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Mathematics

On a Generalization of Calderon-Zygmund's Theorem in Weighted Lebesgue Spaces with Variable Exponent

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ABSTRACT. The paper is devoted to the study of behaviour of the solutions of Poisson equation when the right side belongs to the variable Lebesgue space with weights. The obtained results generalize the well-known Calderon-Zygmund theorem. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *weights, variable Lebesgue spaces, Poisson equation, Sobolev spaces.*

In the present paper we deal with a generalization of the well-known Calderon-Zygmund theorem [1] on the behavior of solution of Poisson equation for the weighted Lebesgue spaces with variable exponent.

In recent years the studies of new directions in nonlinear differential equations theory were related to variable exponent Lebesgue and Sobolev spaces research. These topics are intensively worked on in many scientific centers worldwide, which is mainly caused by their numerous applications in a wide range of fields like mathematical physics, differential equations, boundary value problems, nonlinear elasticity theory, various mathematical models, variational problems, differential equations with variable growth order, etc. These issues are the subject of articles intensively published in international journals of high rank, as well as the subject of reports of participants of regular conferences and symposia. Two international conferences related to the above-mentioned topics were held in Tbilisi during 2003-2005. The same issues were widely presented at the ISAAC International Congress in Italy in 2005, also at the conference on function spaces, differential operators and nonlinear analysis held in Czech Republic in 2004.

Originally, the discussion of variable exponent Lebesgue spaces was initiated by theoretical interest only, particularly by its mathematical curiosity. The matter is that a wide range of well-known important issues, related to classical Lebesgue spaces, lose their own power within the frameworks of the mentioned spaces. For example, the continuity of shift operator, the well-known Young theorem on the convolution of functions does not work in general.

It became clear in recent years that the classical function spaces are not sufficient to solve the contemporary problems of nonlinear elasticity, in modeling of various fluids flow, current variation problems of mechanics. Actual need for study of different integral spaces with variable exponent and variable smoothness, research of boundedness and spectral features of integral and differential operators in these spaces became apparent.

The backgrounds for the variable exponent Lebesgue $L^{p(\cdot)}$ and Sobolev $W^{k,p(\cdot)}$ spaces research were formed in works of W. Orlicz, J. Musielak, N. Nakano, H. Hudzik, S. Samko, O. Kovaèik and J. Rakosnik, D. Edmunds and A. Nekvinda, X. Fan and D. Zhao. Numerous studies were addressed to the smooth functions density problem in these spaces. It is worth mentioning the I. Sharapudinov works related to Fourier operators sequences convergence, proving the Haar function system baseness in $L^{p(\cdot)}$ spaces. The impulse for study of classical integral operators boundedness in variable exponent Lebesgue space came from German mathematician L. Diening, who had proved boundedness of the Hardy-Littlewood maximal functions defined for bounded domains. This direction was further developed in L. Diening and M. Ru•ièka, A. Nekvinda, D. Cruiz-Uribe, A. Fiorenza, C. Neugebauer, C. Perez works. They have studied boundedness of maximal operators, singular integrals, potentials, Sobolev type embedding theorems, including those on the unbounded domains. Chinese and Finnish mathematicians works should be also mentioned, where nonlinear differential equations with $p(x)$-Laplacian, variable exponent variational problems, maximal functions for measured metric spaces were investigated.

The results obtained by the authors of the present paper in last five years are: Cauchy singular integral operator boundedness criteria in variable exponent Lebesgue and Lorentz spaces with power weights is proved; Fredholmness criterion for singular integral operator with piecewise continuous coefficients is obtained; necessary and sufficient condition for maximal functions that governs its boundedness in variable exponent Lebesgue space with power weights is established. Based on these results, a similar problem for Riesz potential is solved. It was found that boundedness condition actually depends on the value of exponent in singular points of power weights. We proved that the Cauchy singular integral operator is bounded in $L^{p(\cdot)}$ space by Dini-Lipschitz $p(x)$ exponent if and only if the curve of integration is regular (Carleson curve).

Let us give some basic definitions. Let Ω be an open set in R^n and $p:\Omega\to[1,\infty)$ be a measurable function on Ω. The weighted Lebesgue space $L^{p(\cdot)}(\Omega,\rho)$ with variable exponent is the Banach function space defined by the norm

$$\|f\|_{L^{p(\cdot)}(\Omega,\rho)}=\inf\left\{\lambda>0:\int_{\Omega}\rho(x)\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\le 1\right\},$$

where $\rho(x)$ is of the form $\rho(x)=w(|x-x_0|)$ with $x_0\in\overline{\Omega}$ and $w(r)$ belongs to a certain class of almost monotonic function defined in terms of Bary-Stechkin class [2] or in terms of their Boyd-type indices. For the details on the weighted Lebesgue spaces with variable exponent we refer to [3].

Definition 1. By $B(\Omega)$ we denote the set of functions $p:\overline{\Omega}\to(1,\infty)$ satisfying the conditions

$$1<\underline{p}\le p(x)\le\overline{p}<\infty \text{ on } \overline{\Omega}$$

and

$$|p(x)-p(y)|\le\frac{A}{-\ln|x-y|} \text{ for all } x,y\in\overline{\Omega} \text{ with } |x-y|\le\frac{1}{2},$$

where $A>0$ does not depend on x and y.

By $C_+[0,l]$, $0<l<\infty$, we denote the class of functions $w(t)$ on $[0,l]$ continuous and positive at every point $t\in(0,l]$ and having a finite or infinite $\lim\limits_{t\to 0}w(t)=w(0)$.

Definition 2. Let $-\infty<\beta<\gamma<\infty$. We define the class $Z_{\beta,\gamma}[0,l]$ as the set of functions in $C_+[0,l]$ satisfying the condition

$$\int_0^h w(x)x^{-1-\beta}dx\le cw(h)h^{-\beta}$$

and

$$\int_h^l w(x)x^{-1-\gamma}dx\le cw(h)h^{-\gamma}$$

where $c=c(w)>0$ does not depend on $h\in(0,l]$.

A nonnegative function φ on $[0,l]$ is said to be almost increasing (or almost decreasing) if there exists a constant $c\ge 1$ such that $\varphi(x)\le c\varphi(y)$ for all $x\le y$ (or $x\ge y$ respectively). Let

$$W_0=\{\varphi\in C_+[0,l],\ \varphi \text{ is almost increasing}\}.$$

Besides W_0 we also need a wider subclass $\widetilde{W}_0$ of functions in $C_+[0,l]$ which become almost increasing after the multiplication by a power function:

$$\widetilde{W}_0[0,l]=\{\varphi:\ \exists a=a(\varphi)\in R^1 \text{ such that } x^a\varphi(x)\in W_0[0,l]\}$$

Definition 3. We define the Zygmund-Bary-Stechkin class

$$\Phi_{\beta,\gamma}=\widetilde{W}_0\cap Z_{\beta,\gamma}.$$

Definition 4. Let $w\in C_+[0,l]$. The numbers

$$m_w=\sup_{x>1}\frac{\ln\left[\underline{\lim}_{h\to 0}\frac{w(xh)}{w(h)}\right]}{\ln x},\quad M_w=\inf_{x>1}\frac{\ln\left[\overline{\lim}_{h\to 0}\frac{w(xh)}{w(h)}\right]}{\ln x}$$

introduced in such a form in [4], will be referred to as the lower and upper index numbers of a functions $w\in W$ (they are close to the Matuszewska-Orlicz indices). In the case $w\in W_0$ we have $0\le m(w)\le M(w)\le\infty$.

Our main result is the following

Theorem 1. *Given an open bounded set $\Omega\subset R^n$, $n\ge 3$ suppose $p\in B(\Omega)$ and a weight function $\rho(|x|)=w(|x-x_0|)$, $x_0\in\overline{\Omega}$ satisfies the condition*

$$w(r)\in\Phi_{\beta_0,\gamma_0}[0,1]$$

where $\beta_0=2p(x_0)-n$, $\gamma_0=n(p(x_0)-1)$ or equivalently

$$w\in\widetilde{W}[0,1],\ \ 2p(x_0)-n<m(w)\le M(w)<n(p(x_0)-1).$$

Then for every function $f\in L^{p(\cdot)}(\Omega,\rho)$, $\overline{p}<\frac{n}{2}$, there exists a function $u\in L^{q(\cdot)}(\Omega,\rho_1)$ with

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{2}{n}$$

and $\rho_1(x)=\rho^{\frac{q(x_0)}{p(x_0)}}(x)$ such that

$$\Delta u=f \text{ a. e. } x\in\Omega.$$

Furthemore, there exists a positive constant c such that

i) $\|u\|_{L^{q(\cdot)}(\Omega,\rho_1)}\le c\|f\|_{L^{p(\cdot)}(\Omega,\rho)}$

for arbitrary $f\in L^{p(\cdot)}(\Omega,\rho)$,

ii) $\|D^1u\|_{L^{r(\cdot)}(\Omega,\rho_2)}\le c\|f\|_{L^{p(\cdot)}(\Omega,\rho)}$

when $\frac{1}{r(x)}=\frac{1}{p(x)}-\frac{1}{n}$, $\rho_2(x)=\rho_1^{\frac{r(x_0)}{p(x_0)}}(x)$

and

iii) $\|D^2u\|_{L^{p(\cdot)}(\Omega,\rho)}\le c\|u\|_{L^{p(\cdot)}(\Omega,\rho)}$.

For the proof of Theorem 1 we need some basic results:

Theorem A [5]. *Given an open bounded set $\Omega\subset R^n$, $n\ge 3$, suppose $p\in B(\Omega)$ and a weight function $\rho(x)=w(|x-x_0|)$, $x_0\in\overline{\Omega}$ where*

$$w(r)\in\widetilde{W}[0,1],\quad \alpha\, p(x_0)-n<m(w)\le M(w)<n(p(x_0)-1).$$

Then the Riesz potential

$$I_\alpha f(x)=\int_\Omega\frac{f(y)}{|x-y|^{n-\alpha}}dy,\quad 0<\alpha<n$$

is bounded from the space $L^{p(\cdot)}(\Omega,\rho)$ *into the space* $L^{q(\cdot)}(\Omega,\rho)$ *when* $\overline{p}<\frac{n}{\alpha}$, $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}$ *and*

$$\rho_1(x)=\rho^{\frac{q(x_0)}{p(x_0)}}(x).$$

Theorem B [6]. *Given an open bounded set* $\Omega\subset R^n$, $n\geq 3$, *suppose* $p\in B(\Omega)$ *and a weight function*

$$\rho(x)=w(|x-x_0|),\ x_0\in\overline{\Omega}$$

where

$$w(r)\in\widetilde{W}[0,1],\ \ -n<m(w)\leq M(w)<n(p(x_0)-1).$$

Given a locally integrable function k defined on $R^n\setminus\{0\}$, *suppose that the Fourier transform of k is bounded, and satisfies the conditions:*

$$|k(x)|\leq\frac{c}{|x|^n},\quad |\nabla k(x)|\leq\frac{c}{|x|^{n+1}}. \tag{*}$$

The singular integral operator K defined by

$$Kf(x)=(k*f)(x)$$

is bounded in $L^{p(\cdot)}(\Omega,\rho)$.

Proof of Theorem 1. Let $f\in L^p(\Omega,\rho)$. Then as Ω is bounded we have $f\in L^{p_0}(\Omega)$ for some p_0, $1<p_0<p<\infty$.

The Newtonian potential

$$u(x)=\int_\Omega\Gamma(x-y)\varphi(y)dy$$

where

$$\Gamma(x)=\frac{c}{\omega_n}|x|^{2-n}$$

satisfies the equation (see [1], p. 230).

$$\Delta u=0$$

Applying Theorem A we have i).

Then for $|\alpha|=1$ we have

$$\left|D^\alpha\Gamma(x)\right|\leq\frac{1}{n\omega_n}|x|^{1-n}.$$

Therefore

$$\left|D^\alpha u(x)\right|\leq\frac{1}{n\omega_n}I_1(|\varphi|)(x).$$

So again applying Theorem A we get ii).

For multiindex α, $|\alpha|=2$, it is well known that $D^\alpha\Gamma$ is a singular integral convolution kernel satisfying the conditions (*). Therefore, the operator

$$K_\alpha g(x)=(D^\alpha\Gamma*g)(x)=D^\alpha(\Gamma*g)(x)$$

is a singular integral operator and by Theorem B we conclude iii).

Introduce the weighted Sobolev space with the norm

$$\|\varphi\| = \|\varphi\|_{L^{q(\cdot)}(\Omega,\rho_1)} + \left\|D^1\varphi\right\|_{L^{r(\cdot)}(\Omega,\rho_2)} + \left\|D^2\varphi\right\|_{L^{p(\cdot)}(\Omega,\rho)}.$$

We denote this space w by $W^{2,\vec{s}}\left(\Omega,\vec{v}\right)$, where

$$\vec{s} = \left(q(\cdot), r(\cdot), p(\cdot)\right) \text{ and } \vec{v} = \left(\rho_1, \rho_2, \rho\right).$$

Then under the conditions of Theorem 1 we have that if $f \in L^{p(\cdot)}(\Omega,\rho)$ then the solution of Poisson equation $u \in W^{2,\vec{s}}\left(\Omega,\vec{v}\right)$.

In the sequel we give some version of the above-mentioned result for unbounded Ω. Along the log-condition we need the following condition at infinity

$$,\ |x-y| \le \frac{1}{2},\ x, y \in \Omega$$

where

$$p_*(x) = p\left(\frac{x}{|x|^2}\right).$$

Then for

$$\rho(x) = \begin{cases} w_0\left(|x-x_0|\right), \text{ if } |x-x_0| \le 1 \\ w_\infty\left(|x-x_0|\right), \text{ if } |x-x_0| > 1 \end{cases}$$

where $w_0 \in \tilde{W}[0,1],\ \ 0 < m(w_0) \le M(w_0) < n[p(x_0)-1],\ \ w_\infty\left(|x|\right) = w_0\left(\frac{1}{|x|}\right) \in \tilde{W}[0,1]$

and

$$\rho_1(x) = \begin{cases} \left[w_0\left(|x-x_0|\right)\right]^{\frac{q(x_0)}{p(x_0)}}, \text{ if } |x-x_0| \le 1 \\ \left[w_\infty\left(|x-x_0|\right)\right]^{\frac{q(\infty)}{p(\infty)}}, \text{ if } |x-x_0| > 1 \end{cases},$$

we have that for arbitrary $f \in L^{p(\cdot)}\left(R^n,\rho\right)$, $\overline{p} < \frac{n}{2}$, there exists a function $u \in L^{q(\cdot)}\left(R^n,\rho_1\right)$, such that

$$\Delta u = f \text{ a. e. } x \in \Omega$$

and the inequalities i), ii) and iii) hold.

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მათემატიკა

კალდერონ-ზიგმუნდის თეორემის განზოგადების შესახებ წონიან ლებეგის სივრცეებში ცვლადი მაჩვენებლებით

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ნაშრომში მოცემულია პუასონის განტოლების ამონახსნის ყოფაქცევის შესახებ კალდერონ-ზიგმუნდის ცნობილი თეორემის განზოგადება ბანახის არასტანდარტული ფუნქციური სივრცეებისათვის. ასეთი სივრცის მაგალითია ცვლადმაჩვენებლიანი ლებეგის წონიანი სივრცე. უკანასკნელ წლებში ცხადი გახდა, რომ კლასიკური ფუნქციური სივრცეები აღარ არის საკმარისი არაწრფივი დრეკადობის თეორიის, უკუმშვად სითხეთა დინების მექანიკის, მექანიკის ვარიაციული პრობლემების, $p(x)$-ლაპლასიანი არაწრფივი დიფერენციალური განტოლებების ამოსახსნელად. ამ გარემოებამ მძლავრი იმპულსი მისცა ცვლადმაჩვენებლიანი ლებეგის და სობოლევის სივრცეების თეორიის განვითარებას. თუ ამ სივრცეებს თავდაპირველად მხოლოდ თეორიული ინტერესი ჰქონდა, ბოლო ათწლეულში მათი საშუალებით შესაძლებელი გახდა მთელი რიგი გამოყენებითი ხასიათის პრობლემების გადაჭრა.

ნაშრომში პუასონის განტოლების ამონახსნისა და მისი წარმოებულების ნორმების შეფასება არსებითად ეყრდნობა პოტენციალთა და სინგულარული ინტეგრალური ოპერატორების შემოსაზღვრულობის შესახებ სტატიის ავტორთა გამოკვლევებს ცვლადმაჩვენებლიან ლებეგის წონიან სივრცეებში.

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