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Maximal and Fractional Operators in Weighted $L^{p(x)}$ Spaces

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Abstract

We study the boundedness of the maximal operator, potential type operators and operators with fixed singularity (of Hardy and Hankel type) in the spaces $L^{p(\cdot)}(\rho, \Omega)$ over a bounded open set in $\mathbb{R}^n$ with a power weight $\rho(x) = |x - x_0|^\gamma$, $x_0 \in \overline{\Omega}$, and an exponent $p(x)$ satisfying the Dini–Lipschitz condition.

1. Introduction

The investigation of the Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent was initiated in [16]. During the last two decades these spaces have been intensively studied, see [3], [6], [8], [10]–[17] and references therein. The interest on these spaces comes from their mathematical curiosity on the one hand and their importance in some applications (see [10], [19], [20]) on the other hand.

As the space $L^{p(\cdot)}(\mathbb{R}^n)$ is not invariant with respect to translations, convolution operators do not behave well in these spaces. For example, the Young theorem is in general not valid in these spaces, see for instance [13, Section 2]. Problems also arise for Mellin convolutions ($n = 1$), since $L^{p(\cdot)}(R^1_+)$ is not invariant with respect to dilations. However, the failure of the Young theorem does not prevent some convolution operators from being bounded operators. Roughly speaking, a convolution operator is bounded in $L^{p(\cdot)}$ if its kernel has singularity at the origin only.

There are two examples, whose importance is difficult to overestimate. One is the convolution with the singular kernel $k(x) = \frac{1}{x}$ ($n = 1$), that is, the well-known singular operator, and the other is the related maximal operator, although the latter is not a convolution.

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For the second operator over open bounded sets the problem of its boundedness was recently solved by L. Diening [1].

In this paper we prove weighted estimates for the maximal operator over bounded open sets and for singular type operators with fixed singularity (of Hardy and Hankel type). We give also weighted estimates for potential type operators of variable order $\alpha(x)$ and show, in particular, that the Sobolev theorem with the limiting exponent

$$\frac{1}{\mu(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$$

is valid for them. We also prove that the potential operator is compact in $L^{p(\cdot)}(\Omega)$.

The main results are formulated in Section 2 as Theorems A–E. Section 3 provides necessary preliminaries and Sections 4–6 contain the proofs of Theorems A–E.

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Notation

Ω is a open bounded set in $\mathbb{R}^n$;

$\mu(\Omega) = |\Omega|$ is the Lebesgue measure of Ω;

$B_r(x) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$;

$|B_r(x)| = \frac{r^n}{n}\,|S^{n-1}|$ is the volume of $B_r(x)$;

$q(x) = \frac{p(x)}{p(x)-1}, \quad 1 < p(x) < \infty, \quad \frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$;

$p_0 = \inf_{x\in\Omega} p(x),\ P = \sup_{x\in\Omega} p(x)$;

$q_0 = \inf_{x\in\Omega} q(x) = \frac{P}{P-1}, \quad Q = \sup_{x\in\Omega} q(x) = \frac{p_0}{p_0-1}$;

c may denote different positive constants.

2. Statement of the Main Results

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 1$, and $p(x)$ a function on $\overline{\Omega}$ satisfying the conditions

$$1 < p_0 \leq p(x) \leq P < \infty, \quad x \in \overline{\Omega} \tag{2.1}$$

and

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \overline{\Omega}. \tag{2.2}$$

The condition (2.2) appears naturally in the theory of the spaces $L^{p(\cdot)}(\Omega)$, see [12]–[15]. In [8] it was shown that this condition is in fact necessary for boundedness of the maximal operator in $L^{p(\cdot)}(\Omega)$. Condition (2.2) also appeared in [9] in case of Hölder spaces $H^{\lambda(x)}$ with variable exponent $\lambda(x)$.

Let

$$M^\beta f(x) = |x - x_0|^\beta \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} \frac{|f(y)|}{|y - x_0|^\beta} \, dy, \tag{2.3}$$

where $x_0 \in \overline{\Omega}$. We write $M = M^0$ in the case where $\beta = 0$.

In the case where $x_0 \in \partial\Omega$ and when considering the necessity of boundedness conditions, we shall make use of a restriction of the type

$$|\Omega_r(x_0)| \sim r^n, \tag{2.4}$$

where $\Omega_r(x_0) = \{y \in \Omega : \ r < |y - x_0| < 2r\}$.

Theorem A *Let $p(x)$ satisfy conditions* (2.1), (2.2). *The operator M^β with $x_0 \in \Omega$ is bounded in $L^{p(x)}(\Omega)$ if and only if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{2.5}$$

If $x_0 \in \partial\Omega$, condition (2.5) *is sufficient for the boundedness of M^β. If $x_0 \in \partial\Omega$ and condition (2.4) is satisfied, then condition (2.5) is also necessary for the boundedness of M^β.*

Let further

$$I^{\alpha(x)} f(x) = \int_\Omega \frac{f(y)}{|x - y|^{n-\alpha(x)}} \, dy, \qquad 0 < \alpha(x) < n. \tag{2.6}$$

Theorem B *Under conditions* (2.1), (2.2) *and the conditions*

$$\inf_{x\in\Omega}\alpha(x)>0 \qquad \text{and} \qquad \sup_{x\in\Omega}\alpha(x)p(x)<n, \tag{2.7}$$

the potential operator $I^{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot)}(\Omega)$ *into* $L^{r(\cdot)}(\Omega)$ *with*

$$\frac{1}{r(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n}\,.$$

Theorem C *Under conditions* (2.1), (2.2) *and the condition* $\inf_{x\in\Omega}\alpha(x)>0$, *the operator* $I^{\alpha(\cdot)}$ *is compact in* $L^{p(\cdot)}(\Omega)$.

For the weighted potential operator

$$I_\beta^{\alpha(x)}f(x)=|x-x_0|^\beta\int_\Omega \frac{f(y)}{|y-x_0|^\beta|x-y|^{n-\alpha(x)}}\,dy, \quad x_0\in\overline{\Omega} \tag{2.8}$$

the following theorem holds.

Theorem D *Under conditions* (2.1), (2.2) *and the condition* $\inf_{x\in\Omega}\alpha(x)>0$, *the operator* $I_\beta^{\alpha(\cdot)}$ *is bounded in* $L^{p(\cdot)}(\Omega)$ *if*

$$-\frac{n}{p(x_0)}<\beta<\frac{n}{q(x_0)}\,. \tag{2.9}$$

Let now $n=1$, $\Omega=(0,\ell)$ with $0<\ell<\infty$ and $x_0=0$. We consider the weighted Hardy-type operators

$$H^\beta f(x)=x^{\beta-1}\int_0^x\frac{f(t)}{t^\beta}\,dt, \quad H_*^\beta f(x)=x^\beta\int_x^\ell\frac{f(t)}{t^{\beta+1}}\,dt \tag{2.10}$$

and the weighted Hankel-type operator

$$\mathcal{H}^\beta f(x)=x^\beta\int_0^\ell\frac{f(t)}{t^\beta(t+x)}\,dt. \tag{2.11}$$

Theorem E *Suppose* $1\le p(x)\le P<\infty$ *for* $x\in[0,\ell]$.

I. *Let conditions* (2.1), (2.2) *be satisfied on a neighbourhood* $[0,d]$ *of the origin,* $d>0$. *If*

$$-\frac{1}{p(0)}<\beta<\frac{1}{q(0)}\,, \tag{2.12}$$

then all the operators H^β, H_*^β *and* $\mathcal{H}^\beta$ *are bounded from* $L^{p(\cdot)}(\Omega)$ *into* $L^{s(\cdot)}(\Omega)$ *with any* $s(x)$ *such that* $1\le s(x)\le S<\infty$ *for* $0\le x\le\ell$,

$$s(0)=p(0) \quad \text{and} \quad |s(x)-p(x)|\le\frac{A}{\ln\frac{1}{x}}\,, \quad 0<x<\delta, \quad \delta>0\,. \tag{2.13}$$

II. *If* $p(0) \leq p(x)$, $0 \leq x \leq d$, *for some* $d > 0$, *then the same statement on boundedness from* $L^{p(\cdot)}(\Omega)$ *into* $L^{s(\cdot)}(\Omega)$ *is true if the requirement of the validity of conditions* (2.1), (2.2) *on* $[0, d]$ *is replaced by the weaker assumption that*

$$p(0) > 1 \quad \textit{and} \quad |s(x) - p(0)| < \frac{A}{\ln \frac{1}{x}}, \quad 0 < x < \min\left(\ell, \frac{1}{2}\right). \tag{2.14}$$

Corollary *Let* $1 \leq p(x) \leq P < \infty$ *on* $[-1, 1]$. *The singular operator with fixed singularity,*

$$S^\beta f = \frac{|x|^\beta}{\pi} \int_0^1 \frac{f(t)}{t - x} \frac{dt}{t^\beta}, \quad x \in [-1, 0]\ ,$$

is bounded from $L^{p(x)}([0,1])$ *into* $L^{p(x)}([-1,0])$ *if*

1. $p(0) > 1$;
2. $p(x)$ *satisfies condition* (2.2) *on* $[-\delta, \delta]$ *for some* $\delta > 0$;
3. $-\frac{1}{p(0)} < \beta < \frac{1}{q(0)}$.

3. Preliminaries

3.1. On the spaces $\mathbf{L}^{p(\cdot)}(\boldsymbol{\Omega})$

The basics on the spaces $L^{p(\cdot)}$ may be found in [3], [6], [11]–[13]. Here we recall only some important facts and definitions.

Let Ω be an open set in $\mathbb{R}^n$ and $p(x)$ a function on $\overline{\Omega}$ such that

$$1 \leq p(x) \leq P < \infty, \quad x \in \overline{\Omega}.$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$I_p(f) := \int_\Omega |f(x)|^{p(x)}\, dx < \infty. \tag{3.1}$$

This is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \ I_p\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

The Hölder inequality holds in the form

$$\int_\Omega |f(x)g(x)|\, dx \leq k\|f\|_{p(\cdot)} \cdot \|g\|_{q(\cdot)} \tag{3.2}$$

with $k = \frac{1}{p_0} + \frac{1}{q_0}$.

The functional $I_p(f)$ and the norm $\|f\|_{p(\cdot)}$ are simultaneously greater than one and simultaneously less than 1:

$$\|f\|_{p(\cdot)}^{P} \le I_p(f) \le \|f\|_{p(\cdot)}^{p_0} \quad \text{if } \|f\|_{p(\cdot)} \le 1 \tag{3.3}$$

and

$$\|f\|_{p(\cdot)}^{p_0} \le I_p(f) \le \|f\|_{p(\cdot)}^{P} \quad \text{if } \|f\|_{p(\cdot)} \ge 1. \tag{3.4}$$

The imbedding

$$L^{p(x)} \subseteq L^{r(x)}, \;\; 1 \le r(x) \le p(x) \le P < \infty$$

is valid if $|\Omega| < \infty$. In that case

$$\|f\|_{r(\cdot)} \le m\|f\|_{p(\cdot)}, \;\; m = a_2 + (1-a_1)|\Omega|, \tag{3.5}$$

where $a_1 = \inf_{x\in\Omega} \frac{r(x)}{p(x)}$ and $a_1 = \sup_{x\in\Omega} \frac{r(x)}{p(x)}$.

We remark that in this paper we deal with $L^{p(\cdot)}$-spaces on open sets in $\mathbb{R}^n$. We shall give some results on boundedness of singular operators with fixed singularity on curves in the complex plane in another paper. Here we only mention that the space $L^{p(\cdot)}(\Gamma)$ on a rectifiable simple curve

$$\Gamma = \big\{t \in \mathbb{C} : \;\; t = t(s), \;\; 0 \le s \le \ell\big\},$$

where s is the arc length, may be introduced in a similar way via the functional

$$I_p(f) = \int_\Gamma |f(t)|^{p(t)}\, |dt| = \int_0^\ell \big|f[t(s)]\big|^{p[t(s)]}\, ds.$$

Condition (2.2) may be imposed either on the function $p(t)$:

$$|p(t_1) - p(t_2)| \le \frac{A}{\ln \frac{1}{|t_1-t_2|}}, \;\; |t_1 - t_2| \le \frac{1}{2}, \;\; t_1, t_2 \in \Gamma \tag{3.6}$$

or on the function $p_*(s) = p[t(s)]$:

$$\big|p_*(s_1) - p_*(s_2)\big| \le \frac{A}{\ln \frac{1}{|s_1-s_2|}}, \;\; |s_1 - s_2| \le \frac{1}{2}, \;\; s_1, s_2 \in [0,\ell]. \tag{3.7}$$

Since $|t(s_1) - t(s_2)| \le |s_1 - s_2|$, (3.6) always implies (3.7). Conversely, (3.7) implies (3.6) if there exists a $\lambda > 0$ such that

$$|s_1 - s_2| \le c\big|t(s_1) - t(s_2)\big|^\lambda.$$

Therefore, conditions (3.6) and (3.7) are equivalent on curves satisfying the so called chord condition, for example.

Let

$$K_\varepsilon f = \frac{1}{\varepsilon^n}\int_\Omega \mathcal{K}\Big(\frac{x-y}{\varepsilon}\Big) f(y)\,dy,$$

where $\mathcal{K}(x)$ has a compact support in $B_R(0)$. In [14], [15] the uniform estimate

$$\big\|K_\varepsilon f\big\|_{L^{p(\cdot)}(\Omega_R)} \le c\big\|f\big\|_{L^{p(\cdot)}(\Omega)}, \tag{3.8}$$

where $\Omega_R = \{x : \operatorname{dist}(x,\Omega) \le R\}$, was proved under the assumption that $p(x)$ is defined in Ω_R and satisfies conditions (2.1), (2.2) on Ω_R.

For the potential type operator $I^{\alpha(x)}$ defined in (2.6), the following statement was proved in [13] in the case of a bounded open set Ω.

Theorem 3.1 *Under assumptions* (2.1), (2.2) *and* (2.7) *the operator* $I^{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot)}(\Omega)$ *into* $L^{r(\cdot)}(\Omega)$,

$$\frac{1}{r(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n},$$

unconditionally if $p(x)$ *is constant, and under the condition that the maximal operator* $M = M^0$ *is bounded in* $L^{p(\cdot)}(\Omega)$ *in the general case.*

3.2. Auxiliary lemmas for averages

Let

$$M_r f(x) = \frac{1}{B_r(x)|}\int_{B_r(x)} |f(y)|\,dy \tag{3.9}$$

denote the mean of the function f over the ball $B_r(x)$. We also need the weighted means

$$M_r^\beta f(x) = \frac{|x-x_0|^\beta}{|B_r(x)|}\int_{B_r(x)} \frac{|f(y)|}{|y-x_0|^\beta}\,dy \tag{3.10}$$

related to the weighted maximal operator (2.3). In (3.9), (3.10) we assume that $f(y) = 0$ for $y \notin \Omega$.

Lemma 3.2 *If* $0 \le \beta < n$, *the inequality*

$$M_r^\beta(1) = \frac{|x-x_0|^\beta}{|B_r(x)|}\int_{B_r(x)} \frac{dy}{|y-x_0|^\beta} \le c \tag{3.11}$$

holds with $c > 0$ *not depending on* x, r *and* x_0.

This lemma is known, but we give its proof in the Appendix for the completeness of the presentation.

Lemma 3.3 *Suppose that $x_0 \in \partial\Omega$ and condition (2.4) is satisfied. If the function $|x-x_0|^\gamma$ is in $L^1(\Omega)$, then necessarily $\gamma > -n$.*

Proof. Suppose that $x_0 \in \partial\Omega$ and $|x-x_0|^\gamma \in L^1(\Omega)$. We have

$$\int_\Omega |x-x_0|^\gamma \, dx \geq \int_{\Omega_r} |x-x_0|^\gamma \, dx = |\xi - x_0|^\gamma |\Omega_r|,$$

where $\xi \in \Omega_r$. Since $|\xi - x_0|^\gamma \sim r^\gamma$, by (2.4) we obtain

$$\int_{\Omega_r} |x-x_0|^\gamma \, dx \geq c r^{\gamma+n}$$

which is only possible if $\gamma + n > 0$. ■

4. The Weighted Maximal Operator in the Space $L^{p(\cdot)}(\Omega)$

In what follows, Ω is an open bounded set in $\mathbb{R}^n$ and $x_0 \in \overline{\Omega}$.

4.1. A pointwise estimate for the weighted means (3.10)

Theorem 4.1 *Let $p(x)$ satisfy conditions* (2.1) *and* (2.2)*. If*

$$0 \leq \beta < \frac{n}{q(x_0)}\,, \tag{4.1}$$

then

$$\left[M_r^\beta f(x)\right]^{p(x)} \leq c \left(1 + \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|^{p(y)} \, dy\right) \tag{4.2}$$

for all $f \in L^{(p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)} \leq 1$, where $c = c(p, \beta)$ is a constant not depending on x, r and x_0.

Proof. From (4.1) and the continuity of $p(x)$ we conclude that there exists a $d > 0$ such that

$$\beta q(x) < n \ \text{ for all } \ |x - x_0| \leq d. \tag{4.3}$$

Without loss of generality we assume that $d \leq 1$. Let

$$p_r(x) = \min_{|y-x| \leq r} p(y)$$

and

$$\frac{1}{q_r(x)} = 1 - \frac{1}{p_r(x)}\,.$$

From (4.1) it is easily seen that

$$\beta q_r(x) < n \ \text{ if } \ |x - x_0| \leq \frac{d}{2} \ \text{ and } \ 0 < r \leq \frac{d}{4}\,. \tag{4.4}$$

1^{o} The case $|x-x_0|\le d/2$ and $0<r\le d/4$ (the main case)

In this case, applying the Hölder inequality with the exponents $p_r(x)$ and $q_r(x)$ to the integral on the right-hand side of the equality

$$\left|M_r\Big(\frac{f(y)}{|y-x_0|^\beta}\Big)\right|^{p(x)} = \frac{c}{r^{np(x)}}\left(\int_{B_r(x)}\frac{|f(y)|}{|y-x_0|^\beta}\,dy\right)^{p(x)}$$

and taking into account (4.3), we get

$$\left|M_r\Big(\frac{f(y)}{|y-x_0|^\beta}\Big)\right|^{p(x)} \le$$

$$\le \frac{c}{r^{np(x)}}\left(\int_{B_r(x)}|f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}}\cdot\left(\int_{B_r(x)}\frac{dy}{|y-x_0|^{\beta q_r(x)}}\right)^{\frac{p(x)}{q_r(x)}}. \tag{4.5}$$

Making use of the estimate (3.11), we obtain

$$\left|M_r\left(\frac{f(y)}{|y-x_0|^\beta}\right)\right|^{p(x)} \le c\,\frac{|x-x_0|^{-\beta p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\left(\int_{B_r(x)}|f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}}.$$

Here

$$\int_{B_r(x)}|f(y)|^{p_r(x)}\,dy \le \int_{B_r(x)}dy + \int_{B_r(x)\cap\{y:\,|f(y)|\ge 1\}}|f(y)|^{p(y)}\,dy,$$

since $p_r(x)\le p(y)$ for $y\in B_r(x)$. Since $p(x)$ is bounded, we see that

$$\left|M_r\left(\frac{f(y))}{|y-x_0|^\beta}\right)\right|^{p(x)} \le c_1\,\frac{|x-x_0|^{-\beta p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\left[r^n+\frac{1}{2}\int_{B_r(x)}|f(y)|^{p(y)}\,dy\right]^{\frac{p(x)}{p_r(x)}}.$$

Since $r\le d/2\le 1/2$ and the second term in the brackets is also less than or equal to $1/2$, we arrive at the estimate

$$\begin{aligned}|M_r^\beta f|^{p(x)} &\le \frac{c}{r^{\frac{np(x)}{p_r(x)}}}\left[r^n+\int_{B_r(x)}|f(y)|^{p(y)}\,dy\right]\\ &\le c\,r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}}\left[1+\frac{1}{r^n}\int_{B_r(x)}|f(y)|^{p(y)}\,dy\right].\end{aligned}$$

From here (4.2) follows, since

$$r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} \le c.$$

Indeed,

$$r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} = e^{\frac{n}{p_r}\,[p(x)-p_r(x)]\,\ln\frac{1}{r}},$$

where

$$\left|\frac{n}{p_r}\,\big[p(x)-p_r(x)\big]\ln\frac{1}{r}\right| \leq n\big|p(x)-p(\xi_r)\big|\ln\frac{1}{r}$$

with $\xi_r \in B_r(x)$, and then by (2.2),

$$\left|\frac{n}{p_r}\,\big[p(x)-p_r(x)\big]\ln\frac{1}{r}\right| \leq nA\frac{\ln\frac{1}{r}}{\ln\frac{1}{|x-\xi_r|}} \leq nA,$$

since $|x-\xi_r| \leq r$.

2° The case $|x-x_0| \geq d/2$, $0 < r \leq d/4$.

This case is trivial, because

$$|y-x_0| \geq |x-x_0| - |y-x| \geq \frac{d}{2} - \frac{d}{4} = \frac{d}{4}\,.$$

Thus $|y-x_0|^\beta \geq \left(\frac{d}{4}\right)^\beta$. Since $|x-x_0|^\beta \leq (\operatorname{diam}\Omega)^\beta$, it follows that

$$M_r^\beta f(x) \leq cM_r f(x),$$

and one may proceed as above for the case $\beta = 0$ (the condition $|x-x_0| \leq \frac{d}{2}$ is not needed in this case).

3° The case $r \geq d/4$

This case is also easy. It suffices to show that the left-hand side of (1.4) is bounded. We have

$$M_r^\beta f(x) \leq \frac{c(\operatorname{diam}\Omega)^\beta}{\left(\frac{d}{4}\right)^n}\left[\int_{|y-x_0|\leq\frac{d}{8}} \frac{|f(y)|}{|y-x_0|^\beta}\,dy + \int_{|y-x_0|\geq\frac{d}{8}} \frac{|f(y)|}{|y-x_0|^\beta}\,dy\right].$$

Here the first integral is estimated via the Hölder inequality with the exponents

$$p_{\frac{d}{8}} = \min_{|y-x_0|\leq\frac{d}{8}} p(y) \quad\text{and}\quad q_{\frac{d}{8}} = p'_{\frac{d}{8}}$$

as in (4.5), which is possible since $\alpha q_{\frac{d}{8}} < n$. The estimate of the second integral is trivial since $|y-x_0| \geq d/8$. ■

Corollary *Let $0 \leq \beta < n/q(x_0)$. If conditions (2.1), (2.2) are satisfied, then*

$$|M^\beta f(x)|^{p(x)} \leq c\left(1 + M\left[|f(\cdot)|^{p(\cdot)}\right](x)\right) \tag{4.6}$$

for all $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)} \leq 1$.

4.2. Boundedness of the weighted maximal operator

Proof of Theorem A

We have to show that

$$\left\|M^\beta f\right\|_{p(\cdot)} \le c$$

in some ball $\|f\|_{p(\cdot)} \le R$, which is equivalent to the inequality

$$I_p(M^\beta f) \le c \ \text{ for } \ \left\|f\right\|_{p(\cdot)} \le R.$$

We observe that

$$|x-x_0|^{\beta p(x)} \sim |x-x_0|^{\beta p(x_0)} \tag{4.7}$$

in case $p(x)$ satisfies the condition (2.2).

I. Sufficiency part. From (4.7) we obtain

$$I_p(M^\beta f) \le c \int_\Omega |x-x_0|^{\beta p(x_0)} \left| M\left(\frac{f(y)}{|y-x_0|^\beta}\right)(x)\right|^{p(x)} dx.$$

Following the idea in [1, p. 25], we represent this as

$$I_p(M^\beta f) \le c \int_\Omega \left(|x-x_0|^{\beta r(x_0)} \left| M\left(\frac{f(y)}{|y-x_0|^\beta}\right)(x)\right|^{r(x)}\right)^{p_0} dx, \tag{4.8}$$

where $r(x) = p(x)/p_0$. In the sequel we distinguish between the cases $\beta \le 0$ and $\beta \ge 0$.

I.a) The case $-n/p(x_0) < \beta \le 0$

Estimate (4.6) with $\beta = 0$ says that

$$|M\psi(x)|^{r(x)} \le c\Big(1 + M\left[\psi^{r(\cdot)}\right](x)\Big) \tag{4.9}$$

for all $\psi \in L^{r(\cdot)}(\Omega)$ with $\|\psi\|_{r(\cdot)} \le 1$. For $\psi(x) = \frac{f(x)}{|x-x_0|^\beta}$ we have

$$\left\|\psi\right\|_{r(\cdot)} \le a_0 \left\| f\right\|_{r(\cdot)}, \quad a_0 = (\operatorname{diam}\Omega)^{|\beta|},$$

where we took into account that $\beta \le 0$. From imbedding (3.5) we obtain

$$\left\|\psi\right\|_{r(\cdot)} \le a_0 \cdot k\left\|f\right\|_{p(\cdot)} \le a_0 k R.$$

Therefore we choose $R = \frac{1}{a_0 k}$. Then $\|\psi\|_{r(\cdot)} \le 1$, so that (4.9) is applicable. From (4.8) we now get

$$I_p(M^\beta f) \le c \int_\Omega \left(|x-x_0|^{\beta r(x_0)} \left[1 + M\left(\left|\frac{f(y)}{|y-x_0|^\beta}\right|^{r(y)}\right)\right]\right)^{p_0} dx.$$

By property (4.7), this yields

$$I_p(M^\beta f) \le c \int_\Omega \left\{ |x-x_0|^{\beta p(x_0)} + \left(|x-x_0|^{\beta r(x_0)} M \left(\frac{f(y)|^{r(y)}}{|y-x_0|^{\beta r(x_0)}} \right) \right)^{p_0} \right\} dx$$
$$\le c + c \int_\Omega \left(M^\gamma \left(|f(\cdot)|^{r(\cdot)} \right)(x) \right)^{p_0} dx,$$

where

$$\gamma = \beta r(x_0) = \frac{\beta p(x_0)}{p_0}\,.$$

As is known [7], the weighted maximal operator M^γ is bounded in L^{p_0} with a constant p_0 if $-\frac{n}{p_0} < \gamma < \frac{n}{p_0'}$, which is satisfied since $-\frac{n}{p(x_0)} < \beta \le 0$. Therefore,

$$I_p(M^\beta f) \le c + c \int_\Omega |f(y)|^{r(y)\cdot p_0}\, dy = c + c \int_\Omega |f(y)|^{p(y)}\, dy < \infty$$

I.b) The case $0 \le \beta < n/q(x_0)$.

We represent the functional $I_p(M^\beta f)$ in the form

$$I_p(M^\beta f) = \int_\Omega \left(\left| M^\beta f(x) \right|^{r(x)} \right)^\lambda dx \tag{4.10}$$

with $r(x) = \frac{p(x)}{\lambda} > 1$, $\lambda > 1$, where λ will be chosen in the interval $1 < \lambda < p_0$. In (4.10), we wish to use the pointwise weighted estimate (4.6):

$$|M^\beta f(x)|^{r(x)} \le c\big[1 + M(f^{r(\cdot)})(x)\big]. \tag{4.11}$$

This estimate is applicable according to Theorem 4.1 if $\|f\|_{r(\cdot)} \le c$ and

$$\beta < \frac{n}{[r(x_0)]'}\,. \tag{4.12}$$

The condition $\|f\|_{r(\cdot)} \le c$ is satisfied since $r(x) \le p(x)$. Condition (4.12) is fulfilled if $\lambda < \frac{n-\beta}{n}\, p(x_0)$. Therefore, under the choice

$$1 < \lambda < \min\left(p_0, \frac{n-\beta}{n}\, p(x_0) \right)$$

we may apply (4.11) to (4.10). This yields

$$I_p(M^\beta f) \le c + c \int_\Omega \left| M(|f|^{r(\cdot)})(x) \right|^\lambda dx \le c + c \int_\Omega \left(|f(x)|^{r(x)} \right)^\lambda dx$$

by the boundedness of the maximal operator M in $L^\lambda(\Omega)$, $\lambda > 1$. Hence

$$I_p(M^\beta f) \le c + c \int_\Omega |f(x)|^{p(x)}\, dx \le c.$$

II. Necessity part. Suppose that M^β is bounded in $L^{p(x)}(\Omega)$. Then, given a function $f(x)$ such that

$$I_p(wf) \le c_1, \quad w(x) = |x - x_0|^\beta, \tag{4.13}$$

we have

$$I_p(wMf) \le c \tag{4.14}$$

(for all f satisfying condition (4.13)).

1) We choose $f(x) = |x - x_0|^\mu$ with $\mu > -\beta - \frac{n}{p(x_0)}$. Then

$$I_p(wf) \le c \int_{|x-x_0|<r} |x - x_0|^{(\beta+\mu)p(x)} dx \le c \int_{|x|<r} |x|^{(\beta+\mu)p(x_0)} dx \ ,$$

where the integral converges, so that we are in the situation (4.13). However,

$$I_p(w\, Mf) \ge c \int_{\Omega\cap B_r(x_0)} |x - x_0|^{\beta p(x_0)} dx \ ,$$

which diverges if $\beta p(x_0) < -n$; here we take into account Lemma 3.3 in the case $x_0 \in \partial\Omega$. Therefore, from (4.14) it follows that $\beta > -\frac{n}{p(x_0)}$.

2) To show the necessity of the right-hand side bound in (2.5), suppose that, on the contrary, $\beta \ge \frac{n}{q(x_0)}$. Let first $\beta > \frac{n}{q(x_0)}$. We choose

$$f(x) = \frac{1}{|x - x_0|^n} \ ,$$

for which $I_p(wf)$ converges but Mf just does not exist. Let now $\beta = \frac{n}{q(x_0)}$. We choose

$$f(x) = \frac{1}{|x - x_0|^n}\left(\ln \frac{1}{|x - x_0|}\right)^\gamma, \quad |x - x_0| \le \frac{1}{2} \, .$$

Then $I_p(wf)$ exists under the choice $\gamma < -\frac{1}{p(x_0)}$, but Mf does not exist when $\gamma > -1$. Thus, taking $\gamma \in \left(-1, -\frac{1}{p(x_0)}\right)$, we arrive at a contradiction.

4.3. Weighted supremal Poisson operator

Let

$$u_f(x, y) = \frac{y}{c_n} \int_{\mathbb{R}^n} \frac{f(x - t)dt}{(|t|^2 + y^2)^{\frac{n+1}{n}}}, \quad y > 0,$$

be the Poisson integral. Here

$$c_n = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \, .$$

The theorem below provides a weighted estimate for the non-tangential supremum of the Poisson integral $u_f(x,y)$. We put

$$\Gamma_a(x) = \{(\xi, y) : |\xi - x| < ay\} \qquad \text{with fixed} \qquad a > 0.$$

Theorem 4.2 *Let $f(x)$ have a compact support in a bounded domain Ω. Under assumptions* (2.1) *and* (2.2), *for the weighted estimate*

$$\Big\| \, |x - x_0|^\beta \sup_{(\xi,y)\in\Gamma_a(x)} |u_f(\xi, y)| \, \Big\|_{L^{p(\cdot)}(\Omega)} \leq c \Big\| \, |x - x_0|^\beta f(x) \, \Big\|_{L^{p(\cdot)}(\Omega)} \tag{4.15}$$

with an interior point $x_0 \in \Omega$ to be valid, it is necessary and sufficient that $-n/p(x_0) < \beta < n/q(x_0)$.

In the case $x_0 \in \partial\Omega$, this condition is sufficient for any x_0 and necessary if x_0 satisfies condition (2.4).

Proof. It suffices to refer to the fact that

$$\sup_{(\xi,y)\in\Gamma_a(x)} |u_f(x, y)| \sim Mf(x)$$

(see [2, p. 45]), and to make use of Theorem A. ■

Remark. A non-weighted estimate of the weaker form

$$\sup_{y>0} \big\| u_f(x, y) \big\|_{L^{p(\cdot)}(\Omega)} \leq c \big\| f \big\|_{L^{p(\cdot)}(\Omega)}$$

follows also from (3.8). In the one-dimensional case, an estimate of the type (4.15) for 2π-periodic functions was obtained in [17].

5. Proof of Theorems B–D

5.1. Proof of Theorem B

This theorem is an immediate consequence of Theorem 3.1 and Theorem A (the latter for the case $\beta = 0$). ■

5.2. Proof of Theorem D

We have

$$\begin{aligned} I_\beta^{\alpha(x)} f(x) = |x - x_0|^\beta \int_{|x-y|>1,\, y\in\Omega} \frac{f(y)dy}{|x - y|^{n-\alpha(x)}|y - x_0|^\beta} + \\ + |x - x_0|^\beta \int_{|x-y|<1,\, y\in\Omega} \frac{f(y)dy}{|x - y|^{n-\alpha(x)}|y - x_0|^\beta} = \\ = A_1 f(x) + A_2 f(x). \end{aligned} \tag{5.1}$$

For the first term we have

$$|A_1 f(x)| \le c|x-x_0|^\beta \int_\Omega \frac{|f(y)dy}{|y-x_0|^\beta} \tag{5.2}$$

and the Hölder inequality (3.2) yields

$$\int_\Omega \frac{|f(y)|dy}{|y-x_0|^\beta} \le c\|f\|_{p(\cdot)} \, \big\| |y-x_0|^{-\beta}\big\|_{q(\cdot)} \le c\{I_q(|y-x_0|^{-\beta})\}^\theta \|f\|, \tag{5.3}$$

where $\theta = \frac{1}{Q}$ if $I_q(\cdots) \le 1$ and $\theta = \frac{1}{q_0}$ otherwise. Obviously,

$$I_q(|y-x_0|^{-\beta}) \le c\int_\Omega |y-x_0|^{-\beta q(x_0)}dy = c < \infty \tag{5.4}$$

by property (4.7) and the condition $\beta q(x_0) < n$. Thus from (5.2)–(5.4) we get

$$|A_1 f(x)| \le c|x-x_0|^\beta \, \|f\|_{p(\cdot)}. \tag{5.5}$$

For the term $A_2 f(x)$ we have

$$|A_2 f(x)| \le |x-x_0|^\beta \sum_{k=0}^{\infty} \int_{2^{-(k+1)}<|x-y|<2^{-k}} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}|y-x_0|^\beta} \;,$$

where it is assumed that $f(x)$ is continued as zero beyond Ω if necessary.

For those x for which $\alpha(x) \le n$, we obtain

$$\begin{aligned} |A_2 f(x)| &\le 2^n|x-x_0|^\beta \sum_{k=0}^{\infty} 2^{k[n-\alpha(x)]} \cdot 2^{-kn}\frac{1}{2^{kn}} \int_{|x-y|<2^{-k}} \frac{|f(y)|dy}{|y-x_0|^\beta} \le \\ &\le 2^n|x-x_0|^\beta \sum_{k=0}^{\infty} 2^{-k\alpha(x)} M\Big(\frac{f(y)}{|y-x_0|^\beta}\Big). \end{aligned}$$

Therefore,

$$|A_2 f(x)| \le cM^\beta f(x) \tag{5.6}$$

with $c = 2^n \sum_{k=0}^\infty 2^{-k\alpha_0}$, $\alpha_0 = \inf_{x\in\Omega} \alpha(x)$.

In the case $\alpha(x) \ge n$, the pointwise estimate of $A_0(x)$ is the same as that for $A_1(x)$. Consequently, for all $x \in \Omega$ by means of (5.5) and (5.6) we obtain

$$\big|I_\beta^{\alpha(x)} f(x)\big| \le cM^\beta f(x) + c|x-x_0|^\beta \|f\|_{p(\cdot)}. \tag{5.7}$$

Therefore,

$$\big\|I_\beta^{\alpha(x)} f\big\|_{p(\cdot)} \le c\big\|M^\beta f\big\|_{p(\cdot)} + c\big\| \, |x-x_0|^\beta \, \big\|_{p(\cdot)} \cdot \big\|f\big\|_{p(\cdot)}.$$

It remains to apply Theorem A to the first term in (5.7) and to notice that $\big\| \, |x-x_0|^\beta\big\|_{p(\cdot)}$ is finite, the latter being obtained as in (5.4). ■

5.3. Proof of Theorem C

From Theorem D we already know that the operator $I^{\alpha(x)}$ is bounded in $L^{p(\cdot)}(\Omega)$. To show its compactness, we represent it as

$$I^{\alpha(x)}f(x) = \int_{|x-y|<\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}} + \int_{|x-y|>\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}} = \tag{5.8}$$
$$= K_\varepsilon f(x) + T_\varepsilon f(x)$$

under the usual assumption that $f(x) \equiv 0$ for $y \notin \Omega$. As in the proof of Theorem D, we have

$$|K_\varepsilon f(x)| \leq \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\varepsilon<|x-y|<2^{-k}\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}} \leq c\varepsilon^{\alpha_0}(Mf)(x) \tag{5.9}$$

with $\alpha_0 = \inf_{x\in\Omega} \alpha(x) > 0$.

The compactness of the operator T_ε may be shown via direct approximation by finite-dimensional operators. Indeed, denote $t_\varepsilon(x,y) \equiv 1$ if $|x-y| \geq \varepsilon$ and $t_\varepsilon = 0$ otherwise. As is known, functions of the form

$$f_n(x,y) = \sum_{k=1}^{n} a_k(x)b_k(y),$$

where $b_k(y) = \chi_{B_k}(y)$, B_k are non-intersecting sets on Ω, and $a_n(x) \in L^Q(\Omega)$, form a dense set in the mixed norm space $L^P[L^Q](\Omega \times \Omega)$ for all constant exponents P and Q, $1 \leq P < \infty$, $1 \leq Q < \infty$. Therefore for the function $t_\varepsilon(x,y)$ with any fixed $\varepsilon > 0$, there exists a sequence of function $k_n(x,y)$ such that

$$\lim_{n\to\infty} \left\| \left\| t_\varepsilon(x,y) - k_n(x,y) \right\|_Q \right\|_P = 0. \tag{5.10}$$

Then the finite-dimensional operators

$$A_n f(x) = \int_\Omega k_n(x,y)\, f(y)\, dy,$$

which are compact in $L^{p(\cdot)}(\Omega)$, approximate the operator T_ε in the operator norm of $L^{p(\cdot)}(\Omega)$ as $n \to \infty$. Indeed, taking into account imbedding (3.5), we obtain

$$\left|(T_\varepsilon - A_n)f(x)\right| \leq c\left\|f\right\|_{p(\cdot)} \left\|k_n(x,\cdot) - t_\varepsilon(x,\cdot)\right\|_{q(\cdot)} \leq c\left\|f\right\|_{p(\cdot)} \left\|k_n(x,\cdot) - t_\varepsilon(x,\cdot)\right\|_Q$$

and then

$$\left\|(T_\varepsilon - A_k)f\right\|_{p(\cdot)} \leq c\left\|f\right\|_{p(\cdot)} \left\| \left\|k_n(x,\cdot) - t_\varepsilon(x,\cdot)\right\|_Q \right\|_{p(\cdot)}.$$

Therefore, by the same imbedding (3.5),

$$\left\|T_\varepsilon - A_k\right\|_{L^{p(\cdot)}\to L^{p(\cdot)}} \leq \left\| \left\|k_n - t_\varepsilon\right\|_Q \right\|_P \to 0$$

in view of (5.10). Consequently, the operators T_ε are compact in $L^{p(\cdot)}(\Omega)$.

It remains to observe that, by (5.8) and (5.9) and by the boundedness of the maximal operator,

$$\left\|I^{\alpha(\cdot)} - T_\varepsilon\right\|_{L^{p(\cdot)}\to L^{p(\cdot)}} = \left\|K^\varepsilon\right\|_{L^{p(\cdot)}\to L^{p(\cdot)}} \leq \varepsilon^{\alpha_0}\left\|M\right\|_{L^{p(\cdot)}\to L^{p(\cdot)}} \to 0\ ,$$

so that $I^{\alpha(x)}$ is a compact operator as well. ■

6. Weighted Estimates for Operators with Fixed Singularity

The following operators may be treated as operators with fixed singularity:

a) the Hardy type operators (2.10) on $[0, l]$;

b) the Hankel operator (2.11) on $[0, l]$;

c) singular operators on a curve Γ_1 with the "outer" variable on another curve Γ_2, the latter having a unique common point with Γ_1; commutators of the singular operator with the operators of multiplication by piece-wise continuous functions.

For such operators, in contrast to the maximal and potential operators, the "global" Dini–Lipschitz condition (2.2) may be replaced by a "local" condition at the point of the fixed singularity.

In this paper we consider the cases a) and b) and deal with the weighted version of the Hardy and Hankel operators. The case of singular operators with fixed singularity on curves in the complex plane is postponed to another paper.

Proof of Theorem E

The case of the Hardy operators

Part I. Suppose as usual that $\left\|f\right\|_{p(\cdot)} \leq 1$. Let $d_0 = \min(d, \delta)$. We have

$$\int_0^l |H^\beta f(x)|^{s(x)} dx \leq \int_0^{d_0} |H^\beta f(x)|^{s(x)} dx + \frac{1}{d_0^a}\int_{d_0}^l \left|\int_0^x \frac{f(t)}{t^\beta} dt\right|^{s(x)} dx, \tag{6.1}$$

where $a = (1-\beta)P$.

The second term may be estimated via the Hölder inequality:

$$\left|\int_0^x \frac{f(t)}{t^\beta}dt\right| \le k\|f\|_{p(\cdot)}\,\|t^{-\beta}\|_{q(\cdot)} \le k\|t^{-\beta}\|_{q(\cdot)} = c \tag{6.2}$$

under the Dini-Lipschitz condition for $p(x)$ on $[0,d]$ and the assumption $\beta q(0) < 1$.

For the first term in (6.1) we observe that the operator H^β is dominated by the weighted maximal operator M^β since

$$\left|\frac{1}{x}\int_0^x f(t)dt\right| \le \frac{1}{x}\int_0^{2x}|f(t)|dt = \frac{1}{x}\int_{x-x}^{x+x}|f(t)|dt \le 2Mf(x).$$

But first we have to pass to the exponent $p(x)$ in the first term. To this end, we observe that

$$\left|H^\beta f(x)\right|^{s(x)-p(x)} \le c, \quad 0 < x \le d_0 \quad \left(\|f\|_{p(\cdot)} \le 1\right), \tag{6.3}$$

where c does not depend on x and f, if $s(x)$ satisfies condition (2.13).

Indeed, by Hölder inequality (3.2),

$$\left|H^\beta f(x)\right| \le kx^{\beta-1}\|f\|_{p(\cdot)}\,\|t^{-\beta}\|_{q(\cdot)} \le c\cdot kx^{\beta-1}\|t^{-\beta}\|_{q(0)} = cx^{\beta-1}. \tag{6.4}$$

Hence

$$\left|H^\beta f(x)\right|^{s(x)-p(x)} \le c^{S-1}x^{(\beta-1)[s(x)-p(x)]}\,, \tag{6.5}$$

which is obviously bounded if $x \ge \frac{1}{2}$. For $0 < x \le \min(d_0, \frac{1}{2})$ from (6.4) we have

$$\left|H^\beta f(x)\right|^{s(x)-p(x)} \le c^{S-1}e^{(1-\beta)[s(x)-p(x)]\ln\frac{1}{x}} \le c_1 < \infty$$

by (2.13). Therefore, in view of (6.3),

$$\int_0^{d_0}\left|H^\beta f(x)\right|^{s(x)}dx \le c\int_0^{d_0}\left|H^\beta f(x)\right|^{p(x)}dx. \tag{6.6}$$

It remains to apply Theorem A on $[0,d_0]$. Then

$$\int_0^d\left|H^\beta f(x)\right|^{s(x)}dx \le c$$

by (6.1), (6.2) and (6.6).

The operator $H_*^\beta = (H^{-\beta})^*$ may be regarded as the operator adjoint to $H^{-\beta}$ treated in $L^{q(\cdot)}([0,l])$. However, we admit the possibility for $q(x)$ to be unbounded beyond a neighborhood of the point $x = 0$, and hence we should first proceed as in (6.1):

$$\begin{aligned} I_s(H_*^\beta f) &\le \int_0^{d_0} \left|H_*^\beta f(x)\right|^{s(x)} dx + c\int_{d_0}^{l} \left(\int_x^l |f(t)|dt\right)^{p(x)} dx \le \\ &\le \int_0^{d_0} \left|H_*^\beta f(x)\right|^{s(x)-p(x)} \cdot \left|H_*^\beta f(x)\right|^{p(x)} dx + c \end{aligned} \tag{6.7}$$

assuming that $\|f\|_{p(\cdot)} \le 1$. Similarly to (6.5),

$$\left|H_*^\beta f(x)\right|^{s(x)-p(x)} \le c, \quad 0 < x \le d_0$$

which is shown as in (6.4), (6.5), since

$$\left|H_*^\beta f(x)\right| \le x^{\beta-1}\int_0^l \frac{f(t)}{t^\beta} dt \qquad \text{etc.}$$

Then from (6.7),

$$I_s(H_*^\beta f) \le c\int_0^{d_0} (H_*^\beta f(x))^{p(x)} + c. \tag{6.8}$$

It remains to use the duality argument for $H_*^\beta = (H^{-\beta})^x$.

Part II. We need only to estimate anew the first term in (6.1). In the case $p(0) \le p(x)$, $0 \le x \le d$, we can avoid the passage to the maximal operator by observing that, similarly to (6.3),

$$\left|H^\beta f(x)\right|^{s(x)-p(0)} \le c \quad \left(\|f\|_{p(\cdot)} \le 1\right)$$

under the second condition in (2.13). Then the first term in (6.1) is dominated by

$$c\int_0^d \left|H^\beta f(x)\right|^{p(0)} dx \le c\int_0^d |f(x)|^{p(0)} dx$$

by virtue of the boundedness of the weighted Hardy operator H^β in $L^{p(0)}$ with $p(0) > 1$ and $-\frac{1}{p(0)} < \beta < \frac{1}{q(0)}$. Therefore,

$$\int_0^d \left|H^\beta f(x)\right|^{p(0)} dx \le c\int_0^d |f(x)|^{p(x)} dx$$

by imbedding (3.5).

For the operator H_*^β we may again proceed as in (6.7), (6.8) and use the boundedness of H_*^β in $L^{p(0)}$. ■

The case of the Hankel operator

Let for simplicity $f(x)$ be non-negative. We have

$$\mathcal{H}^\beta f(x) \le x^{\beta-1}\int_0^x \frac{f(t)}{t^\beta} + x^\beta \int_x^l \frac{f(t)}{t^{\beta+1}}\,dt,$$

that is,

$$\mathcal{H}^\beta f(x) \le H^\beta f(x) + H_*^\beta f(x). \tag{6.9}$$

Consequently, the boundedness of $\mathcal{H}^\beta$ follows immediately from that of the operators H^β and H_*^β.

Proof of the Corollary to Theorem E

We have

$$\int_{-1}^0 \left|S^\beta f(x)\right|^{p(x)} dx = \int_0^1 \left|\int_0^1 \left(\frac{x}{t}\right)^\beta \frac{f(t)}{t+x}\right|^{p(-x)} dx.$$

Thus, it suffices to make use of Theorem E for the Hankel operator $\mathcal{H}^\beta$, choosing $s(x) = p(-x)$ in that theorem. The condition

$$|s(x) - p(x)| = |p(-x) - p(x)| \le \frac{A}{\ln\frac{1}{|x|}}, \quad 0 < |x| \le \delta$$

of Theorem A is obviously satisfied. ■

Appendix

Proof of inequality (3.11)

Let

$$J_r(x) = \int_{|y-x|<r} \frac{dy}{|y-x_0|^\alpha} = \int_{|y-(x-x_0)|<r} \frac{dy}{|y|^\alpha}. \tag{6.10}$$

Without loss of generality we may assume that $x_0 = 0$. The change of variables $y = |x|\xi$ gives

$$J_r(x) = |x|^{n-\alpha} \int_{|\xi - \frac{x}{|x|}|<\frac{r}{|x|}} \frac{d\xi}{|\xi|^\alpha} = |x|^{n-\alpha}\int_{|u-e_1|<\frac{r}{|x|}} \frac{du}{|u|^\alpha}, \tag{6.11}$$

where $e_1 = (1, 0, \dots, 0)$ and in the last equation we made the rotation change of variables

$$\xi = \omega_x(u), \quad |\xi| = |u|,$$

where $\omega_x(u)$ is the rotation of $\mathbb{R}^n$ such that $\omega_x(e_1) = \frac{x}{|x|}$.

From (6.11) we have

$$J_r(x) = |x|^{n-\alpha} g\Big(\frac{r}{|x|}\Big), \qquad g(t) = \int_{|y-e_1|<t} \frac{dy}{|y|^\alpha}, \quad t > 0. \tag{6.12}$$

To estimate $g(t)$, we distinguish between the three cases,

$$0 < t \le \frac{1}{2}, \qquad t \ge 2 \qquad \text{and} \qquad \frac{1}{2} \le t \le 2.$$

In the case $0 < t \le \frac{1}{2}$ we have

$$|y| = |y - e_1 + e_1| \ge 1 - |y - e_1| \ge 1 - t \ge \frac{1}{2},$$

so that

$$g(t) \le 2^\alpha \int_{|g-e_1|<t} dy = 2^\alpha |B_r(e_1)| = 2^\alpha |B_r(x)|. \tag{6.13}$$

If $t \ge 2$, we obtain

$$g(t) = \int_{|y-e_1|<2} \frac{dy}{|y|^\alpha} + \int_{2<|y-e_1|<t} \frac{dy}{|y|^\alpha} = c + \int_{2<|y|<t} \frac{dy}{|y+e_1|^\alpha}.$$

Here

$$|y + e_1| \ge |y| - 1 \ge |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

Therefore

$$g(t) \le c + 2^\alpha \int_{2<|y|<t} \frac{dy}{|y|^\alpha} = c + 2^\alpha |s^{n-1}| \int_0^1 \rho^{n-1-\alpha} d\rho =$$

$$= c + c_1 t^{n-\alpha} \le c_2 t^{n-\alpha}. \tag{6.14}$$

Finally, if $\frac{1}{2} \le t \le 2$, we have $g(t) \le g(r) = c_3$. Thus, by (6.13), (6.14)

$$g(t) \le c \begin{cases} t^n, & 0 < t < 1, \\ t^{n-\alpha}, & t \ge 1. \end{cases}$$

Now we obtain from (6.12) that

$$J_r(x) \le c \begin{cases} r^n |x|^{-\alpha}, & r \le |x|, \\ r^{n-\alpha}, & r \ge x \le c r^n |x|^{-\alpha}. \end{cases}$$

Hence (3.11) follows. ■

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