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ON THE APPROXIMATION IN WEIGHTED LEBESGUE SPACES

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Abstract. In this paper we deal with the estimation of the best approximation and generalized modulus of continuity of derivatives of periodic functions in weighted reflexive Lebesgue spaces.

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1. Introduction and the Main Results

In this paper the approximation problems for periodic functions are investigated in weighted Lebesgue spaces with the Muckenhoupt weights. For this case we obtain inverse type inequality for the derivatives of 2π periodic function in terms of generalized modulus of continuity. The introduction of such structural characteristic of functions was caused by the failure of continuity of shift operator in weighted spaces.

In unweighted Lebesgue spaces the inequalities for classical modulus of continuity and the best approximations of derivatives were derived in the papers [1], [2]. For the approximation in weighted Lebesgue spaces we refer to [3], [4], [5].

Let $\mathbf{T}$ denote the interval $(-\pi, \pi)$. A positive almost everywhere, integrable function $w : \mathbf{T} \to (0, \infty)$ is called as a weight function. With any given weight w we associate the w-weighted Lebesgue space $L^p_w(\mathbf{T})$ consisting of all measurable functions f on $\mathbf{T}$ such that

$$\|f\|_{L^p_w(\mathbf{T})} = \|fw\|_{L^p(\mathbf{T})} < \infty.$$

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Let $1<p<\infty$ and $1/p+1/q=1$. A weight function w belongs to the Muckenhoupt class $A_p(\mathbf{T})$ if

$$\left(\frac{1}{|I|}\int_I w^p(x)dx\right)^{1/p}\left(\frac{1}{|I|}\int_I w^{-q}(x)dx\right)^{1/q}\leq C$$

with a finite constant C independent of I, where I is any subinterval of $\mathbf{T}$ and $|I|$ denotes the length of I.

Let $1<p<\infty$ and $w\in A_p(\mathbf{T})$. We define an operator on $L^p_w(\mathbf{T})$ by

$$\mathcal{A}_h(g)(x)=\frac{1}{2h}\int_{x-h}^{x+h}g(t)dt,\qquad 0<h<\pi$$

It is known that the operator γ_h is bounded uniformly with respect to h in $L^p_w(\mathbf{T})$, when $w\in A_p(\mathbf{T})$, $1<p<\infty$. The modulus of continuity $\Omega(g,\cdot)_{L^p_w}$ of $g\in L^p_w(\mathbf{T})$ is defined by

$$\Omega(g,\delta)_{L^p_w}=\sup\left\{\|g-\mathcal{A}_h\|_{L^p_w}\,,\ 0<h\leq\delta\right\}.$$

It is easily proved that $\Omega(g,\cdot)_{L^p_w}$ is a continuous, nonnegative, nondecreasing function satisfying this conditions

$$\lim_{\delta\to 0}\Omega(g,\delta)_{L^p_w}=0,\quad \Omega(g_1+g_2,\cdot)_{L^p_w}\leq\Omega(g_1,\cdot)_{L^p_w}+\Omega(g_2,\cdot)_{L^p_w}.$$

The best approximation of $f\in L^p_w(\mathbf{T})$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{L^p_w}=\inf\left\{\|f-T_n\|_{L^p_w}:T_n\in\Pi_n\right\}.$$

Theorem 1. *Let $f\in L^p_w(\mathbf{T})$, $1<p<\infty$, and $w\in A_p(\mathbf{T})$. If*

$$\sum_{k=1}^{\infty}k^{r\gamma-1}E_k^{\gamma}(f)_{L^p_w}<\infty$$

for some natural number r and $\gamma=\min\{2,p\}$, then there exists the absolutely continuous derivative $f^{(r-1)}(x)$, $f^{(r)}\in L^p_w(\mathbf{T})$ and the estimate

$$E_n(f^{(r)})_{L^p_w}\leq c\left\{n^rE_n(f)_{L^p_w}+\left(\sum_{k=n+1}^{\infty}k^{r\gamma-1}E_k^{\gamma}(f)_{L^p_w}\right)^{1/\gamma}\right\}$$

holds with a constant c independent of n and f.

Theorem 2. *Let $f\in L^p_w(\mathbf{T})$, $1<p<\infty$, and $w\in A_p(\mathbf{T})$. If*

$$\sum_{k=1}^{\infty}k^{r\gamma-1}E_k^{\gamma}(f)_{L^p_w}<\infty$$

for some natural number r *and* $\gamma = \min\{2, p\}$, *then there exists the absolutely continuous derivative* $f^{(r-1)}(x)$, $f^{(r)} \in L_w^p(\mathbf{T})$ *and the estimate*

$$\Omega(f^{(r)}, 1/n)_{L_w^p} \leq \frac{c}{n^2}\left(\sum_{k=1}^{n} k^{(r+2)\gamma-1} E_{k-1}^{\gamma}(f)_{L_w^p}\right)^{1/\gamma} + \\ + c\left(\sum_{k=n+1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L_w^p}\right)^{1/\gamma}$$

holds with a constant c *independent of* n *and* f.

Corollary. *If*

$$E_n(f)_{L_w^p} = O\left(\frac{1}{n^{r+2}}\right)$$

then for $\gamma = \min\{2, p\}$

$$\Omega(f^{(r)}, 1/n)_{L_w^p} = O\left(\frac{(\ln n)^{1/\gamma}}{n^2}\right).$$

This corollary shows that the smoothness of function in weighted Lebesgue spaces depends not only on the rate of convergence of the best approximation but also on the metric of the space. For unweighted case see [1], [2].

2. Auxiliary Results

Lemma 1. *Let* $\{f_n\}$ *be a sequence such that every* f_n *are absolutely continuous and let* $w \in A_p[a, b]$. *If the sequence* $\{f_n\}$ *converges to the function* f *in* $L_w^p[a, b]$ *norm and the sequence of first derivatives* $\{f_n'\}$ *converges to some function* g *in* $L_w^p[a, b]$ *norm, then* f *is absolutely continuous and* $f'(x) = g(x)$ *almost everywhere.*

Proof. Since $\|f_n - f\|_{L_w^p} \to 0$ there exists a subsequence $\{f_{n_k}\}$ of the sequence $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ almost everywhere. Let x_0 is a point of convergence. By using Holder's inequality, we get

$$\left|\int_{x_0}^{x} f_{n_k}'(t)dt - \int_{x_0}^{x} g(t)dt\right| \leq \left\|f_{n_k}' - g\right\|_{L_w^p} \left(\int_{x_0}^{x} w^{-\frac{q}{p}} dt\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since $w \in A_p$ and $\|f_n' - g\|_{L_w^p} \to 0$ we have

$$\left|\int_{x_0}^{x} f_{n_k}'(t)dt - \int_{x_0}^{x} g(t)dt\right| \to 0.$$

Therefore, we obtain

$$\int_{x_0}^{x} g(t)dt = \lim_{k\to\infty}\int_{x_0}^{x} f'_{n_k}(t)dt = \lim_{k\to\infty}\left(f_{n_k}(x) - f_{n_k}(x_0)\right) = f(x) - f(x_0)$$

almost everywhere. This completes the proof. $\square$

Lemma 2. *Let $1 < p < \infty$ and let $w \in A_p(\mathbf{T})$. Suppose that $f \in L^p_w(\mathbf{T})$ and*

$$\sum_{n=1}^{\infty} n^{r\gamma-1} E_n^{\gamma}(f)_{L^p_w} < \infty, \tag{1}$$

where $\gamma = \min\{2, p\}$. Then

$$\lim_{n\to\infty} n^r E_n(f)_{L^p_w} = 0.$$

Proof. Since the sequence $\left\{E_n(f)_{L^p_w}\right\}$ is decreasing, we have

$$n^{r\gamma} E_n^{\gamma}(f)_{L^p_w} \le c \sum_{k=[\frac{n}{2}]}^{n} k^{r\gamma-1} E_k^{\gamma}(f)_{L^p_w} \to 0. \qquad \square$$

Lemma 3. *Under the conditions of Lemma 2, the trigonometric series*

$$\sum_{k=1}^{\infty} k^r B_{k,r}(x) \tag{2}$$

converges in L^p_w where $B_{k,r}(x) = a_k \cos(k + r\frac{\pi}{2}) + b_k \sin(k + r\frac{\pi}{2})$ and a_k, b_k are the Fourier coefficients of the function f.

Proof. Let n and p be the arbitrary natural numbers, $p > n$. Suppose $2^{m-1} < n < 2^m$ and $2^{s-1} < p < 2^s$. We have

$$\sum_{k=n}^{p} k^r B_{k,r}(x) = \sum_{k=n}^{2^m-1} k^r B_{k,r}(x) + \sum_{k=2^m}^{2^{s-1}} k^r B_{k,r}(x) + \\ + \sum_{k=2^{s-1}+1}^{p} k^r B_{k,r}(x). \tag{3}$$

Applying the weighted version of the Littlewood-Paley's theorem (see [7]), we obtain

$$\left\| \sum_{k=2^m}^{2^{s-1}} k^r B_{k,r}(x) \right\|_{L^p_w} \le c \left\| \left(\sum_{\mu=m}^{s-1} \left| \sum_{k=2^\mu}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p_w}.$$

For $1 < p \le 2$ we have

$$\left(\int_{\mathbf{T}} \left(\sum_{\mu=m}^{s-1} \left| \sum_{k=2^\mu}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right|^2 \right)^{\frac{p}{2}} w(x)dx \right)^{\frac{1}{p}} \le$$

$$\leq \left(\int\limits_{\mathbf{T}} \left(\sum_{\mu=m}^{s-1} \left| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right|^p \right) w(x)dx \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\sum_{\mu=m}^{s-1} \int\limits_{\mathbf{T}} \left| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right|^p w(x)dx \right)^{\frac{1}{p}} =$$

$$= \left(\sum_{\mu=m}^{s-1} \left\| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right\|_{L_w^p}^p \right)^{\frac{1}{p}}.$$

Now suppose that $p > 2$. Applying Minkowskii's inequality, we get

$$\left(\int\limits_{\mathbf{T}} \left(\sum_{\mu=m}^{s-1} \left| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right|^2 \right)^{\frac{p}{2}} w(x)dx \right)^{\frac{1}{p}} \leq$$

$$\leq \left(\sum_{\mu=m}^{s-1} \left(\int\limits_{\mathbf{T}} \left(\left| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right|^2 \right)^{\frac{p}{2}} w(x)dx \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\sum_{\mu=m}^{s-1} \left\| \left(\sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right)^2 \right\|_{L_w^{\frac{p}{2}}} \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\sum_{\mu=m}^{s-1} \left\| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right\|_{L_w^p}^2 \right)^{\frac{1}{2}}.$$

Thus

$$\left\| \sum_{k=2^m}^{2^s-1} k^r B_{k,r}(x) \right\|_{L_w^p} \leq \left(\sum_{\mu=m}^{s-1} \left\| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right\|_{L_w^p}^{\gamma} \right)^{\frac{1}{\gamma}}. \tag{4}$$

Using Abel's transform, we get

$$\left\| \sum_{k=2^{\mu}}^{2^{\mu+1}-1} k^r B_{k,r}(x) \right\|_{L_w^p} \leq \sum_{k=2^{\mu}}^{2^{\mu+1}-2} \left((k+1)^r - k^r \right) \left\| T_k - T_{2^{\mu+1}-1} \right\|_{L_w^p} +$$
$$+ \left(2^{\mu}-1\right)^r \left\| T_{2^{\mu}-1} - T_{2^{\mu-1}} \right\|_{L_w^p}$$

where T_k denotes $k-th$ partial sum of the trigonometric series

$$\sum_{k=1}^{\infty} B_{k,r}(x). \tag{5}$$

Note that the series (5) is the Fourier trigonometric series or its conjugate series multiplied by $+1$ or -1 depending on r. Thanks to the theorem Hunt-Muckenhoupt-Wheeden [6], we obtain that the best approximation by trigonometric polynomials in $L_w^p(\mathbf{T})$ with $w \in A_p(\mathbf{T})$ has the same order as deviation by the partial sum of the Fourier series. It means that for $\varphi \in L^p_w$

$$\left\|\varphi - S_n(\varphi)\right\|_{L^p_w} \le cE_n(\varphi)_{L^p_w}$$

with a positive constant c independent on φ and n. Note that by the theorem mentioned above, for the best approximations of the given function and its conjugate in L^p_w, we have

$$c_1 E_n(f)_{L^p_w} \le E_n(\widetilde{f})_{L^p_w} \le c_2 E_n(f)_{L^p_w}$$

where c_1 and c_2 don't depend on n and f.

Taking into account all the mentioned above we get

$$\left\|\sum_{k=2^\mu}^{2^{\mu+1}-1} k^r B_{k,r}(x)\right\|_{L^p_w} \le c2^{\mu r} E_{2^\mu-1}(f)_{L^p_w}$$

and then from this inequality and (4), we have

$$\left\|\sum_{k=2^m}^{2^s-1} k^r B_{k,r}(x)\right\|^\gamma_{L^p_w} \le c\sum_{\mu=m}^{s-1} 2^{\gamma\mu r} E^\gamma_{2^\mu-1}(f)_{L^p_w}.$$

On the other hand

$$\sum_{k=\left[\frac{n}{2}\right]}^{s} k^{r\gamma-1} E^\gamma_k(f)_{L^p_w} \ge \sum_{\mu=m}^{s-1} \sum_{k=2^{\mu-1}+1}^{2^\mu-1} k^{r\gamma-1} E^\gamma_k(f)_{L^p_w} \ge$$
$$\ge c\sum_{\mu=m}^{s} 2^{r\gamma\mu} E^\gamma_{2^\mu-1}(f)_{L^p_w}.$$

Thus

$$\left\|\sum_{k=2^m}^{2^s-1} k^r B_{k,r}(x)\right\|^\gamma_{L^p_w} \le \sum_{k=\left[\frac{n}{2}\right]}^{s} k^{r\gamma-1} E^\gamma_k(f)_{L^p_w}.$$

From the condition (1), we obtain that

$$\lim_{\substack{s\to\infty\\ m\to\infty}} \left\|\sum_{k=2^m}^{2^s-1} k^r B_{k,r}(x)\right\|_{L^p_w} = 0.$$

On the other hand, using Bernstein's inequality in L^p_w (see [3]), we get

$$\left\|\sum_{k=n}^{2^m-1} k^r B_{k,r}(x)\right\|_{L^p_w} \le c\,2^{mr} \left\|\sum_{k=n}^{2^m-1} B_{k,r}(x)\right\|_{L^p_w} \le$$
$$\le c\,2^{mr} E_{n-1}(f)_{L^p_w} \le c(n-1)^r E_{n-1}(f)_{L^p_w}.$$

By Lemma 2, the right side tends to zero and therefore the left side tends also to zero when n tends to infinity. By the similar way we can see that

$$\left\| \sum_{k=2^{s-1}+1}^{p} k^r B_{k,r}(x) \right\|_{L^p_w}$$

tends to zero when p tends to infinity.

Now, from (3) we conclude that the series (2) converges in L^p_w. $\square$

Lemma 4. *Under the conditions of Lemma 2 there exists the absolutely continuous derivative $f^{(r-1)}$, and $f^{(r)} \in L^p_w$ and we have the equality*

$$f^{(r)}(x) = \sum_{n=1}^{\infty} n^r B_{n,r}(x)$$

almost everywhere.

Proof. By Lemma 3 the series (2) converges in L^p_w to some function g. From the openness of A_p class it follows that it converges in L^{p_0}, for some p, $1 < p_0 < p$. From the last we obtain that the sequence of norms of partial sums of the series (2) is bounded and consequently the sequence of norms of Cesaro means is bounded. Thus, the series (2) is the Fourier series of function g. Note that by the condition (1)

$$\sum_{n=1}^{\infty} n^{s\gamma-1} E_n^{\gamma}(f)_{L^p_w} < \infty$$

for arbitrary $1 < s < r$. Using Lemma 1 we conclude that $f^{(r-1)}$ is absolutely continuous and

$$g(x) = f^{(r)}(x)$$

almost everywhere. By Hunt-Carleson theorem the series (2) converges everywhere and so we have

$$f^{(r)}(x) = \sum_{n=1}^{\infty} n^r B_{n,r}(x)$$

almost everywhere. $\square$

3. Proofs of the Theorems

The proof of Theorem 1. Let $2^m < n < 2^{m+1}$. We have

$$\left\| f^{(r)} - S_n(f^{(r)}) \right\|_{L^p_w} \leq \left\| S_{2^{m+2}}(f^{(r)}) - S_n(f^{(r)}) \right\|_{L^p_w} + \\ + \sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}}(f^{(r)}) - S_{2^k}(f^{(r)}) \right\|_{L^p_w}.$$

By the weighted version of Bernstein's inequality, we have

$$\left\|S_{2^{m+2}}(f^{(r)})-S_n(f^{(r)})\right\|_{L^p_w}=\left\|S^r_{2^{m+2}}(f)-S^r_n(f)\right\|_{L^p_w}\le \tag{6}$$
$$\le c\,2^{(m+2)r}E_{2^{m+2}}(f)_{L^p_w}\le cn^rE_n(f)_{L^p_w}.$$

Applying the weighted version of Littlewood-Paley's theorem we obtain

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}(f^{(r)})-S_{2^k}(f^{(r)})\right]\right\|_{L^p_w}\le$$
$$\le c\left(\sum_{k=m+2}^{\infty}\left\|\sum_{\mu=2^k+1}^{2^{k+1}}\mu^rB_{\mu,r}(x)\right\|^{\gamma}_{L^p_w}\right)^{\frac{1}{\gamma}}.$$

Using the estimation of the inside norm derived by Lemma 3, we get

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}(f^{(r)})-S_{2^k}(f^{(r)})\right]\right\|_{L^p_w}\le c\left(\sum_{k=m+2}^{\infty}2^{kr\gamma}E^{\gamma}_{2^k-1}(f)_{L^p_w}\right)^{\frac{1}{\gamma}}.$$

Therefore we have

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}(f^{(r)})-S_{2^k}(f^{(r)})\right]\right\|_{L^p_w}\le c\left(\sum_{k=n+1}^{\infty}k^{r\gamma-1}E^{\gamma}_{k}(f)_{L^p_w}\right)^{\frac{1}{\gamma}}. \tag{7}$$

Summarizing the formulas (6) and (7) we deduce the desired inequality. □

The proof of Theorem 2. Let $2^{m-1}\le n<2^m$, $n=\left[\frac{1}{2h}\right]$, $\frac{1}{2^{m+1}}\le h<\frac{1}{2^m}$. Let $\sigma_h=I-\mathcal{A}_h$ We have

$$\sigma_h(f^{(r)})(x)=\sigma_h(f^{(r)}-S^{(r)}_{2^m})(x)+\sigma_h(S^{(r)}_{2^m})(x) \tag{8}$$

Note that $S^{(r)}_{2^m}$ is r-th derivative of partial sum of function f. It means that it is partial sum of series (2). Using the boundedness of operator σ_h uniformly with respect to h in L^p_w, we get

$$\left\|\sigma_h(f^{(r)}-S^{(r)}_{2^m})\right\|_{L^p_w}\le c\left\|f^{(r)}-S^{(r)}_{2^m}\right\|_{L^p_w}\le cE_{2^m}(f^{(r)})$$

By Theorem 1 for $2^{m-1}\le n<2^m$, we have the estimate

$$E_{2^m}(f^{(r)})_{L^p_w}\le c\left\{2^{mr}E_{2^m}(f)_{L^p_w}+\left(\sum_{k=2^m+1}^{\infty}k^{r\gamma-1}E^{\gamma}_k(f)_{L^p_w}\right)^{1/\gamma}\right\}\le$$
$$\le c\left\{2^{mr}E_{2^m}(f)_{L^p_w}+\left(\sum_{k=n+1}^{\infty}k^{r\gamma-1}E^{\gamma}_k(f)_{L^p_w}\right)^{1/\gamma}\right\}.$$

Since the sequence $\{E_k(f)\}$ is decreasing, it is clear that

$$2^{mr}E_{2^m}(f)_{L^p_w}\le c\,n^rE_n(f)_{L^p_w}\le\frac{c}{n^2}\left(\sum_{k=1}^{n}k^{(r+2)\gamma-1}E^{\gamma}_{k-1}(f)_{L^p_w}\right)^{1/\gamma}. \tag{9}$$

Therefore, we have

$$\left\|\sigma_h(f^{(r)}-S_{2^m}^{(r)})\right\|_{L_w^p}\le\frac{c}{n^2}\left(\sum_{k=1}^{n}k^{(r+2)\gamma-1}E_{k-1}^{\gamma}(f)_{L_w^p}\right)^{1/\gamma}+$$
$$+c\left(\sum_{k=n+1}^{\infty}k^{r\gamma-1}E_k^{\gamma}(f)_{L_w^p}\right)^{1/\gamma}. \tag{10}$$

Then

$$\left\|\sigma_h(S_{2^m}^{(r)})\right\|_{L_w^p}=\left\|\sum_{k=1}^{2^m}k^rB_{k,r}(x)\left(1-\frac{\sin kh}{kh}\right)\right\|_{L_w^p}\le$$
$$\le\left\|\sum_{\mu=1}^{m}\sum_{k=2^{\mu-1}}^{2^\mu-1}k^rB_{k,r}(x)\left(1-\frac{\sin kh}{kh}\right)\right\|_{L_w^p}+$$
$$+\left\|2^{mr}B_{2^m,r}(x)\left(1-\frac{\sin 2^mh}{2^mh}\right)\right\|_{L_w^p}=I_1+I_2$$

For the first term, by weighted version of Littlewood-Paley theorem, we have, as in the proof of the previous theorem

$$I_1\le\left(\sum_{\mu=1}^{m}\left\|\sum_{k=2^{\mu-1}}^{2^\mu-1}k^rB_{k,r}(x)\left(1-\frac{\sin kh}{kh}\right)\right\|_{L_w^p}^{\gamma}\right)^{1/\gamma}$$

for $\gamma=\min\{2,p\}$. Applying Abel's transform, we get

$$\sum_{k=2^{\mu-1}}^{2^\mu-1}k^rB_{k,r}(x)\left(1-\frac{\sin kh}{kh}\right)=$$
$$=\sum_{k=2^{\mu-1}}^{2^\mu-2}\left\{k^r\left(1-\frac{\sin kh}{kh}\right)-(k+1)^r\left(1-\frac{\sin(k+1)h}{(k+1)h}\right)\right\}\times$$
$$\times(T_k-T_{2^{\mu-1}-1})+$$
$$+(2^\mu-1)^r\left(1-\frac{\sin(2^\mu-1)h}{(2^\mu-1)h}\right)(T_{2^\mu-1}-T_{2^{\mu-1}-1})$$

and

$$\left\|\sum_{k=2^{\mu-1}}^{2^\mu-1}k^rB_{k,r}(x)\left(1-\frac{\sin kh}{kh}\right)\right\|_{L_w^p}\le$$
$$\le\frac{c}{h^r}\sum_{k=2^{\mu-1}}^{2^\mu-2}h^r\left|k^r\left(1-\frac{\sin kh}{kh}\right)-(k+1)^r\left(1-\frac{\sin(k+1)h}{(k+1)h}\right)\right|\times$$
$$\times E_{2^{\mu-1}-1}(f)_{L_w^p}+2^{\mu(r+2)}h^2E_{2^{\mu-1}-1}(f)_{L_w^p} \tag{11}$$

On the other hand, as the function $x^r\left(1-\frac{\sin x}{x}\right)$ is increasing, we have

$$\frac{c}{h^r}\sum_{k=2^{\mu-1}}^{2^{\mu}-2} h^r\left|k^r\left(1-\frac{\sin kh}{kh}\right)-(k+1)^r\left(1-\frac{\sin(k+1)h}{(k+1)h}\right)\right|\leq$$
$$\leq \frac{c}{h^r}2^{\mu r}h^r\left(1-\frac{\sin(2^{\mu}-1)h}{(2^{\mu}-1)h}\right)\leq c2^{\mu(r+2)}h^2.$$

From this and (11) we have

$$\left\|\sum_{k=2^{\mu-1}}^{2^{\mu}-1} k^r B_{k,r}(x)\left(1-\frac{\sin kh}{kh}\right)\right\|_{L^p_w}\leq c2^{\mu(r+2)}h^2E_{2^{\mu-1}-1}(f)_{L^p_w}.$$

Therefore, we obtain

$$I_1\leq ch^2\left(\sum_{\mu=1}^{m}2^{\mu(r+2)\gamma}E^{\gamma}_{2^{\mu-1}-1}(f)_{L^p_w}\right)^{1/\gamma}\leq$$
$$\leq\frac{c}{n^2}\left(\sum_{k=1}^{n}k^{(r+2)\gamma-1}E^{\gamma}_{k-1}(f)_{L^p_w}\right)^{1/\gamma}.$$

On the other hand,

$$I_2\leq c\,2^{m(r+2)}h^2(|a_{2^m}|+|b_{2^m}|)\leq$$
$$\leq c\,2^{m(r+2)}h^2E_{2^m}(f)_{L^p_w}\leq c2^{mr}h^2E_{2^m}(f)_{L^p_w}.$$

Taking into account (9), we get

$$I_2\leq\frac{c}{n^2}\left(\sum_{k=1}^{n}k^{(r+2)\gamma-1}E^{\gamma}_{k-1}(f)_{L^p_w}\right)^{1/\gamma}.$$

Therefore, we have

$$\left\|\sigma_h(S^{(r)}_{2^m})\right\|_{L^p_w}\leq\frac{c}{n^2}\left(\sum_{k=1}^{n}k^{(r+2)\gamma-1}E^{\gamma}_{k-1}(f)_{L^p_w}\right)^{1/\gamma}. \tag{12}$$

From (8), (10) and (12) we deduce the desired estimate. □

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