min\limits_{t\in\gamma} p(t) = 1$, for which conditions (15) are fulfilled. In [12], in connection with the investigation of problems dealing with the approximation of functions from $L^{p(\cdot)}(\gamma)$, it was considered the sets V_r, $r \in N_0 = \{0, 1, 2, \dots\}$ of those function f from $L^{p(\cdot)}(\gamma)$ for which

$$\int\limits_0^{\delta_0} \frac{\Omega(t,\delta)}{\delta} \Big(\ln\frac{1}{\delta}\Big)^r d\delta < \infty, \text{ where } \delta_0 > 0,$$

and

$$\Omega(f,\delta)=\sup_{h\leq\delta}\left\|\int\limits_{s-h}^{s+h} f(e^{i\sigma})d\sigma - f(s)\right\|_{p(\cdot)}.$$

It has been proved that the Cauchy operator $S: b \to Sb$ transfers V_{r+1} into V_r, and $S(V_0)\subset L^{p(\cdot)}(\gamma)$.

Consequently, the following theorem is valid.

Theorem 6. *If $b \in V_0$, then problem* (14) *is solvable in the class $\widetilde{h}^{p(\cdot)}$.*

4.4. On classes of functions V and $\widetilde{h}^{p(\cdot)}(\gamma;V)$. The Dirichlet problem in the class $\widetilde{h}^{p(\cdot)}(\gamma;V)$. The above-mentioned work [12] considers also the set

$$V=\bigcap_{z\in N_0} V_r.$$

which is invariant with respect to the operator S, i.e.,

$$S(V)=V_l.$$

Let us consider the Dirichlet problem in the following statement: find the function $u(w)$ from the set

$$\widetilde{h}^{p(\cdot)}(\gamma;V)=\left\{u:\exists\,\Phi\in H^{p(\cdot)}\Phi(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\Phi^+(\tau)d\tau}{\tau-w},\right.$$
$$\left.\Phi^+\in V,\quad u(w)=\operatorname{Re}\Phi(w)\right\},$$

which satisfies the boundary condition

$$u^+(t)=b(t).$$

In this case the theorem below is valid.

Theorem 7. *If $b \in V$, then the Dirichlet problem is solvable in the class $\widetilde{h}^{p(\cdot)}(\gamma;U)$.*

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