Proc. A. Razmadze Math. Inst. 129(2002), 145–149

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Maximal and Fractional Operators in Weighted $\mathbf{L^{p(x)}}$ Spaces

(Reported on April 25, 2002)

Introduction

The Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent were an object of interest during the last two decades, see the papers [1]–[9], the first investigation of these spaces being undertaken in [1]. The interest to these papers, apart from the mathematical curiosity is aggrivated by the importance in some applications, (see [11]–[13]).

The space $L^{p(\cdot)}(R^n)$ not being invariant with respect to the translation, the convolution operators do not behave well in these spaces. For example, the Young theorem is not in general valid in these spaces, (see [8], Section 2). The same problem arises for Mellin convolutions ($n = 1$), since $L^{p(\cdot)}(R^1_+)$ is not invariant with respect to dilations. However, the failure of the Young theorem does not prevent from boundedness of some convolution operators. Roughly speaking, the convolution operator to be bounded in $L^{p(\cdot)}$, its kernel may have singularity only at the origin.

For the maximal functions over open bounded sets the boundedness problem in $L^{p(\cdot)}$ was recently solved by L. Diening [10].

In this paper we prove weighted estimates for maximal operators over bounded open sets and for singular operators with fixed singularity (of Hardy and Hankel type). We give also weighted estimates for potential type operators of variable order $\alpha(x)$ and show, in particular, that the Sobolev theorem with the limiting exponent

$$\frac{1}{\mu(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$$

is valid for them. We also prove that the potential operator is compact in $L^{p(\cdot)}(\Omega)$.

1. Notation

Ω is a open bounded set in R^n;
$\mu(\Omega) = |\Omega|$ is the Lebesgue measure of Ω;
$B_r(x) = \{y \in R^n : \ |y - x| < r\}$;
$|B_r(x)| = \frac{r^n}{n}\,|s^{n-1}|$ is the volume of $B_r(x)$;
$q(x) = \frac{p(x)}{p(x)-1}$, $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$, $1 \le p(x) < \infty$;
$p_0 = \inf\limits_{x\in\Omega} p(x)$, $P = \sup\limits_{x\in\Omega} p(x)$;
$q_0 = \inf\limits_{x\in\Omega} q(x) = \frac{P}{P-1}$, $Q = \sup\limits_{x\in\Omega} q(x) = \frac{p_0}{p_0-1}$;
c may denote different positive constants.

2000 *Mathematics Subject Classification:* 42B20, 42B25.
Key words and phrases. Maximal functions, weighted Lebesgue spaces, variable exponent, potential operators, integral operators with fix singularity.

2. On the spaces $L^{p(\cdot)}(\Omega)$

The basics on the spaces $L^{p(\cdot)}$ may be found in [1]–[3], [5]–[9]. Here we recall only some important facts and definitions.

Let Ω be a domain in R^n and $p(x)$ a function on Ω such that

$$1 \le p(x) \le P < \infty, \quad x \in \overline{\Omega}.$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$I_p(f) := \int_\Omega |f(x)|^{p(x)}\,dx < \infty. \tag{2.1}$$

This is a Banach space with respect to the norm

$$\big\|f\big\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \ I_p\Big(\frac{f}{\lambda}\Big) \le 1\right\}.$$

The Hölder inequality holds in the form

$$\int_\Omega |f(x)g(x)|\,dx \le k\big\|f\big\|_{p(\cdot)} \cdot \big\|g\big\|_{q(\cdot)} \tag{2.2}$$

with $n = \frac{1}{p_0} + \frac{1}{q_0}$. The modular $I_p(f)$ and the norm $\|f\|_{p(\cdot)}$ are simultaneously greater than one a simultaneously less than 1:

$$\big\|f\big\|^P_{p(\cdot)} \le I_p(f) \le \big\|f\big\|^{p_0}_{p(\cdot)}, \ \text{ if } \ \big\|f\big\|_{p(\cdot)} \le 1 \tag{2.3}$$

and

$$\big\|f\big\|^{p_0}_{p(\cdot)} \le I_p(f) \le \big\|f\big\|^{P}_{p(\cdot)}, \ \text{ if } \ \big\|f\big\|_{p(\cdot)} \ge 1. \tag{2.4}$$

The imbedding

$$L^{p(x)} \subseteq L^{r(x)}, \ \ 1 \le r(x) \le p(x) \le P < \infty$$

is valid in the case $|\Omega| < \infty$ and

$$\big\|f\big\|_{r(\cdot)} \le m\big\|f\big\|_{p(\cdot)}, \ \ m = a_2 + (1 - a_1)|\Omega|, \tag{2.5}$$

where $a_1 = \inf\limits_{x\in\Omega} \frac{r(x)}{p(x)}$ and $a_1 = \sup\limits_{x\in\Omega} \frac{r(x)}{p(x)}$.

3. Main Results

Let Ω be an open bounded set in R^n, $n \ge 1$, and $p(x)$ a function on $\overline{\Omega}$ satisfying the conditions

$$1 < p_0 \le p(x) \le P < \infty, \quad x \in \overline{\Omega}, \tag{3.1}$$

and

$$|p(x) - p(y)| \le \frac{A}{\ln\frac{1}{|x-y|}}, \ \ |x - y| \le \frac{1}{2}, \ \ x, y \in \overline{\Omega}. \tag{3.2}$$

The condition (3.2) appears naturally in the theory of the spaces $L^{p(\cdot)}(\Omega)$.

Let

$$M^\beta f(x) = |x - x_0|^\beta \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} \frac{|f(y)|}{|y - x_0|^\beta}\,dy, \tag{3.3}$$

where $x_0 \in \overline{\Omega}$. In the case where $x_0 \in \partial\Omega$, we shall need the condition

$$|\Omega_r(x_0)| \sim r^n, \tag{3.4}$$

where $\Omega_r(x_0) = \{y \in \Omega : \ r < |y - x_0| < 2r\}$. We write

$$M = M^0.$$

Theorem A. *Let $p(x)$ satisfy conditions (3.1), (3.2). The operator M^β with $x_0 \in \overline{\Omega}$ is bounded in $L^{p(x)}(\Omega)$ if and only if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}\,. \tag{3.5}$$

When $x_0 \in \partial\Omega$, condition (3.5) is sufficient in the case of any point x_0 and necessary in the case of x_0 satisfying condition (3.4).

Let further

$$I^{\alpha(x)}f(x) = \int\limits_{\Omega} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\,dy, \ \ 0 < \alpha(x) < n. \tag{3.6}$$

Theorem B. *Under conditions (3.1), (3.2) and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0 \ \ \textit{and} \ \ \sup_{x\in\Omega} \alpha(x)p(x) < n, \tag{3.7}$$

the potential operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Omega)$ into $L^{r(\cdot)}(\Omega)$ with $\frac{1}{r(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$.

Theorem C. *Under conditions (3.1), (3.2) and the condition $\inf\limits_{x\in\Omega} \alpha(x) > 0$, the operator $I^{\alpha(\cdot)}$ is compact in $L^{p(\cdot)}(\Omega)$.*

For the weighted potential operator

$$I^{\alpha(x)}_\beta f(x) = |x-x_0|^\beta \int\limits_{\Omega} \frac{f(y)}{|y-x_0|^\beta |x-y|^{n-\alpha(x)}}\,dy, \ \ x_0 \in \overline{\Omega}, \tag{3.8}$$

the following theorem holds.

Theorem D. *Under conditions (3.1), (3.2) and the condition $\inf\limits_{x\in\Omega} \alpha(x) > 0$, the operator $I^{\alpha(\cdot)}_\beta$ is bounded in $L^{p(\cdot)}(\Omega)$ if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}\,. \tag{3.9}$$

Let now $n = 1$, $\Omega = (0, \ell)$ with $0 < \ell < \infty$ and $x_0 = 0$. We consider the following weighted Hardy-type operators

$$H^\beta f(x) = x^{\beta-1} \int\limits_0^x \frac{f(t)}{t^\beta}\,dt, \quad H^\beta_* f(x) = x^\beta \int\limits_x^\ell \frac{f(t)}{t^{\beta+1}}\,dt \tag{3.10}$$

and the weighted Hankel-type operator

$$\mathcal{H}^\beta f(x) = x^\beta \int\limits_x^\ell \frac{f(t)}{t^\beta (t+x)}\, dt. \tag{3.11}$$

Theorem E. *Let $1 \le p(x) \le P < \infty$ for $x \in [0,\ell]$.*

I. *Let conditions* (3.1), (3.2) *be satisfied on a neighbourhood $[0,d]$ of the origin, $d > 0$. Then all the operators H^β, H_*^β and $\mathcal{H}^\beta$ are bounded from $L^{p(\cdot)}(\Omega)$ into $L^{s(\cdot)}(\Omega)$ with any $s(x)$ such that $1 \le s(x) \le s < \infty$, $0 \le x \le \ell$, and*

$$s(0) = p(0) \ \ \textit{and} \ \ |s(x) - p(x)| \le \frac{A}{\ln \frac{1}{x}}\,, \quad 0 < x < \delta, \quad \delta > 0, \tag{3.12}$$

if

$$-\frac{1}{p(0)} < \beta < \frac{1}{q(0)}\,. \tag{3.13}$$

II. *If $p(0) \le p(x)$, $0 \le x \le d$, for some $d > 0$, then the same statement on boundedness from $L^{p(\cdot)}(\Omega)$ into $L^{s(\cdot)}(\Omega)$ is true if the requirtments of validity of conditions* (3.1), (3.2) *on $[0,d]$ is replaced by a weaken assumption that*

$$p(0) > 1 \ \ \textit{and} \ \ |s(x) - p(0)| < \frac{A}{\ln \frac{1}{x}}\,, \quad 0 < x < \min\left(\ell, \frac{1}{2}\right). \tag{3.14}$$

Corollary. *Let $1 \le p(x) \le P < \infty$ on $[-1,1]$. The singular operator with fixed singularity*

$$S^\beta f = \frac{|x|^\beta}{\pi} \int\limits_0^1 \frac{f(t)}{t-x} \frac{dt}{t^\beta}\,, \quad x \in [-1,0],$$

is bounded from $L^{p(x)}([0,1])$ into $L^{p(x)}([-1,0])$ if

1) $p(0) > 1$;
2) $p(x)$ satisfies condition (3.2) on $[-\delta,\delta]$ for some $\delta > 0$;
3) $-\frac{1}{p(0)} < \beta < \frac{1}{q(0)}$.

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