Proceedings of A. Razmadze
Mathematical Institute
Vol. **131** (2003), 61–78

SINGULAR INTEGRAL EQUATIONS IN THE LEBESGUE SPACES WITH VARIABLE EXPONENT

V. KOKILASHVILI AND S. SAMKO

Abstract. For the singular integral operators with piecewise continuous coefficients there is proved the criterion of Fredholmness and formula for index in the generalized Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ on a finite closed Lyapunov curve Γ or a curve of bounded rotation. The obtained criterion shows that Fredholmness in this space and the index depend on values of the function $p(t)$ at the discontinuity points of the coefficients of the operator, but do not depend on values of $p(t)$ at points of their continuity.

რეზიუმე. ნაშრომში დადგენილია ფრედჰოლმურობის კრიტერიუმი და ინდექსის ფორმულა უბან-უბან უწყვეტი კოეფიციენტებიანი სინგულარული ინტეგრალური ოპერატორებისათვის სასრულ ჩაკეტილ ლიაპუნოვის Γ წირებზე განსაზღვრული ლებეგის განზოგადებულ $L^{p(\cdot)}(\Gamma)$ სივრცეებში. ნაჩვენებია, რომ ამ სივრცეებში ფრედჰოლმურობა და ინდექსი დამოკიდებულია $p(t)$ ფუნქციის მნიშვნელობებზე ოპერატორის კოეფიციენტთა წყვეტის წერტილებში და არ არის დამოკიდებული $p(t)$ ფუნქციის მნიშვნელობებზე მათი უწყვეტობის წერტილებში.

1. Introduction

We consider the singular integral operator

$$A\varphi(t) := u(t)\varphi(t) + \frac{v(t)}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)\,d\tau}{\tau - t} = f(t), \quad t \in \Gamma, \tag{1.1}$$

or

$$A = aP_+ + bP_-, \quad a = u + v, \quad b = u - v, \tag{1.2}$$

where $P_\pm = \frac{1}{2}(I \pm S)$ are the projectors, generated by the singular integral operator

$$S\varphi(t) = \frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)\,d\tau}{\tau - t},$$

2000 *Mathematics Subject Classification.* 45E05, 46E30.

Key words and phrases. Variable exponent, Lebesgue spaces, singular operator, Fredholm operators, essential spectrum.

in the generalized Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ with a variable exponent $p(t)$ satisfying the logarithmic smoothness condition. The coefficients u and v are assumed to be piece-wise continuous and Γ is a finite closed curve in the complex plane.

We obtain necessary and sufficient conditions for the operator A to be Fredholm in the space $L^{p(\cdot)}(\Gamma)$ and give a formula for the index under some natural assumptions on $p(x)$, see Theorem A. The obtained criterion shows that Fredholmness of the operator A in the space $L^{p(t)}(\Gamma)$ and its index depend on values of the function $p(t)$ at the discontinuity points of the coefficients $a(t)$ and $b(t)$, but do not depend on values of $p(t)$ at points of continuity.

The generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent and operators in these spaces are intensively studied nowadays. One may see an evident rise of interest to these spaces and to the corresponding Sobolev type spaces $W^{m,p(\cdot)}$ during the last decade, especially the last years. The increase in studying both the spaces $L^{p(\cdot)}$ or $W^{m,p(\cdot)}$ themselves and the operator theory in these spaces is observed. As is known, this interest is aroused, apart from mathematical curiosity, by possible applications to models with the so called non-standard local growth (in fluid mechanics, elasticity theory, in differential equations, see for example [23], [7] and references therein).

The development of the operator theory in the spaces $L^{p(\cdot)}$ encountered essential difficulties from the very beginning. For example, in the case of the spaces $L^{p(\cdot)}(\mathbb{R}^n)$, the convolution operators in general are not bounded in these spaces, the Young theorem not being valid in the general case. A convolution operator may be bounded in this space if, roughly speaking, its kernel has singularity only at the origin, see [24]. One of the problems which were open for a long time, was the boundedness of the maximal operator and of singular operators in these spaces. The boundedness of the maximal operator was recently proved by L. Diening [5], [6] in case of bounded domains $\Omega \subset \mathbb{R}^n$ or in the case of $\Omega = \mathbb{R}^n$, but $p(x)$ constant at infinity. Recently, D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer [4] proved the boundedness of the maximal operator on unbounded domains when the exponent $p(x)$ is not necessary constant at infinity.

There is also an evident progress in this direction for singular operators. L. Diening and M. Ružička [7] proved the boundedness of Calderon-Zygmund type operators in these spaces. As is known, for application, the weighted boundedness of singular operators is required. The weighted estimates with power weights were proved by the authors for the maximal operator on bounded domains in [5]–[19] see also [14] and for singular type operators in [20], [16], [18], [17].

In this paper we give an application of the weighted results obtained in [20], [16], [18], [17] to the theory of Fredholm solvability of singular integral equations (1.1) with piece-wise continuous coefficients. As is well known

to researches in this field, to investigate such equations in this or other function space, one should know exactly what are necessary and sufficient conditions of weighted singular operator in this space. These conditions being known, to obtain the criterion of Fredholmness, one should follow the known scheme of investigation of singular operators in already studied situations, for example in the spaces $L_p(\Gamma)$, $p = \text{const}$. This scheme may be rewritten in terms of an arbitrary Banach space of functions defined on Γ, subject to some natural axioms. We do this in Section 4. As a model of the scheme to follow we use the Gakhov–Muskhelishvili–Khvedelidze–Gohberg–Krupnik scheme of investigation of singular operators with piece-wise continuous coefficients.

The theory of singular integral operators itself was intensively developed last decades and was generalized, in particular, to the case of general weights (Muckenhoupt weights) and Carleson curves, both generalizations leading to new effects, see [1]–[3], [28] and references therein. We do not touch such generalizations in the $L^{p(\cdot)}$-setting in this paper. Basing on the approaches developed in [1]–[3], [28] one can consider the operator A in these more general situations, as soon as necessary and sufficient conditions of the boundedness of the singular operator S in the spaces $L^{p(\cdot)}(\Gamma, \rho)$ with a general weight ρ and/or a Carleson curve Γ are known. For the time being, this boundedness is a challenging open problem in both the cases. This problem being solved, this would pave the way to obtain results on Fredholmness of singular integral operators in more general situations.

For "bad" curves and general weights this open problem is tightly connected with other open questions. As is already known, on Lyapunov curves the assumption on p(t) guaranteering the boundedness of the singular operator is the logarithmic smoothness condition, see (2.6), which is necessary in a sense; at the least, it is surely necessary for the maximal operator.

Can the boundedness of the singular integral operator on a Carleson curve be proved under this condition? Or can it be proved if $p(t)$ is even infinitely differentiable, but variable. Or on the whole class of Carleson curves the boundedness may be true only for constant p ? All these questions are open.

The paper is organized as follows. The main statement on Fredholmness of the operator A is given in Theorem A. In this statement for the spaces $L^{p(\cdot)}(\Gamma)$, the curve Γ is assumed to be a Lyapunov curve or a curve of bounded rotation without cusps.

However, in fact we formulate a more general statement on Fredholmness of the operator A in an abstract Banach space of functions on Γ, satisfying some natural axioms. This statement, as already mentioned above, appears as a result of an abstract Banach space reformulation of the Gohberg-Krupnik scheme of investigation of singular operators with piece-wise continuous coefficients. For the completeness of the presentation

and the reader's convenience we expose this reformulation with proofs in Section 4.

The theorem on Fredholmness of the operator A in the spaces $L^{p(\cdot)}(\Gamma)$ is obtained as a corollary to that abstract Banach space scheme, see Section 5. It is possible to extend the Fredholmness theorem for the operator A also for piecewise Lyapunov curves or curves of bounded variation with arbitrary cusps, but we do not dwell on this extension in this paper.

We remind the basics for the Lebesgue spaces with variable exponents in Subsection 2.1; the reader is referred for details to the papers [27], [21], [26], [25] in the case of the spaces $L^{p(\cdot)}(\Omega)$, $\Omega \in \mathbb{R}^n$ and to the papers [19], [20], [16] in the case of the spaces $L^{p(\cdot)}(\Gamma)$ on curves.

Notation:

Γ is a finite closed rectifiable Jordan curve on a complex plane;

ℓ is its length;

D^+ is the interior of the curve Γ and D^- is its exterior;

$PC(\Gamma)$ is the class of piece-wise continuous functions on Γ with a finite number of jumps;

$\operatorname{Ind}_X A$ is the index of the Fredholm operator A in a Banach space X;

$\operatorname{ind} a$ is the winding number of a continuous function a on a closed curve Γ; $\alpha_X(A)$ and $\beta_X(A)$ are deficiency numbers of a Fredholm operator A in the Banach space X;

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$; $\frac{1}{q(t)} = 1 - \frac{1}{p(t)}$.

2. Preliminaries

a). On $L^{p(\cdot)}$-spaces. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $p(x)$ a measurable function on Ω such that $1 < p_0 \le p(x) \le P < \infty, \quad x \in \Omega$ and

$$\big|p(x) - p(y)\big| \le \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x-y| \le \frac{1}{2}, \quad x, y \in \Omega. \tag{2.1}$$

We refer to ([14], Appendix A) for examples of non-holderian functions satisfying condition (2.1). By $L^{p(\cdot)}(\Omega)$ we denote the space of functions $f(x)$ on Ω such that

$$A_p(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

This is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf \Big\{ \lambda > 0 : A_p\Big(\frac{f}{\lambda}\Big) \le 1 \Big\}. \tag{2.2}$$

Under condition (2.5) the space $L^{p(\cdot)}$ coincides with the space

$$\left\{f(x) : \Big| \int\limits_{\Omega} f(x)\varphi(x)\,dx\Big| < \infty \quad \text{for all} \quad \varphi(x) \in L^{q(\cdot)}(\Omega)\right\} \tag{2.3}$$

where $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$, up to equivalence of the norms

$$\|f\|_{L^{p(\cdot)}} \sim \sup_{\|\varphi\|_{L^{q(\cdot)}} \le 1} \Big| \int\limits_{\Omega} f(x)\varphi(x)\,dx\Big| \sim \sup_{A_q(\varphi)\le 1} \Big| \int\limits_{\Omega} f(x)\varphi(x)\,dx\Big|, \tag{2.4}$$

see [21], Theorem 2.3 or [26], Theorem 3.5.

Let ρ be a measurable almost everywhere positive integrable function. The weighted Lebesgue space $L^{p(\cdot)}_{\rho} = L^{p(\cdot)}(\Omega,\rho)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot)}_{\rho}} = \|\rho f\|_{L^{p(\cdot)}} < \infty.$$

The space $L^{p(\cdot)}(\Omega,\rho)$ is a Banach space.

The space $L^{p(\cdot)}(\Gamma)$ on a rectifiable simple curve

$$\Gamma = \big\{t \in \mathbb{C} : \ \ t = t(s), \ \ 0 \le s \le \ell\big\},$$

where s is the arc length, may be introduced in a similar way via the functional

$$I_p(f) = \int\limits_{\Gamma} \big|f(t)|^{p(t)}\,|dt| = \int\limits_{0}^{\ell} \big|f[t(s)]\,\big|^{p[t(s)]}\,ds.$$

We assume that

$$1 < p_0 \le p(t) \le P < \infty, \quad t \in \Gamma. \tag{2.5}$$

Condition (2.1) may be imposed either on the function $p(t)$:

$$\big|p(t_1) - p(t_2)\big| \le \frac{A}{\ln\frac{1}{|t_1-t_2|}}, \ \ |t_1 - t_2| \le \frac{1}{2}, \ \ t_1, \ \ t_2 \in \Gamma \tag{2.6}$$

or on the function $p_*(s) = p[t(s)]$:

$$\big|p_*(s_1) - p_*(s_2)\big| \le \frac{A}{\ln\frac{1}{|s_1-s_2|}}, \ \ |s_1 - s_2| \le \frac{1}{2}, \ \ s_1, \ \ s_2 \in [0,\ell]. \tag{2.7}$$

Since $|t(s_1)-t(s_2)| \le |s_1-s_2|$, condition (2.6) always implies (2.7). Inversely, (2.7) implies (2.6),if there exists $\lambda > 0$ such that $|s_1 - s_2| \le c|t(s_1) - t(s_2)|^{\lambda}$. Therefore, conditions (2.6) and (2.7) are equivalent, for example on curves with the so called chord condition.

We shall deal with the weighted space

$$L^{p(\cdot)}(\Gamma,\rho) = \big\{f : \|f[t(s)]\rho(s)\|_{L^{p(s)}} < \infty\big\}$$

where

$$\rho(s)=\prod_{k=1}^{n}|t(s)-t(c_k)|^{\beta_k}\approx\prod_{k=1}^{n}|s-c_k|^{\beta_k} \tag{2.8}$$

where $c_k\in[0,\ell]$, $k=1,2,\dots,n$.

We remind the Hölder inequality

$$\Big|\int_{\Gamma}f(t)g(t)\,dt\Big|\le c\|f\|_{L^{p(\cdot)}}\|g\|_{L^{q(\cdot)}} \tag{2.9}$$

for the spaces with variable exponent. From (2.9) the imbedding follows

$$L^{p(\cdot)}(\Gamma,|t-t_0|^{\gamma})\subset L^1(\Gamma),\quad\text{if}\quad\gamma<\frac{1}{q(t_0)}. \tag{2.10}$$

b). Two theorems on the spaces $L^{p(\cdot)}(\Gamma)$. In [18], [17] the following statements were proved.

Theorem 2.1. *Let Γ be a Lyapunov curve or a curve of bounded rotation without cusps and let $p(s)$ satisfy conditions* (2.5) *and* (2.7). *The operator S is bounded in the space $L^{p(\cdot)}_{\rho}(\Gamma)$ with the weight function* (2.8) *if and only if*

$$-\frac{1}{p(c_k)}<\beta_k<\frac{1}{q(c_k)},\quad k=1,2,\dots,n. \tag{2.11}$$

Theorem 2.2. *Let $p(x)$ satisfy the condition $1\le p(x)\le P<\infty$, $x\in\mathbb{R}^n$ and let $\rho(x)\ge 0$ be such that $|\{x\in\mathbb{R}^n:\ \rho(x)=0\}|=0$ and*

$$w(x)=[\rho(x)]^{p(x)}\in L^1_{loc}(\mathbb{R}^n). \tag{2.12}$$

Then $C_0^{\infty}(\mathbb{R}^n)$ is dense in the space $L^{p(\cdot)}(\mathbb{R}^n,\rho)$.

Similarly, the following analogue of Theorem 2.2 can be proved.

Theorem 2.3. *Let Γ be a Jordan curve. The set $C^{\infty}(\Gamma)$ (and even the set of rational functions on Γ) is dense in $L^{p(\cdot)}(\Gamma,\rho)$ under the assumptions $1\le p(t)\le P<\infty$, $t\in\Gamma$ and $|\{t\in\Gamma:\ \rho(t)=0\}|=0$ and $[\rho(t)]^{p(t)}\in L^1(\Gamma)$.*

3. Statement of the main result for the spaces $L^{p(\cdot)}(\Gamma)$

Let $h(t)\in PC(\Gamma)$ and $t_1,t_2,\dots,t_n$ be the points of discontinuity of $h(t)$.

Definition 3.1. Following the known definition ([10], p. 63), we say that a function $h(t)\in PC(\Gamma)$ is $p(\cdot)$-nonsingular, if

$$\inf_{t\in\Gamma}|h(t)|>0 \tag{3.1}$$

and at all the points of discontinuity of $h(t)$ the following condition is satisfied:

$$\arg\frac{h(t_k-0)}{h(t_k+0)} \neq \frac{2\pi}{p(t_k)} (\mathrm{mod}\, 2\pi), \quad k=1,2,\dots,n. \tag{3.2}$$

For a non-vanishing function $h(t) \in PC(\Gamma)$ we denote

$$\theta(t_k) = \frac{1}{2\pi} \int\limits_{t_k+0}^{t_{k+1}-0} d\arg h(t). \tag{3.3}$$

Definition 3.2. Let $h(t) \in PC(\Gamma)$ be a $p(\cdot)$-nonsingular function. The integer

$$\mathrm{ind}_{p(\cdot)}\, a = \sum_{k=1}^{n} \Big[\theta(t_k) - \frac{1}{2\pi}\arg\frac{h(t_k-0)}{h(t_k+0)}\Big], \tag{3.4}$$

where the values of $\frac{1}{2\pi}\arg\frac{h(t_k-0)}{h(t_k+0)}$ are chosen in the interval

$$-\frac{1}{q(t_k)} < \frac{1}{2\pi}\arg\frac{h(t_k-0)}{h(t_k+0)} < \frac{1}{p(t_k)} \tag{3.5}$$

where $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$, is called the $p(\cdot)$-index of the function a.

Basing on Lemma 2.7 from [11], it is easy to see that $\mathrm{ind}_{p(\cdot)}\, a$ is the same as the Gohberg–Krupnik p-index defined as the winding number of the curve, obtained from the image $h(\Gamma)$ of the curve Γ by supplementing it at its discontinuities by the corresponding circular arcs in the well known way (see for instance, [10], p. 63–64); the only difference is now in the fact that the angle of the arc is defined by the exponent $p(t_k)$ varying from one discontinuity point to another.

Theorem A. *Let Γ be a closed Lyapunov curve or a curve of bounded rotation without cusps and let $p(t), t \in \Gamma$, satisfy assumptions* (2.5) *and* (2.7). *The operator $A = aP_+ + bP_-$ with $a, b \in PC(\Gamma)$ is Fredholm in the space $L^{p(\cdot)}(\Gamma)$ if and only if*

$$\inf_{t\in\Gamma} |a(t)| \neq 0, \quad \inf_{t\in\Gamma} |b(t)| \neq 0 \tag{3.6}$$

and the function $\frac{a(t)}{b(t)}$ is $p(\cdot)$-nonsingular. Under these conditions

$$\mathrm{Ind}_{L_{p(\cdot)}}\ A = -\,\mathrm{ind}_{p(\cdot)}\,\frac{a}{b}. \tag{3.7}$$

Theorem A is proved in Section 5.

From Theorem A it follows that the essential spectrum of the operator $aP_+ + P_-$ with $a \in PC(\Gamma)$ in the space $L^{p(\cdot)}(\Gamma)$ (the set of points on complex plane for which an operator is not Fredholm) is described similarly to the case of constant p, as the union of the images $a(\Gamma)$ and the well known

circular arcs $\nu_{p(t_k)}(a(t_k-0), a(t_k+0))$, connecting the points $a(t_k-0)$ and $a(t_k+0)$ and having the angle $\frac{2\pi}{p(t_k)}$ depending on the point t_k.

4. Singular integral operators in Banach function spaces $X(\Gamma)$

The theory of singular integral equations with coefficients in$PC(\Gamma)$ is well known, for example, in the Lebesgue weighted spaces $L_p(\Gamma,\rho)$ (see for instance, [10] and in other spaces of integrable functions. A natural question is the following. Let $X(\Gamma)$ be an arbitrary Banach function space on Γ. Under what axioms on the space $X(\Gamma)$ the result on Fredholmness of the singular operator is formulated in the terms similar to those used in Theorem A, that is, in the terms of X-nonsingular functions and X-index, properly defined.

We give some answer to that question below. In this connection we observe that the idea of singling out the bounds for the weight functions (used in Axioms 1 and 2) as the base of construction of Fredholm criterion is well known in the theory of singular integral operators, see [28]; [1]–[3], Ch. 2, [12]. In the context of Carleson curves and general weights this idea led to the notion of the so called indicator set of the space at the point $t_0 \in \Gamma$, see [3], p. 72. We show that it is possible to axiomatize this idea so that the Gohberg-Krupnik approach known for $L_p(\Gamma,\rho)$-spaces on Lyapunov curves, may be presented for an arbitrary Banach function space under two natural axioms.

4.1. **Banach function spaces, suitable for singular operators.** Let $X = X(\Gamma)$ be any Banach space of functions on a closed simple Jordan rectifiable curve Γ satisfying the following assumptions

$$C(\Gamma) \subset X(\Gamma) \subset L_1(\Gamma), \tag{4.1}$$

$$\|a\ f\|_X \le \sup_{t\in\Gamma} |a(t)| \cdot \|f\|_X \quad \text{for any} \quad a \in L_\infty(\Gamma), \tag{4.2}$$

$$\text{the operator} \quad S \quad \text{is bounded in} \quad X(\Gamma), \tag{4.3}$$

$$C^\infty(\Gamma) \quad \text{is dense in} \quad X(\Gamma). \tag{4.4}$$

Assumptions (4.1)–(4.4) will be used to formulate the statement on Fredholmness in the case of continuous coefficients. For the case of piece-wise coefficients we shall also need the following Axioms 1 and 2.

Axiom 1. *For the space $X(\Gamma)$ there exist two functions $\alpha(t)$ and $\beta(t)$, $0<\alpha(t)<1$, $0<\beta(t)<1$, such that the operator*

$$|t-t_0|^{\gamma(t_0)} S |t-t_0|^{-\gamma(t_0)} I, \quad t_0 \in \Gamma \tag{4.5}$$

is bounded in the space $X(\Gamma)$ for all $\gamma(t_0)$ such that

$$-\alpha(t_0) < \gamma(t_0) < 1-\beta(t_0) \tag{4.6}$$

and is unbounded in $X(\Gamma)$ if $\gamma(t_0) \notin (-\alpha(t_0), 1-\beta(t_0))$.

The functions $\alpha(t)$ and $\beta(t)$ will be called *index functions* of the space $X(\Gamma)$.

In the case $X(\Gamma) = L^{p(\cdot)}(\Gamma, \rho) = \{f : |t-t_0|^{\mu} f(t) \in L^{p(\cdot)}(\Gamma)\}$ we have

$$\alpha(t) = \beta(t) = \frac{1}{p(t)} + \mu, \tag{4.7}$$

which follows from Theorem 2.1.

Let $X(\Gamma, |t-t_0|^{\gamma}) = \{f : |t-t_0|^{\gamma} f(t) \in X(\Gamma)\}$.

Axiom 2. *For any $\gamma < 1 - \beta(t_0)$ the imbedding $X(\Gamma, |t-t_0|^{\gamma}) \subset L^1(\Gamma)$ is valid and $C^{\infty}(\Gamma)$ is dense in $X(\Gamma, |t-t_0|^{\gamma})$, whatsoever $t_0 \in \Gamma$ is.*

Lemma 4.1. *Let the space $X(\Gamma)$ satisfy conditions (4.1)–(4.2) and t_1, $t_2, \ldots, t_n \in \Gamma$. Then*

$$\prod_{k=1}^{n} |t-t_k|^{\gamma_k} \in X(\Gamma) \tag{4.8}$$

for all $\gamma_k > -\alpha_k$, $k = 1, 2, \ldots, n$.

Proof. Let first $n = 1$. If $\gamma_1 \geq 0$, the inclusion (4.8) is obvious because of the imbedding $C(\Gamma) \subset X(\Gamma)$.

Let $\gamma_1 \leq 0$. Since $1 \in X(\Gamma)$, from Axiom 1 it follows that $|t-t_1|^{\gamma_1} S(|\tau - t_1|^{-\gamma_1})(t) \in X(\Gamma)$. As $-\gamma_1 \geq 0$, we have that $S(|\tau - t_1|^{-\gamma_1})(t)$ is a continuous function non-vanishing at the point $y = t_1$, as is known. Then $|t-t_1|^{\gamma_1} \in X(\Gamma)$, by property (4.2) taken into account.

The case $n > 1$ reduces to the case $n = 1$ by introducing a unity partition on Γ: $1 \equiv \sum\limits_{j=1}^{n} \omega_j(t)$ with $\omega_j(t) \in C^{\infty}(\Gamma)$ and $\omega_j(t) \equiv 0$ in a small neigbourhood of the point t_j. Then

$$\prod_{k=1}^{n} |t-t_k|^{\gamma_k} = \sum_{j=1}^{n} |t-t_j|^{\gamma_j} a_j(t) \tag{4.9}$$

with $a_j(t) \in C_{\infty}(\Gamma)$, so that $\prod\limits_{k=1}^{n} |t-t_k|^{\gamma_k} \in X$ in view of the case $n = 1$ and (4.2). □

Let now

$$X(\Gamma, \rho) = \big\{f : \rho(t) f(t) \in X(\Gamma)\big\}, \quad \rho(t) = \prod_{k=1}^{n} |t-t_k|^{\gamma_k}, \tag{4.10}$$
$$t_1, \ldots, t_n \in \Gamma.$$

Lemma 4.2. *Let $X(\Gamma)$ be a Banach function space satisfying conditions* (4.1)–(4.2) *and Axioms* 1–2. *Then the space $X(\Gamma,\rho)$ satisfies conditions* (4.1)–(4.2) *as well, if*

$$-\alpha_(t_k) < \gamma_k < 1 - \beta(t_k), \quad k = 1, \dots, n.$$

Proof. To verify properties (4.1)–(4.2) for the space $X(\Gamma,\rho)$, we observe that $\rho \cdot C(\Gamma) \subset X(\Gamma)$ by Lemma 4.1, which means that $C(\Gamma) \subset X(\Gamma,\rho)$. The imbedding $X(\Gamma,\rho) \subset L^1(\Gamma)$ is easily derived from Axiom 2 (introduce the unity partition).

Property (4.2) for $X(\Gamma,\rho)$ obviously follows from its validity for $X(\Gamma)$. Property (4.3) is in fact postulated in Axiom 1, the passage from the single weight $|t-t_k|^{\gamma_k}$ to the weight $\rho(t)$ in (4.10) being justified by the standard us of a unity partition, as in (4.9). Finally, property (4.2) is also in fact postulated in Axiom 1 since the space $X(\Gamma,\rho)$ is the algebraic sum of the spaces $X(\Gamma,|t-t_k|), k = 1, 2, ..., n$. □

4.2. **X-nonsingular functions and X-index of a PC-function.** Here we present an abstract Banach space reformulation of the notions of p-nonsingularity and p-index [10]. A development of these notions in the context of Carleson curves related to the notion of the indicator set may be found in [3], Proposition 7.3 and Theorem 7.4.

For a function $a \in PC(\Gamma)$ we put as usual

$$\gamma(t) = \frac{1}{2\pi i} \ln \frac{a(t-0)}{a(t+0)} \tag{4.11}$$

and

$$\omega(t) = \prod_{k=1}^{n} (t - z_0)_k^{\gamma(t_k)} \tag{4.12}$$

where $z_0 \in D^+$, t_k are the points of discontinuity of a and the functions $\omega_k(z) = (z - z_0)_k^{\gamma(t_k)}$ stand for univalent analytic functions in the complex plane with the cut passing from z_0 to infinity through the point $t_k \in \Gamma$. The function

$$a_1(t) = \frac{a(t)}{\omega(t)} \tag{4.13}$$

is continuous on Γ independently of the choice of

$$\Re\gamma(t_k) = \frac{1}{2\pi} \arg \frac{a(t_k-0)}{a(t_k+0)}. \tag{4.14}$$

Following Definitions 3.1 and 3.2, we introduce the following definitions.

Definition 4.3. Let $X(\Gamma)$ be a Banach function space satisfying Axiom 1. A function $a \in PC(\Gamma)$ is called X-nonsingular if $\inf\limits_{t\in\Gamma} |a(t)| > 0$ and

$$\frac{1}{2\pi} \arg \frac{a(t_k-0)}{a(t_k+0)} \notin [\alpha(t_k), \beta(t_k)] + \mathbb{Z} \tag{4.15}$$

where $[\cdots]+\mathbb{Z}$ stands for the set of $\bigcup_{\xi\in[\cdots]}\{\xi,\xi\pm1,\xi\pm2,\dots\}$, and $\alpha(t)$ and $\beta(t)$ are the index functions of the space X.

Definition 4.4. Let $X(\Gamma)$ satisfy Axiom 1 and $a\in PC(\Gamma)$ be X-nonsingular. The integer

$$\operatorname{ind}_X a=\sum_{k=1}^{n}\big[\theta(t_k)-\Re\gamma(t_k)\big], \tag{4.16}$$

where $\theta(t_k)$ are increments (3.3) and $\Re\gamma(t_k)$ are chosen in the interval

$$\beta(t_k)-1<\Re\gamma(t_k)<\alpha(t_k), \tag{4.17}$$

will be referred to as X-index of the function a.

4.3. The case of $a\in C(\Gamma)$.

Theorem B. *Let $X(\Gamma)$ be any Banach function space satisfying assumptions* (4.1)–(4.4). *The operator $A=aP_++bP_-$ with $a,\,b\in C(\Gamma)$ is Fredholm in the space X if and only if $a(t)\neq0, b(t)\neq0$ for all $t\in\Gamma$. In this case* $\operatorname{Ind}_X A=\operatorname{ind}_X\frac{b}{a}:=\varkappa$.

Proof. The proof is completely standard and follows the well known arguments, but we give short proofs for completeness.

1st step (*compactness of the commutators $aS-SaI$, $a\in C(\Gamma)$*). These commutators are compact in $X(\Gamma)$. Indeed, it is known that any function $a(t)$ continuous on Γ may be approximated in $C(\Gamma)$ by a rational function $r(t)$, whatsoever Jordan curve Γ we have, as is known from the famous Mergelyan's result, see for instance, [8], p. 169. Therefore, since the singular operator S is bounded in $X(\Gamma)$ by assumption (4.3), we obtain that the commutator $aS-SaI$ is approximated in the operator norm in X by the commutator $rS-SrI$ which is finite-dimensional operator, and consequently compact in $X(\Gamma)$. Therefore, $aS-SaI$ is compact.

2st step (*sufficiency*). By compactness of the commutators we have $(aP_++bP_-)(bP_++aP_-)=ab\,I+T$, where T is a compact operator, so the operator (aP_++bP_-) has a regularizer. Consequently, it is Fredholm.

3rd step (*the operator $A_\varkappa=P_++t^\varkappa P_-$*). Let $0\in D^+$. The operator $A_\varkappa$ is right invertible in $X(\Gamma)$, if $\varkappa\geq0$ and left invertible if $\varkappa\leq0$ and has the deficiency numbers $\alpha_X(A_\varkappa)=\varkappa$ and $\beta_X(A_\varkappa)=0$ if $\varkappa\geq0$ and $\alpha_X(A_\varkappa)=0$ and $\beta_X(A_\varkappa)=|\varkappa|$ if $\varkappa\leq0$. Indeed, the operator $A_\varkappa$ is Fredholm in $X(\Gamma)$ by the sufficiency part of Theorem B (the previous step). The one-sided invertibility follows from the relations

$$A_\varkappa A_{-\varkappa}=I,\quad\text{if}\quad\varkappa\geq0,\quad A_{-\varkappa}A_\varkappa=I,\quad\text{if}\quad\varkappa\leq0$$

well known on spaces of "nice" functions and valid on $X(\Gamma)$ by (4.3)–(4.4). To obtain the information on the deficiency numbers in the space $X(\Gamma)$, we

observe that $H^\lambda(\Gamma) \subset C(\Gamma) \subset X(\Gamma)$ by (4.1) and that $\alpha_{H^\lambda}(A_\varkappa) = \varkappa$ in case $\varkappa \geq 0$ ([22]). Therefore, $\alpha_X(A_\varkappa) \geq \varkappa$. Since $X(\Gamma) \subset L_1(\Gamma)$, we also have $\alpha_X(A_\varkappa) \leq \varkappa$. The case $\varkappa \leq 0$ is treated similarly.

4th step (*the operator* $N = (t-\lambda)P_+ + P_-$). The operator N is invertible in $X(\Gamma)$, if $\lambda \in D^-$ and is Fredholm with $\operatorname{Ind}_X N = -1$, if $\lambda \in D^+$. Indeed, the invertibility in the case when $\lambda \in D^-$ is checked directly: $N_1 N = NN_1 = I$, where $N_1 = \frac{1}{t-\lambda}P_+ + P_-$, with conditions (4.3)–(4.4) taken into account. The case when $\lambda \in D^+$ follows from the 3rd step, since $(t-\lambda)P_+ + P_- = (t-\lambda)\left[P_+ + (t-\lambda)^{-1}P_-\right]$.

5th step (*necessity*). Suppose that $a(t_0) = 0$ for some $t_0 \in \Gamma$ and the operator A is Fredholm. By compactness of the commutators $aS - SaI$ (the 1st step), we have the relations

$$aP_+ + bP_- = (P_+ + bP_-)(aP_+ + P_-) + T_1 = (aP_+ + P_-)(P_+ + bP_-) + T_2$$

where T_1 and T_2 are compact operators in $X(\Gamma)$. So $aP_+ + P_-$ is Fredholm and $a(t_0) = 0$. We may approximate the function a in $C(\Gamma)$ by rational functions a_ε such that $a_\varepsilon(t_0) = 0$. Then the operators $a_\varepsilon P_+ + P_-$ with ε small enough, are Fredholm. To arrive at a contradiction, we follow [9], p. 174, and represent a_ε as $a_\varepsilon(t) = (t-t_0)s(t)$. Then

$$a_\varepsilon P_+ + P_- = (sP_+ + P_-)[(t-t_0)P_+ + P_-] = [(t-t_0)P_+ + P_-](sP_+ + P_-) + T,$$

where T is a compact operator. Therefore, the operator $(t-t_0)P_+ + P_-$ has a regularizer and is a Fredholm operator, which is impossible in view of the statement of the 4th step and the known property of the stability of index of Fredholm operator.

6th step (*index formula*). As in [9], p. 103, we approximate the function $c(t) = \frac{a(t)}{b(t)}$ by a rational function $r(t)$ so that

$$c(t) = r(t)[1 + m(t)] \quad \text{with} \quad \max_{t\in\Gamma} |m(t)| < \frac{1}{\|P_+\|_X}. \tag{4.18}$$

Let $r(t) = t^{-\varkappa}\frac{\chi_+(t)}{\chi_-(t)}$ be the factorization of the function $r(t)$. Since $\|m\|_{C(\Gamma)} < 1$, we have $\operatorname{ind}(1+m) = 0$ and then $\operatorname{ind} r = \operatorname{ind} c = -\varkappa$.

In the case $\varkappa \leq 0$, the representation is valid:

$$A = b\chi_-(I + mP_+)\Big(\frac{1}{\chi_+}P_+ + \frac{1}{\chi_-}P_-\Big)(t^{-\varkappa}P_+ + P_-) \tag{4.19}$$

with the reference to conditions (4.3)–(4.4). The operator $I + mP_-$ is invertible since $\|mP_+\|_X < 1$ by (4.18) and (4.3). Since the operator $\frac{1}{\chi_+}P_+ + \frac{1}{\chi_-}P_-$ is also obviously invertible in X, from (4.19) we obtain $\operatorname{Ind}_X A = \operatorname{ind}_X(t^{-\varkappa}P_+) + P_- = \varkappa$ according to the statement at the 3rd step. □

4.4. The case of $a \in PC(\Gamma)$.

Theorem C. *Let $X(\Gamma)$ be any Banach function space satisfying assumptions (4.1)–(4.4) and Axioms 1–2. The operator $A = aP_+ + bP_-$ with $a, b \in PC(\Gamma)$ is Fredholm in the space X if*

$$\inf_{t\in\Gamma} |a(t)| \neq 0, \quad \inf_{t\in\Gamma} |b(t)| \neq 0 \tag{4.20}$$

and

$$\text{the function } \frac{a(t)}{b(t)} \text{ is } X - \text{nonsingular.} \tag{4.21}$$

In this case

$$\operatorname{Ind}_X A = -\operatorname{ind}_X \frac{a}{b}. \tag{4.22}$$

Condition (4.20) is also necessary for the operator A to be Fredholm in X. If the index functions $\alpha(t)$ and $\beta(t)$ of the space X coincide at the points t_k of discontinuity of the coefficients $a(t), b(t)$:

$$\alpha(t_k) = \beta(t_k), \quad k = 1, 2, \ldots, n, \tag{4.23}$$

then condition (4.21) is necessary as well.

Proof. Because of condition (4.20) we may assume that $b(t) \equiv 1$ (the necessity of (4.20) for both a and b simultaneously is shown similarly to the case $b(t) \equiv 1$).

Sufficiency. Let

$$\omega(t) = \frac{\omega^+(t)}{\omega^-(t)}, \quad \omega^+(t) = \prod_{k=1}^{n} (z - t_k)^{\gamma(t_k)}, \quad \omega^-(t) = \prod_{k=1}^{n} \left(\frac{z - t_k}{z - z_0}\right)^{\gamma(t_k)}$$

be the well known factorization of the function (4.12). We remind that $\Re\gamma(t_k)$ are chosen according to (4.17). We make use of the well known representation

$$aP_+ + P_- = \frac{1}{\omega^-}(a_1P_+ + P_-)\omega^-(\omega P_+ + P_-), \tag{4.24}$$

where a_1 is function (4.13), see for instance, [11], p. 22. The function a_1 is in $C(\Gamma)$ by the choice of the values $\gamma(t_k)$. Relation (4.24) being valid for instance in case of "nice" functions is extended to the space $X(\Gamma)$ by condition (4.4), since both the operator $\omega P_+ + P_-$ and $\frac{1}{\omega^-}(a_1P_+ + P_-)\omega^-$ are bounded in $X(\Gamma)$, the former by condition (4.3) and the latter by Lemma 4.2. The operator $\frac{1}{\omega^-}(a_1P_+ + P_-)\omega^-$ is Fredholm in $X(\Gamma)$ by Theorem B and Lemma 4.2 and its index in $X(\Gamma)$ is equal to $\operatorname{ind} a_1$ which is nothing else, but $ind_X a$. Thus (4.22) is obtained.

It remains to show that the operator $\omega P_+ + P_-$ is invertible in the space $X(\Gamma)$ thanks to the choice (4.17). This is checked in the familiar way: $N(\omega P_+ + P_-) = (\omega P_+ + P_-)N$, where $N = \frac{1}{\omega^-}(\frac{1}{\omega}P_+ + P_-)\omega^-$. The operator K is bounded under the choice (4.17) in the space $X(\Gamma)$ by Lemma 4.2.

Necessity. Let the operator A be Fredholm in X. We first assume that $a(t_k \pm 0) \neq 0, k = 1, 2, \ldots, n$. We have to show that $a(t) \neq 0$ for all other points and that the required conditions on the jumps are satisfied.

1st step (*reduction to a simpler operator*). Since $a(t_k \pm 0) \neq 0$, the function $\omega(t)$ is well defined and the function $a_1(t) = \frac{a(t)}{\omega(t)}$ is continuous. As the commutators $aS - SaI, a \in C(\Gamma)$ are compact in the space $X(\Gamma)$ (see the 1st step in the proof of Theorem B), we have

$$A = (\omega P_+ + P_-)(a_1 P_+ + P_-) + T \tag{4.25}$$

From Fredholmness of the operator A we conclude by the Yood theorem (see f.e. [11], p. 4, Property 1.11) that the operator $\omega P_+ + P_-$ is a Φ_--operator.

2nd step (*necessity of the conditions on jumps for the operator* $\omega P_+ + P_-$). The following lemma reformulates a statement well known for example for $L^p(\Gamma, \rho)$-spaces for the case of the abstract spaces $X(\Gamma)$.

Lemma 4.3. *Let* $a(t_k \pm 0) \neq 0, k = 1, 2, \ldots, n$ *and the space* $X(\Gamma)$ *satisfy conditions* (4.1)–(4.4) *and Axioms* 1–2 *and let* $\alpha(t_k) = \beta(t_k), k = 1, 2, \ldots, n$. *The operator* $\Psi = \omega P_+ + P_-$ *with* ω *defined in* (4.12), *is a* Φ_+- *or* Φ_--*operator in the space* $X(\Gamma)$ *if and only if*

$$\Re\gamma_k \neq \alpha(t_k) \pmod 1 \quad \textit{for all} \quad k = 1, 2, \ldots, n. \tag{4.26}$$

Proof. By the sufficiency part of Theorem C, condition (4.26) is sufficient. To prove the necessity, suppose that $\Re\gamma_k = \alpha(t_k) + r$ for some $r = 0, \pm 1, \pm 2, \ldots$ and for some k, say $k = 1$, but that the operator Ψ is a Φ_+- or Φ_--operator. Let first $\Re\gamma_k \neq \alpha_k \pmod 1$ for all other $k = 2, 3, \ldots, n$. We put $\Psi_{\pm\varepsilon} = \omega_{\pm\varepsilon} P_+ + P_-$, $\varepsilon > 0$, where $\omega_{\pm\varepsilon} = (t - z_0)_1^{\pm\varepsilon} \omega(t)$. This new function has the new exponents $\gamma_1^{\pm\varepsilon} = \gamma_1 \pm \varepsilon$. We choose ε small enough, so that $\Re\gamma_1 \pm \varepsilon - \alpha_1$ is not an integer. Then, by the sufficiency part of Theorem C, the operators Ψ_ε and $\Psi_{-\varepsilon}$ are Fredholm operators in the space $X(\Gamma, \rho)$. The calculation of the index by formula (4.22) gives

$$\operatorname{Ind}_X[(t - z_0)^\nu P_+ + P_-] = [\alpha(t_1) - \Re\nu] \quad \text{in case} \quad \Re\nu \neq \alpha(t_1) + m, \tag{4.27}$$

where $m = 0, \pm 1, \pm 2, \ldots$ and $[\cdots]$ on the right-hand side stands for the entire part of a number. Then

$$\operatorname{Ind}_X \Psi_\varepsilon - \operatorname{Ind}_X \Psi_{-\varepsilon} =$$
$$= [\Re\gamma(t_1) + \varepsilon - \alpha(t_1)] - [\Re\gamma(t_1) - \varepsilon - \alpha(t_1)] = [\varepsilon] - [-\varepsilon] = 1. \tag{4.28}$$

But on the other hand, $\|\Psi_{\pm\varepsilon} - \Psi\|_X \leq c \sup_{t \in \Gamma} |(t - z_0)^{\pm\varepsilon} - 1| \leq c_1 \varepsilon$ which contradicts (4.2) by stability theorem for $\Phi_\pm$-operators in Banach spaces.

This proves the lemma for the case $k = 1$. If condition on (4.26) is violated for several $k = n_1, \ldots, n_m$, the arguments are similar: the operators

$\Psi_{\pm\varepsilon}$ must then be introduced with the functions

$$\omega_{\pm\varepsilon}(t) = \prod_{i=1}^{m}(t-z_0)_i^{\pm\varepsilon}\omega(t). \qquad \square$$

3rd step (*necessity of the conditions for the operator* N). Since the operator $P_+ + \omega P_-$ is a Φ_--operator (see the 1st step), by Lemma 4.3, conditions (4.26) are satisfied. Consequently, by the sufficiency part of our theorem, the operator $P_+ + \omega P_-$ is a Fredholm operator in the space $X(\Gamma)$. As is well known, if any two of the linear operators A, B and AB are Fredholm then the remaining one is Fredholm as well (see, f.e. [11], p. 4, Property 1.12). Therefore, from (4.25) we conclude that the operator $a_1P_+ + P_-$ is Fredholm in X. Then by Theorem B, $a_1(t) \neq 0$ and consequently $a(t) \neq 0$, $t \in \Gamma$.

4rd step. *It remains to lift the assumptions* $a(t_k \pm 0) \neq 0, b(t_k \pm 0) \neq 0$. Suppose that some of the numbers $a(t_k \pm 0)$ are equal to zero and the operator A is Fredholm in $X(\Gamma)$. There exists a complex number ε with an arbitrarily small modulus and a point t_0 close to t_k such that $a(t_k \pm 0)+\varepsilon \neq 0$, but $a(t_0)+\varepsilon = 0$. Let $A_\varepsilon = (a+\varepsilon)P_+ + P_-$. Evidently, $\|A_\varepsilon - A\| = \|\varepsilon I\| = \varepsilon$. Therefore, by the stability theorem for Fredholm operators, we obtain that the operator A_ε is Fredholm for sufficiently small ε. This contradicts the preceding part. $\square$

5. Proof of Theorem A

Proof. To show that the statements of Theorem A may be obtained from Theorem C as a particular case, we have to verify that the space $L^{p(\cdot)}(\Gamma)$ is the space of the type $X(\Gamma)$ under the assumptions of Theorem A. To this end we have to check conditions (4.1)–(4.4) and Axioms 1–2 of Subsection 4.1.

Condition (4.1) is obvious by assumption (2.5). Condition (4.2) is evident.

Condition (4.3) follows from Theorem 2.1.

Condition (4.4), that is, denseness of $C^\infty(\Gamma)$ in $L^{p(\cdot)}(\Gamma)$, follows from Theorem 2.3.

The validity of Axiom 1 for the space $X(\Gamma) = L^{p(\cdot)}(\Gamma)$ follows from Theorem 2.1 according to (4.7). The imbedding $L^{p(\cdot)}(\Gamma, |t-t_0|^\gamma) \subset L^1(\Gamma)$ for $\gamma < 1 - \beta(t_0)$, required by Axiom 2, follows from (2.10) since $\beta(t_0) = \frac{1}{p(t_0)}$ according to (4.7). Finally, the denseness of $C^\infty(\Gamma)$ in the spaces $X(\Gamma, |t-t_0|^\gamma)$ for $t_0 \in \Gamma$ follows as a particular case from Theorem 2.2. $\square$

Remark 5.1. Following the same scheme, it is not difficult to prove that the operator $A = aP_+ + bP_-$ with $a, b \in PC(\Gamma)$ has the same solvability picture in the spaces with variable exponent as in the spaces with constant p, that

is, $\dim\ker A = \varkappa = \operatorname{ind}_{p(\cdot)} a$, $\dim\operatorname{coker} A = 0$, if $\varkappa \geq 0$, and $\dim\ker A = 0$, $\dim\operatorname{coker} A = |\varkappa|$, if $\varkappa \leq 0$.

We also note that, basing on (4.7), one can also easily obtain a similar corollary from Theorem C for the case of the weighted spaces $L^{p(\cdot)}(\Gamma, \rho)$ with the power weight fixed to a finite number of points on Γ.

References

1. A. Böttcher and Yu. Karlovich, Toeplitz and singular integral operators on general Carleson Jordan curves. *In: Singular Integral Operators and Related Topics,* (*Tel Aviv,* 1995), 90, *Oper. Theory Adv. Appl.* 119–152, *Birkhäuser, Basel*, 1996.

2. A. Böttcher and Yu. Karlovich, Carleson Curves, Muckenhoupt weight, and Toeplitz operators. *Basel, Boston, Berlin, Birkhäuser Verlag*, 1997.

3. A. Böttcher abd Yu. Karlovich, Toeplitz operators with PC symbols on general Carleson Jordan curves with arbitrary Muckenhoupt weights. *Trans Amer. Math. Soc.* **351**(1999), No. 8, 3143–3196.

4. D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer, The maximal function on variable L^p spaces. *Preprint. Intituto per le Applicaziopni del Calculo "Mauro Picone", Sezione di Napoli,* 249, 2002.

5. L. Diening, Maximal functions on generalized Lebesgue spaces $L^{p(x)}$. *Preprint Mathematische Fakultät, Albert-Ludwigs-Universitẗ Freiburg,* No. 2, 2002.

6. L. Diening, Maximal functions on generalized Lebesgue spaces $L^{p(x)}$. *Math. Inequal. Appl.* (to appear).

7. L. Diening and M. Ružička, Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics. *Preprint Mathematische Fakultät, Albert-Ludwigs-Universitẗ Freiburg*, No. 21, 2002, 1–20.

8. D. Gaier, Vorlesungen über approximation im komplexen. *Birkhäuser, Basel, Boston, Stutgart*, 1980.

9. I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations, I. Introduction. *Oper. Theory Adv. Appl.*, 53, *Basel–Boston, Birkhäuser–verlag*, 1992.

10. I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations, II. General theory and applications. *Oper. Theory Adv. Appl.*, 54, *Basel–Boston, Birkhäuser–Verlag*, 1992.

11. N. K. Karapetiants and S. G. Samko, Equations with involutive operators. *Birkhäuser, Boston*, 2001.

12. A. Yu. Karlovich, Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces. *Integral Equations Operator Theory* **32**(1998), No. 4, 436–481.

13. B. V. Khvedelidze, The method of Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one variable. (Russian) *Itogi Nauki. i Tekhniki, Sovrem. Probl. Mat.* **7**(1977), 5–162.

14. V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$-spaces. *Rev. Mat. Iberoamericana* (accepted for publication).

15. V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Proc. A. Razmadze Math. Inst.* **129**(2002), 145–149.

16. V. Kokilashvili and S. Samko, Singular integrals and potentials in some Banach spaces with variable exponent. *Proc. A. Razmadze Math. Inst.* **129**(2002), 150–155.

17. V. Kokilashvili and S. Samko, Singular integrals and Weighted Lebesgue spaces with variable exponent. *Georgian Math. J.* **10**(2003), No. 1, 145–146.

18. V. Kokilashvili and S. Samko, Singular integrals and Weighted Lebesgue spaces with variable exponent. *Preprint. Instituto Superior Técnico, Departamento de Matemática*, No. 30, 1–16, *Lisbon, December*, 2002.

19. V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Preprint. Instituto Superior Técnico, Departamento de Matemática,* No. 13, 1–26, *Lisbon, May*, 2002.

20 V. Kokilashvili and S. Samko, Singular integrals and potentials in some Banach spaces with variable exponent. *Preprint. Instituto Superior Técnico, Departamento de Matemática,* No. 24, 1–14, *Lisbon, September*, 2002.

21. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.* **41(116)**(1991), No. 4, 592–618.

22. N. I. Muskhelishvili, Singular integral equations. (Russian) *Nauka, Moscow*, 1968

23. M. Ružička, Electrorheological fluids: modeling and mathematical theory. *Lecture Notes in Mathematics,* 1748. *Springer, Berlin,* 2000.

24. S. G. Samko, Convolution and potential type operators in $L^{p(x)}(R^n)$. *Integral Transform. Spec. Funct.* **7**(1998), No. 3-4, 261–284.

25. S. G. Samko, Convolution type operators in $L^{p(x)}$. *Integral Transform. Spec. Funct.* **7**(1998), No. 1–2, 123–144.

26. S. G. Samko, Differentiation and integration of variable order and the spaces $L^{p(x)}$. *Operator theory for complex and hypercomplex analysis* (*Mexico City,* 1994), 203–219, *Contemp. Math.,* 212, *Amer. Math. Soc., Providence, RI,* 1998.

27. I. I. Sharapudinov, The topology of the space $\mathcal{L}^{p(t)}([0, 1])$. (Russian) *Mat. Zametki* **26**(1979), No. 4, 613–632.

28. I. M. Spitkovsky, Singular integral operators with PC symbols on the spaces with general weights. *J. Funct. Anal.* **105**(1992), No. 1, 129–143.

(Received 26.12.2002)

Authors' addresses:

V. Kokilashvili
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0193
Georgia

S. Samko
University of Algarve
Unidade de Ciências Exactas e Humanas
Campus de Cambelas
Faro, 8000
Portugal