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SOME APPROXIMATION PROBLEMS FOR (α, ψ)-DIFFERENTIABLE FUNCTIONS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

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We prove direct and inverse theorems for (α, ψ)-differentiable functions in weighted variable exponent Lebesgue spaces. We also define a Besov type space and obtain some properties of this space. Bibliography: 29 titles.

1 Statement of the Problem

Variable exponent Lebesgue spaces $L^{p(x)}$ were mentioned in the literature for the first time by Orlicz [1]. These spaces were systematically studied by Nakano [2, 3]. In the appendix of [2, p. 284], Nakano explicitly indicated variable exponent Lebesgue spaces as an example of modular spaces. Also, under the condition

$$\operatorname*{ess\,sup}_{x\in\boldsymbol{T}} p(x) < \infty,$$

the space $L^{p(x)}$ is a particular case of Musielak–Orlicz spaces [4]. Topological properties of $L^{p(x)}$ were studied by Sharapudinov [5] (cf. also [6]–[8] and the monograph [9]). The spaces $L^{p(x)}$ have many applications in elasticity theory, fluid mechanics, differential operators [10, 11], nonlinear Dirichlet boundary value problems [6], nonstandard growth, and variational calculus [12]. For $p(x) := p$, $1 < p < \infty$, the space $L^{p(x)}$ coincides with the classical Lebesgue space L^p. Unlike L^p, the space $L^{p(x)}$ is not $p(\cdot)$-continuous and is not invariant under translations [6]. This fact causes some difficulties for defining the smoothness moduli. Using the Steklov means, Gadjieva [13] introduced the smoothness moduli in the case of weighted Lebesgue spaces. These moduli

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turned out to be also suitable for the weighted spaces $L^p(x)$. For example, some inequalities on trigonometric approximation in the weighted spaces $L^p(x)$ were proved in [14]–[19]. We note that the inverse inequalities were obtained by S. Stechkin for the space C and by A. Timan and M. Timan for the spaces L^p $(1 \leqslant p < \infty)$. We emphasize the results of Stepanets [20]–[23], in particular, a Bernstein type inequality in unweighted classical Lebesgue spaces was proved in [23] for the derivatives in general sense. Stepanets developed the approximation theory for functions in the spaces C and L^p that are differentiable in the general sense.

In [19], the authors proved the following assertion.

Theorem 1.1 (cf. [19])**.** *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, $\psi \in \mathfrak{M}_0$, $r \in (0, \infty)$, $f \in L^{p(\cdot)}_\omega$ and*

$$\sum_{\nu=1}^{\infty} \frac{E_\nu(f)_{p(\cdot),\omega}}{\nu\psi(\nu)} < \infty, \tag{1.1}$$

then there exists a constant $c > 0$, depending only on ψ, r, and p, such that

$$\Omega_r(f^\psi_\alpha, \frac{1}{n})_{p(\cdot),\omega} \leqslant c\left\{\frac{1}{n^r}\sum_{\nu=1}^{n}\frac{\nu^r E_\nu(f)_{p(\cdot),\omega}}{\nu\psi(\nu)} + \sum_{\nu=n+1}^{\infty}\frac{E_\nu(f)_{p(\cdot),\omega}}{\nu\psi(\nu)}\right\}. \tag{1.2}$$

In this paper, we improve Theorem 1.1. We show that r can be replaced with $2r$ on the right-hand side of (1.2). For this purpose, we refine the converse inequality.

Theorem 1.2 (cf. [15])**.** *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $f \in L^{p(\cdot)}_\omega$, and $r \in \mathbb{R}^+$, then*

$$\Omega_r\Big(f, \frac{1}{n+1}\Big)_{p(\cdot),\omega} \leqslant \frac{c}{(n+1)^r}\sum_{\nu=0}^{n}\frac{(\nu+1)^r E_\nu(f)_{p(\cdot),\omega}}{\nu+1}, \quad n = 0, 1, 2, 3, \dots,$$

where the constant $c > 0$ depends only on r and p.

We also give a characterization of weighted variable exponent Besov spaces [24].

Let a function $\omega : \boldsymbol{T} \to [0, \infty]$ be a weight on $\boldsymbol{T}$. Let $\mathscr{P}$ denote the class of Lebesgue measurable functions $p(x) : \boldsymbol{T} \to (1, \infty)$ such that

$$1 < p_* := \operatorname*{ess\,inf}_{x\in\boldsymbol{T}} p(x) \leqslant p^* := \operatorname*{ess\,sup}_{x\in\boldsymbol{T}} p(x) < \infty.$$

Then we introduce the class $L^{p(x)}$ of 2π-periodic measurable functions $f : \boldsymbol{T} \to \mathbb{R}$ such that

$$\int_{\boldsymbol{T}} |f(x)|^{p(x)} dx < \infty$$

for $p \in \mathscr{P}$. It is known that $L^{p(x)}$ is a Banach space [6] equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\alpha > 0 : \int_{\boldsymbol{T}} \Big|\frac{f(x)}{\alpha}\Big|^{p(x)} dx \leqslant 1\right\}.$$

We denote by $L_{\omega}^{p(\cdot)}$ the class of Lebesgue measurable functions $f : \boldsymbol{T} \to \mathbb{R}$ such that $\omega f \in L^{p}(x)$. The weighted variable exponent Lebesgue space $L_{\omega}^{p(\cdot)}$ is a Banach space equipped with the norm $\|f\|_{p(\cdot),\omega} := \|\omega f\|_{p(\cdot)}$.

For a given $p \in \mathscr{P}$ we denote by $A_{p(\cdot)}$ the class of weights ω satisfying the condition [25]

$$\|\omega \chi_Q\|_{p(\cdot)} \|\omega^{-1} \chi_Q\|_{p'(\cdot)} \leqslant C|Q|$$

for all balls Q in $\boldsymbol{T}$. Here, $p'(x) := p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$. The variable exponent $p(x)$ is said to be *log-Hölder continuous* on $\boldsymbol{T}$ if there exists a constant $c \geqslant 0$ such that

$$|p(x_1) - p(x_2)| \leqslant \frac{c}{\log\left(e + 1/\left|x_1 - x_2\right|\right)} \quad \text{for all } x_1, x_2 \in \boldsymbol{T}.$$

We denote by $\mathscr{P}^{\log}(\boldsymbol{T})$ the class of exponents $p \in \mathscr{P}$ such that $1/p : \boldsymbol{T} \to [0,1]$ is log-Hölder continuous on $\boldsymbol{T}$.

If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $f \in L_{\omega}^{p(\cdot)}$, then it was proved in [25] that the $L_{\omega}^{p(\cdot)}$-norm of the Hardy–Littlewood maximal function $\mathscr{M}$ is bounded if and only if $\omega \in A_{p(\cdot)}$.

We set $f \in L_{\omega}^{p(\cdot)}$ and

$$\mathscr{A}_h f(x) := \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t)dt, \quad x \in \boldsymbol{T}.$$

If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $\mathscr{A}_h$ is bounded in $L_{\omega}^{p(\cdot)}$. Consequently if $x, h \in \boldsymbol{T}$ and $0 \leqslant r$, we define, via the binomial expansion,

$$\sigma_h^r f(x) := (I - \mathscr{A}_h)^r f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} (\mathscr{A}_h)^k,$$

where $f \in L_{\omega}^{p(\cdot)}$, Γ is the Gamma function, and I is the identity operator.

For $0 \leqslant r$ we define the *fractional moduli of smoothness* for $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}$ and $f \in L_{\omega}^{p(\cdot)}$ by the formula

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \sup_{0<h_i,t\leqslant\delta} \left\| \prod_{i=1}^{[r]} (I - \mathscr{A}_{h_i}) \sigma_t^{\{r\}} f \right\|_{p(\cdot),\omega}, \quad \delta \geqslant 0,$$

where

$$\Omega_0(f,\delta)_{p(\cdot),\omega} := \|f\|_{p(\cdot),\omega}, \qquad \prod_{i=1}^{0} (I - \mathscr{A}_{h_i}) \sigma_t^r f := \sigma_t^r f, \quad 0 < r < 1,$$

and $[r]$ denotes the integer part of a real number r and $\{r\} := r - [r]$.

If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $\omega^{p(x)} \in L^1(\boldsymbol{T})$. This implies that the set of trigonometric polynomials is dense [26] in the space $L_{\omega}^{p(\cdot)}$. On the other hand, if $p \in \mathscr{P}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $L_{\omega}^{p(\cdot)} \subset L^1(\boldsymbol{T})$.

For a given $f \in L_{\omega}^{p(\cdot)}$ we consider the *Fourier series*

$$f(x) \backsim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx)$$

and the *conjugate Fourier series*

$$\widetilde{f}(x) \backsim \sum_{k=1}^{\infty}(a_k(f)\sin kx - b_k(f)\cos kx).$$

We say that a function $f \in L_{\omega}^{p(\cdot)}$, $p \in \mathscr{P}$, $\omega \in A_{p(\cdot)}$, has a (α, ψ)-derivative f_{α}^{ψ} if for a given sequence $\psi(k)$, $k = 1, 2, \ldots$, and a number $\alpha \in \mathbb{R}$ the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)}\Big(a_k(f)\cos k\Big(x + \frac{\alpha\pi}{2k}\Big) + b_k(f)\sin k\Big(x + \frac{\alpha\pi}{2k}\Big)\Big)$$

is the Fourier series of the function f_{α}^{ψ}. For $\psi(k) = k^{-\alpha}$, $k = 1, 2, \ldots$, $\alpha \in \mathbb{R}^{+}$, we have the fractional derivative $f^{(\alpha)}$ of f in the sense of Weyl [27]. For $\psi(k) = k^{-\alpha}\ln^{-\beta} k$, $k = 1, 2, \ldots$, $\alpha, \beta \in \mathbb{R}^{+}$ we have the power logarithmic–fractional derivative $f^{(\alpha,\beta)}$ of f (cf. [28]).

Let $\mathfrak{M}$ be the set of functions $\psi(v)$ that are convex downwards for any $v \geqslant 1$ and satisfy the condition $\lim_{v\to\infty}\psi(v) = 0$. We associate every function $\psi \in \mathfrak{M}$ with a pair of functions $\eta(t) = \psi^{-1}(\psi(t)/2)$, $\mu(t) = t/(\eta(t) - t)$ and $\overline{\eta}(t) = \psi^{-1}(2\psi(t))$. We set $\mathfrak{M}_0 := \{\psi \in \mathfrak{M} : 0 < \mu(t) \leqslant K\}$. These classes were intensively studied in [20]–[22].

Definition 1.3. A function $\psi(t)$ is said to be *quasiincreasing* (respectively, *quasidecreasing*) on $(0, \infty)$ if there exists a constant c such that $\psi(t_1) \leqslant c\psi(t_2)$ (respectively, $\psi(t_1) \geqslant c\psi(t_2)$) for any $t_1, t_2 \in (0, \infty)$, $t_1 \leqslant t_2$.

Definition 1.4. Let φ be a nondecreasing function on $(0, \infty)$ such that $\varphi(0) = 0$ and

(i) there exists $\beta > 0$ such that $\varphi(t)t^{-\beta}$ is quasiincreasing,

ii) there exists $\beta_1 > 0$ such that $k > \beta_1$ and $\varphi(t)t^{\beta_1 - k}$ is quasidecreasing.

The class of such functions is denoted by $U(k)$.

The properties of this class were studied, for example, in [29].

Definition 1.5. Suppose that $\varphi \in U(k)$ and $1 \leqslant \gamma < \infty$. The collection $B_{p(\cdot),\gamma}^{k,\varphi}$ of functions $f \in L_{\omega}^{p(\cdot)}$ satisfying the condition

$$\int_0^1 \Omega_k^{\gamma}(f, t)_{p(\cdot),\omega}\,\varphi^{\gamma}(1/t)\,t^{-1}dt < +\infty$$

is referred to as the *weighted variable exponent Besov spaces.*

The norm in $B_{p(\cdot),\gamma}^{k,\varphi}$ can be defined by the formula

$$\|f\|_{p(\cdot),\gamma}^{k,\varphi} = \|f\|_{p(\cdot),\omega} + \left\{\int_0^1 \Omega_k^{\gamma}(f, t)_{p(\cdot),\omega}\varphi^{\gamma}(1/t)t^{-1}dt\right\}^{1/\gamma}. \tag{1.3}$$

We refer to [24] for more information about Besov spaces.

In this paper, we prove the following inequalities of trigonometric approximation.

Theorem 1.6. *Suppose that $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, $r \in \mathbb{R}^+$ and $f \in L_\omega^{p(\cdot)}$. Then for every natural number n the following estimate holds:*

$$\Omega_r\Big(f, \frac{1}{n}\Big)_{p(\cdot),\omega} \leqslant \frac{c}{n^{2r}}\left\{E_0(f)_{p(\cdot),\omega} + \sum_{k=1}^{n} \frac{k^{2r} E_k(f)_{p(\cdot),\omega}}{k}\right\},$$

where the constant $c > 0$ is independent of n.

Theorem 1.7. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, $\psi \in \mathfrak{M}_0$, $r \in (0, \infty)$, $f \in L_\omega^{p(\cdot)}$, and (1.1) is satisfied, then there exist constants $c, C > 0$, depending only on ψ, r, and p, such that*

$$\Omega_r\Big(f_\alpha^\psi, \frac{1}{n}\Big)_{p(\cdot),\omega} \leqslant \frac{c}{n^{2r}} \sum_{\nu=1}^{n} \frac{\nu^{2r} E_\nu(f)_{p(\cdot),\omega}}{\nu\psi(\nu)} + C \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f)_{p(\cdot),\omega}}{\nu\psi(\nu)}.$$

Theorem 1.8. *Suppose that $1 \leqslant \gamma < +\infty$, $\varphi \in U(k)$, $k \in \mathbb{R}^+$, and $f \in L_\omega^{p(\cdot)}$. Then there exist constants $c, C > 0$ such that*

$$c\int_0^1 \Omega_k^\gamma(f,t)_{p(\cdot),\omega}\, \varphi^\gamma(1/t)\, t^{-1} dt \leqslant \sum_{i=0}^{\infty} E_{2^i}^\gamma(f)_{p(\cdot),\omega}\, \varphi^\gamma(2^i) \leqslant C \int_0^1 \Omega_k^\gamma(f,t)_{p(\cdot),\omega}\, \varphi^\gamma(1/t)\, t^{-1} dt.$$

Theorem 1.9. *Suppose that $1 \leqslant \gamma < +\infty$ and $\varphi \in U(k)$. The space $B_{p(\cdot),\gamma}^{k,\varphi}$ is a Banach space with respect to the norm (1.3).*

Theorem 1.10. *Suppose that $1 \leqslant \gamma < +\infty$, $\varphi \in U(k)$, and $f \in B_{p(\cdot),\gamma}^{k,\varphi}$. Then*

$$\lim_{h \to 0} \|f - \mathscr{A}_h f\|_{p(\cdot),\gamma}^{k,\varphi} = 0.$$

In particular, Theorem 1.8 implies the following assertion.

Corollary 1.11. *Suppose that $1 \leqslant \gamma < +\infty$, $f \in L_\omega^{p(\cdot)}$, $\varphi(x) := x^\alpha$, and $k := 1 + [\alpha]$. Then there exist constants $c, C > 0$ such that*

$$c\int_0^1 \Omega_{1+[\alpha]}^\gamma(f,t)_{p(\cdot),\omega}\, t^{-\alpha\gamma-1} dt \leqslant \sum_{i=0}^{\infty} E_{2^i}^\gamma(f)_{p(\cdot),\omega}\, 2^{i\alpha\gamma} \leqslant C \int_0^1 \Omega_{1+[\alpha]}^\gamma(f,t)_{p(\cdot),\omega}\, t^{-\alpha\gamma-1} dt.$$

Theorem 1.12. *Suppose that $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$, and $\beta := \max\{2, p^*\}$. If $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \to 0$ as $k \to \infty$, then for every $n = 0, 1, 2, 3, \ldots$ there exists a constant $c > 0$ independent of n such that*

$$\Omega_r\Big(f_\alpha^\psi, \frac{1}{n}\Big)_{p(\cdot),\omega} \geqslant \frac{c}{n^{2r}}\left(\sum_{\nu=1}^{n} \frac{\nu^{2\beta r} E_\nu^\beta(f)_{p(\cdot),\omega}}{\nu\psi^\beta(\nu)}\right)^{1/\beta}. \tag{1.4}$$

Theorem 1.12 is a refinement of the following assertion.

Theorem 1.13 (cf. [19])**.** *Let $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, $r \in \mathbb{R}^+$ and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \to 0$ as $k \to \infty$, then for every $n = 1, 2, 3, \ldots$ there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot),\omega} \leqslant c\psi(n)\Omega_r\Big(f_\alpha^\psi, \frac{1}{n}\Big)_{p(\cdot),\omega}.$$

Indeed,

$$\frac{c}{n^{2r}}\left(\sum_{\nu=1}^{n} \frac{\nu^{2\beta r} E_\nu^\beta(f)_{p(\cdot),\omega}}{\nu\psi^\beta(\nu)}\right)^{1/\beta} \geqslant \frac{E_n(f)_{p(\cdot),\omega}}{\psi(n)}.$$

On the other hand, the term on the left-hand side of (1.4) is often important: it defines the order of estimation from below. For the sake of simplicity, we set $r = 1$ and $\psi(n) := n^{-\alpha}$. Then for

$$E_\nu\,(f)_{p(\cdot),\omega} \sim \nu^{-2-\alpha}$$

the left-hand side of (1.4) is $\sim n^{-2}\,(\ln n)^{1/\beta}$ and (1.4) implies

$$\Omega_1\Big(f, \frac{1}{n}\Big)_{p(\cdot),\omega} \geqslant \frac{c}{n^2}(\ln n)^{1/\beta}. \tag{1.5}$$

On the other hand,

$$\left(\sum_{\nu=n+1}^{\infty} \nu^{\alpha\beta-1} E_\nu^\beta(f)_{p(\cdot),\omega}\right)^{1/\beta} \sim n^{-2} \quad \text{and} \quad \Omega_1\Big(f, \frac{1}{n}\Big)_{p(\cdot),\omega} \geqslant \frac{c}{n^2}.$$

Thus, the estimate (1.5) is better.

Remark 1.14. It was M. Timan who first noted the influence of the metric on the direct and inverse inequalities in the classical Lebesgue spaces L^p $(1 < p < \infty)$.

In the particular case $\psi(k) = k^{-\alpha}\ln^{-\beta} k$, $k = 1, 2, \ldots$, $\alpha, \beta \in \mathbb{R}^+$, from Theorem 1.7 we obtain the following new result for power logarithmic-fractional derivatives.

Theorem 1.15. *If $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\boldsymbol{T})$ for some $p_0 \in (1, p_*)$, $\alpha, \beta, r \in \mathbb{R}^+$, and*

$$\sum_{\nu=1}^{\infty} \frac{\nu^\alpha \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega}}{\nu} < \infty,$$

then there exist constants $c, C > 0$, depending only on α, β, r, and p, such that

$$\Omega_r\Big(f^{(\alpha,\beta)}, \frac{1}{n}\Big)_{p(\cdot),\omega} \leqslant \frac{c}{n^{2r}} \sum_{\nu=1}^{n} \frac{\nu^{2r+\alpha} \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega}}{\nu} + C \sum_{\nu=n+1}^{\infty} \frac{\nu^\alpha \ln^\beta \nu E_\nu(f)_{p(\cdot),\omega}}{\nu}.$$

In the particular case $\alpha, r \in \mathbb{Z}^+$ and $\beta = 0$, Theorem 1.15 was announced in [18].

Theorem 1.16. *Suppose that $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_*)$, $\alpha, \beta, r \in \mathbb{R}^+$, $f, f^{(\alpha,\beta)} \in L_\omega^{p(\cdot)}$, and $\beta := \max\{2, p^*\}$. Then for every $n = 1, 2, 3, \ldots$ there exists a constant $c > 0$ independent of n such that*

$$\Omega_r\Big(f^{(\alpha,\beta)}, \frac{1}{n}\Big)_{p(\cdot),\omega} \geqslant \frac{c}{n^{2r}}\left(\sum_{\nu=1}^{n} \frac{\nu^{2\beta r} E_\nu^\beta(f)_{p(\cdot),\omega}}{\nu\psi^\beta(\nu)}\right)^{1/\beta}.$$

2 Proof of the Main Results

We begin with the following assertion.

Theorem 2.1 (cf. [19])**.** *Suppose that* $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ *for some* $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, *and* $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. *If* $\psi(k)$, $(k \in \mathbb{N})$ *is an arbitrary nonincreasing sequence of nonnegative numbers such that* $\psi(k) \to 0$ *as* $k \to \infty$, *then for every* $n = 0, 1, 2, 3, \dots$ *there exists a constant* $c > 0$ *independent of* n *such that*

$$E_n(f)_{p(\cdot),\omega} \leqslant c\psi(n) E_n(f_\alpha^\psi)_{p(\cdot),\omega}.$$

The following Lemma was proved in the previous paper by the authors [19, Corollary 2.1], where we essentially used the idea due to Stepanets and Kushpel' [23].

Lemma 2.2. *If* $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ *for some* $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, $\psi(k)$, $(k \in \mathbb{N})$ *is an arbitrary nonincreasing sequence of nonnegative numbers, and* $T_n \in \mathscr{T}_n$, *then*

$$\|(T_n)_\alpha^\psi\|_{p(\cdot),\omega} \leqslant c(\psi(n))^{-1} \|T_n\|_{p(\cdot),\omega}.$$

Theorem 2.3 (cf. [19])**.** *If* $p \in \mathscr{P}^{\log}(\boldsymbol{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ *for some* $p_0 \in (1, p_*)$, $\alpha \in \mathbb{R}$, $\psi \in \mathfrak{M}_0$, $f \in L_\omega^{p(\cdot)}$, *and* (1.1) *is satisfied, then* $f_\alpha^\psi \in L_\omega^{p(\cdot)}$ *and*

$$E_n(f_\alpha^\psi)_{p(\cdot),\omega} \leqslant c \left(\frac{E_n(f)_{p(\cdot),\omega}}{\psi(n)} + \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f)_{p(\cdot),\omega}}{\nu \psi(\nu)} \right),$$

where the constant $c > 0$ *depends only on* α *and* p.

Proof of Theorem 1.6. We choose m satisfying $2^m \leqslant n \leqslant 2^{m+1}$. By the subadditivity of Ω_r, we have

$$\Omega_r(f, \delta)_{p(\cdot),\omega} \leqslant \Omega_r(f - T_{2^{m+1}}, \delta)_{p(\cdot),\omega} + \Omega_r(T_{2^{m+1}}, \delta)_{p(\cdot),\omega} \tag{2.1}$$

and

$$\Omega_r(f - T_{2^{m+1}}, \delta)_{p(\cdot),\omega} \leqslant c \|f - T_{2^{m+1}}\|_{p(\cdot),\omega} \leqslant c E_{2^{m+1}}(f)_{p(\cdot),\omega}. \tag{2.2}$$

By [15, Corollary 2.5], we have

$$\begin{aligned}
\Omega_r(T_{2^{m+1}}, \delta)_{p(\cdot),\omega} &\leqslant c\delta^{2r} \|T_{2^{m+1}}^{(2r)}\|_{p(\cdot),\omega} \\
&\leqslant c\delta^{2r} \left\{ \|T_1^{(2r)} - T_0^{(2r)}\|_{p(\cdot),\omega} + \sum_{i=1}^{m} \|T_{2^{i+1}}^{(2r)} - T_{2^i}^{(2r)}\|_{p(\cdot),\omega} \right\} \\
&\leqslant c\delta^{2r} \left\{ E_0(f)_{p(\cdot),\omega} + \sum_{i=1}^{m} 2^{(i+1)2r} E_{2^i}(f)_{p(\cdot),\omega} \right\} \\
&\leqslant c\delta^{2r} \left\{ E_0(f)_{p(\cdot),\omega} + 2^{2r} E_1(f)_{p(\cdot),\omega} + \sum_{i=1}^{m} 2^{(i+1)2r} E_{2^i}(f)_{p(\cdot),\omega} \right\}.
\end{aligned}$$

Using the inequality

$$2^{(i+1)2r} E_{2^i}(f)_{p(\cdot),\omega} \leqslant 2^{4r} \sum_{k=2^{i-1}+1}^{2^i} k^{2r-1} E_k(f)_{p(\cdot),\omega}, \quad i \geqslant 1, \tag{2.3}$$

we get

$$\Omega_r(T_{2^{m+1}},\delta)_{p(\cdot),\omega} \leqslant c\delta^{2r}\left\{E_0(f)_{p(\cdot),\omega} + 2^{2r}E_1(f)_{p(\cdot),\omega} + 2^{4r}\sum_{k=2}^{2^m} k^{2r-1}E_k(f)_{p(\cdot),\omega}\right\}$$

$$\leqslant c\delta^{2r}\left\{E_0(f)_{p(\cdot),\omega} + \sum_{k=1}^{2^m} k^{2r-1}E_k(f)_{p(\cdot),\omega}\right\}. \tag{2.4}$$

Since

$$E_{2^{m+1}}(f)_{p(\cdot),\omega} \leqslant \frac{2^{4r}}{n^{2r}} \sum_{k=2^{m-1}+1}^{2^m} \frac{k^{2r}E_k(f)_{M,\omega}}{k},$$

we obtain the required relation from (2.1)–(2.4). □

Proof of Theorem 1.7. Using Theorems 1.6 and 2.3, we find

$$\Omega_r\Big(f_\alpha^\psi,\frac{1}{n}\Big)_{p(\cdot),\omega} \leqslant \frac{c}{n^{2r}}\sum_{\nu=1}^{n}\frac{\nu^{2r}E_\nu(f_\alpha^\psi)_{p(\cdot),\omega}}{\nu},$$

which implies the required inequality

$$\Omega_r\Big(f_\alpha^\psi,\frac{1}{n}\Big)_{p(\cdot),\omega} \leqslant \frac{c}{n^{2r}}\sum_{\nu=1}^{n}\frac{\nu^{2r}E_\nu(f)_{p(\cdot),\omega}}{\nu\psi(\nu)} + C\sum_{\nu=n+1}^{\infty}\frac{E_\nu(f)_{p(\cdot),\omega}}{\nu\psi(\nu)}.$$ □

Proof of Theorem 1.8. Let

$$\int_0^1 \Omega_k^\gamma(f,t)_{p(\cdot),\omega}\,\varphi^\gamma(1/t)\,t^{-1}dt < +\infty.$$

Using Jackson inequality [15, Theorem 1.4]

$$E_n(f)_{p(\cdot),\omega} \leqslant c\Omega_k\Big(f,\frac{1}{n}\Big)_{p(\cdot),\omega},$$

we find

$$\sum_{i=0}^{n} E_{2^i}^\gamma(f)_{p(\cdot),\omega}\varphi^\gamma(2^i) \leqslant c\sum_{i=0}^{n}\Omega_k^\gamma\Big(f,\frac{1}{2^i}\Big)_{p(\cdot),\omega}\varphi^\gamma(2^i) \leqslant c\int_0^n \Omega_k^\gamma\Big(f,\frac{1}{2^u}\Big)_{p(\cdot),\omega}\varphi^\gamma(2^u)du$$

$$= \frac{c}{\ln 2}\ln 2\int_0^n \Omega_k^\gamma\Big(f,\frac{1}{2^u}\Big)_{p(\cdot),\omega}\varphi^\gamma(2^u)du \leqslant \frac{c}{\ln 2}\int_0^1 \Omega_k^\gamma(f,t)_{p(\cdot),\omega}\varphi^\gamma(1/t)t^{-1}dt < +\infty.$$

Hence

$$\sum_{i=0}^{\infty} E_{2^i}^\gamma(f)_{p(\cdot),\omega}\,\varphi^\gamma\left(2^i\right) \leqslant c\int_0^1 \Omega_k^\gamma(f,t)_{p(\cdot),\omega}\,\varphi^\gamma(1/t)\,t^{-1}dt.$$

For the other direction, we set $T_1 \in \mathscr{T}_1$, $E_1(f)_{p(\cdot),\omega} = \|f - T_1\|_{p(\cdot),\omega}$, $f(x) - T_1(x) = F(x)$, and

$$\sum\nolimits_{i=0}^{\infty} E_{2^i}^\gamma(f)_{p(\cdot),\omega}\varphi^\gamma(2^i) < +\infty.$$

Then

$$\int_0^1 \Omega_k^\gamma(F,t)_{p(\cdot),\omega}\varphi^\gamma(1/t)t^{-1}dt = \ln 2\int_0^\infty \Omega_k^\gamma\Big(F,\frac{1}{2^u}\Big)_{p(\cdot),\omega}\varphi^\gamma(2^u)du$$

$$\leqslant c\sum_{i=0}^\infty \varphi^\gamma(2^i)\Omega_k^\gamma\Big(F,\frac{1}{2^i}\Big)_{p(\cdot),\omega}.$$

On the other hand,

$$f(x) = T_1(x) + \sum_{i=1}^\infty \{T_{2^i}(x) - T_{2^{i-1}}(x)\}$$

and we get

$$\|\sigma_{2^{-m}}^k F\|_{p(\cdot),\omega} = \left\|\sigma_{2^{-m}}^k\left(\sum_{i=1}^\infty \{T_{2^i}(x) - T_{2^{i-1}}(x)\}\right)\right\|_{p(\cdot),\omega}$$

$$= \left\|\sum_{i=1}^\infty \sigma_{2^{-m}}^k(T_{2^i}(x) - T_{2^{i-1}}(x))\right\|_{p(\cdot),\omega} \leqslant \sum_{s=1}^\infty \|\sigma_{2^{-m}}^k Q_s\|_{p(\cdot),\omega},$$

where $Q_s(x) := T_{2^s}(x) - T_{2^{s-1}}(x)$. Hence, by [15, Lemma 2.6], we have

$$\|\sigma_{2^{-m}}^k F\|_{p(\cdot),\omega} \leqslant \sum_{s=1}^\infty \|\sigma_{2^{-m}}^k Q_s\|_{p(\cdot),\omega} \leqslant 2^{-mk}\sum_{s=1}^\infty \|Q_s^{(k)}(x)\|_{p(\cdot),\omega}$$

$$= 2^{-mk}\sum_{s=1}^{m+1} \|Q_s^{(k)}(x)\|_{p(\cdot),\omega} + 2^{-mk}\sum_{s=m+2}^\infty 2^{sk}\|Q_s(x)\|_{p(\cdot),\omega}$$

$$\leqslant 2^{-mk}\sum_{s=1}^{m+1} \|Q_s^{(k)}(x)\|_{p(\cdot),\omega} + 2^{-mk}2^{(m+2)k}\sum_{s=m+2}^\infty \|Q_s(x)\|_{p(\cdot),\omega}$$

$$\leqslant c\left\{2^{-mk}\sum_{s=0}^m 2^{sk}E_{2^s}(f)_{p(\cdot),\omega} + 2^k\sum_{s=m+1}^\infty E_{2^s}(f)_{p(\cdot),\omega}\right\}.$$

Then

$$\int_0^1 \Omega_k^\gamma(F,t)_{p(\cdot),\omega}\varphi^\gamma(1/t)t^{-1}dt \leqslant c\left\{\sum_{m=0}^\infty \varphi^\gamma(2^m)2^{-m\gamma k}\left[\sum_{s=0}^m 2^{sk}E_{2^s}(f)_{p(\cdot),\omega}\right]^\gamma\right.$$

$$\left. + \sum_{m=0}^\infty \varphi^\gamma(2^m)2^{k\gamma}\left[\sum_{s=m+1}^\infty E_{2^s}(f)_{p(\cdot),\omega}\right]^\gamma\right\} =: c(I_1 + I_2).$$

We estimate I_1. By Definition 1.4 (ii), we have

$$I_1 = \sum_{m=0}^\infty \varphi^\gamma(2^m)\,2^{-m\gamma k}\left[\sum_{s=0}^m 2^{sk}E_{2^s}(f)_{p(\cdot),\omega}\right]^\gamma$$

$$= \sum_{m=0}^{\infty} \varphi^{\gamma}(2^m) 2^{-m\gamma k} \left[\sum_{s=0}^{m} E_{2^s}(f)_{p(\cdot),\omega} 2^{sk} \frac{2^{-s(k-\alpha)}}{\varphi(2^s)} \varphi(2^s) 2^{s(k-\alpha)} \right]^{\gamma}$$

$$\leqslant C \sum_{m=0}^{\infty} \varphi^{\gamma}(2^m) 2^{-m\gamma k} \left[\sum_{s=0}^{m} E_{2^s}(f)_{p(\cdot),\omega} 2^{sk} \frac{2^{-s(k-\alpha)}}{\varphi(2^m)} \varphi(2^s) 2^{m(k-\alpha)} \right]^{\gamma}$$

$$= C \sum_{m=0}^{\infty} 2^{-m\gamma\alpha} \left[\sum_{s=0}^{m} E_{2^s}(f)_{p(\cdot),\omega} 2^{\alpha s} \varphi(2^s) \right]^{\gamma}$$

$$\leqslant C \sum_{m=0}^{\infty} \left[\sum_{s=0}^{m} E_{2^s}(f)_{p(\cdot),\omega} \varphi(2^s) \right]^{\gamma} \leqslant \sum_{s=0}^{\infty} E_{2^s}^{\gamma}(f)_{p(\cdot),\omega} \varphi^{\gamma}(2^s).$$

For estimating I_2 we use Definition 1.4 (i):

$$I_2 = \sum_{m=0}^{\infty} \varphi^{\gamma}(2^m) \left[\sum_{s=m+1}^{\infty} E_{2^s}(f)_{p(\cdot),\omega} \right]^{\gamma} = \sum_{m=0}^{\infty} \varphi^{\gamma}(2^m) \left[\sum_{s=m+1}^{\infty} E_{2^s}(f)_{p(\cdot),\omega} \frac{\varphi(2^s)}{\varphi(2^s)} \frac{2^{s\beta}}{2^{s\beta}} \right]^{\gamma}$$

$$\leqslant C \sum_{m=0}^{\infty} \varphi^{\gamma}(2^m) \frac{2^{m\beta\gamma}}{\varphi^{\gamma}(2^m) 2^{(m+1)\beta\gamma}} \left[\sum_{s=m+1}^{\infty} E_{2^s}(f)_{p(\cdot),\omega} \varphi(2^s) \right]^{\gamma}$$

$$\leqslant C \sum_{m=0}^{\infty} \left[\sum_{s=m+1}^{\infty} E_{2^s}(f)_{p(\cdot),\omega} \varphi(2^s) \right]^{\gamma} \leqslant C \sum_{s=0}^{\infty} E_{2^s}^{\gamma}(f)_{p(\cdot),\omega} \varphi^{\gamma}(2^s).$$

Summarizing the above estimates, we obtain the inequality

$$\int_0^1 \Omega_k^{\gamma}(f - T_1, t)_{p(\cdot),\omega} \varphi^{\gamma}(1/t) t^{-1} dt \leqslant C \sum_{s=0}^{\infty} E_{2^s}^{\gamma}(f)_{p(\cdot),\omega} \varphi^{\gamma}(2^s).$$

Hence

$$\int_0^1 \Omega_k^{\gamma}(f, t)_{p(\cdot),\omega} \varphi^{\gamma}(1/t) t^{-1} dt \leqslant C \sum_{s=0}^{\infty} E_{2^s}^{\gamma}(f)_{p(\cdot),\omega} \varphi^{\gamma}(2^s). \qquad \square$$

Proof of Theorem 1.9. We follow the arguments of [24]. For a given $F \in L_{\omega}^{p(\cdot)}$ we denote by $t_k(F) \in \mathscr{T}_k$ the best approximating polynomial for F. Then for arbitrary functions φ and ψ in $L_{\omega}^{p(\cdot)}$ we have

$$|E_k(\varphi) - E_k(\psi)| \leqslant \|\varphi - \psi\|_{p(\cdot),\omega}. \tag{2.5}$$

Indeed,

$$E_k(\psi)_{p(\cdot),\omega} \leqslant \|\psi - t_k(\varphi)\|_{p(\cdot),\omega} = \|\psi - \varphi + \varphi - t_k(\varphi)\|_{p(\cdot),\omega} \leqslant \|\psi - \varphi\|_{p(\cdot),\omega} + E_k(\varphi)_{p(\cdot),\omega}.$$

On the other hand

$$E_k(\varphi)_{p(\cdot),\omega} \leqslant \|\psi - \varphi\|_{p(\cdot),\omega} + E_k(\psi)_{p(\cdot),\omega}.$$

Thus we have (2.5).

Let $\|f_m - f_n\|_{p(\cdot),\gamma}^{k,\varphi} \to 0$ as $m \to \infty$, $n \to \infty$. Consequently, for every $\varepsilon > 0$ and N we have

$$\|f_m - f_n\|_{p(\cdot),\omega} + \left(\sum_{i=0}^{N} E_{2^i}^{\gamma} (f_m - f_n)_{p(\cdot),\omega} \varphi^{\gamma} \left(2^i\right) \right)^{1/\gamma} < \varepsilon$$

if $m, n > M(\varepsilon)$, where $M(\varepsilon)$ is an increasing integer-valued function such that $M(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $\{f_j\}$ is a Cauchy sequence in the Banach space $L_{\omega}^{p(\cdot)}$, there exists $f \in L_{\omega}^{p(\cdot)}$ such that $\|f_m - f\|_{p(\cdot),\omega} \to 0$ as $m \to \infty$. We fix N and pass to the limit as $m \to \infty$. Then

$$\|f - f_n\|_{p(\cdot),\omega} + \left(\sum_{i=0}^{N} E_{2^i}^{\gamma} (f - f_n)_{p(\cdot),\omega} \varphi^{\gamma} \left(2^i\right) \right)^{1/\gamma} \leqslant \varepsilon, \quad n > M(\varepsilon).$$

Again passing to the limit as $N \to \infty$, we get

$$\|f - f_n\|_{p(\cdot),\omega} + \left(\sum_{i=0}^{\infty} E_{2^i}^{\gamma} (f - f_n)_{p(\cdot),\omega} \varphi^{\gamma} \left(2^i\right) \right)^{1/\gamma} \leqslant \varepsilon, \quad n > M(\varepsilon).$$

Thus, we can conclude that $f \in B_{p(\cdot),\gamma}^{k,\varphi}$ and

$$\lim_{n \to \infty} \|f - f_n\|_{p(\cdot),\gamma}^{k,\varphi} = 0. \qquad \square$$

Proof of Theorem 1.10. Let $f \in B_{p(\cdot),\gamma}^{k,\varphi}$. Since $\mathscr{A}_h$ is bounded in $L_{\omega}^{p(\cdot)}$, we have $\mathscr{A}_h f \in L_{\omega}^{p(\cdot)}$ and

$$\|f - \mathscr{A}_h f\|_{p(\cdot),\omega} \to 0 \quad \text{as } h \to 0. \tag{2.6}$$

For any $\delta \in (0,1)$ we have

$$\begin{aligned}
&\int_0^1 \Omega_k^{\gamma} (\mathscr{A}_h f, t)_{p(\cdot),\omega} \varphi^{\gamma} (1/t)\, t^{-1} dt \\
&\leqslant \int_0^{\delta} \Omega_k^{\gamma} (\mathscr{A}_h f, t)_{p(\cdot),\omega} \varphi^{\gamma} (1/t)\, t^{-1} dt + \int_{\delta}^{1} \Omega_k^{\gamma} (\mathscr{A}_h f, t)_{p(\cdot),\omega} \varphi^{\gamma} (1/t)\, t^{-1} dt \\
&\leqslant \int_0^{\delta} \Omega_k^{\gamma} (\mathscr{A}_h f, t)_{p(\cdot),\omega} \varphi^{\gamma} (1/t)\, t^{-1} dt + (1-\delta)\, \varphi^{\gamma} (1/\delta)\, \delta^{-1} \sup_{u<h} \Omega_k^{\gamma} (\mathscr{A}_u f, 1)_{p(\cdot),\omega} \\
&\leqslant \int_0^{\delta} \Omega_k^{\gamma} (f, t)_{p(\cdot),\omega} \varphi^{\gamma} (1/t)\, t^{-1} dt + (1-\delta)\, \varphi^{\gamma} (1/\delta)\, \delta^{-1} \sup_{u<h} \|\mathscr{A}_u f\|_{p(\cdot),\omega}^{\gamma} =: I_1 + I_2.
\end{aligned}$$

Since $f \in B_{p(\cdot),\gamma}^{k,\varphi}$ we have $I_1 < \infty$. On the other hand, for fixed δ

$$I_2 \leqslant (1-\delta)\, \varphi^{\gamma} (1/\delta)\, \delta^{-1} \sup_{u<h} \|f\|_{p(\cdot),\omega}^{\gamma} = C(\delta)\, \|f\|_{p(\cdot),\omega}^{\gamma} < \infty.$$

Hence $\mathscr{A}_h f \in B^{k,\varphi}_{p(\cdot),\gamma}$. Again, for any $\delta \in (0,1)$ we obtain

$$\int_0^1 \Omega_k^\gamma (\mathscr{A}_h f - f, t)_{p(\cdot),\omega}\, \varphi^\gamma (1/t)\, t^{-1} dt$$

$$\leqslant 2^\gamma \int_0^\delta \Omega_k^\gamma (f, t)_{p(\cdot),\omega}\, \varphi^\gamma (1/t)\, t^{-1} dt + \int_\delta^1 \Omega_k^\gamma (\mathscr{A}_h f - f, t)_{p(\cdot),\omega}\, \varphi^\gamma (1/t)\, t^{-1} dt$$

$$\leqslant 2^\gamma \int_0^\delta \Omega_k^\gamma (f, t)_{p(\cdot),\omega}\, \varphi^\gamma (1/t)\, t^{-1} dt + (1-\delta)\, \varphi^\gamma (1/\delta)\, \delta^{-1} \sup_{u<h} \Omega_k^\gamma (\mathscr{A}_u f - f, 1)_{p(\cdot),\omega} =: I_1' + I_2'.$$

Since $f \in B^{k,\varphi}_{p(\cdot),\gamma}$, the quantity I_1' can be arbitrarily small with the choice of δ. Then for fixed δ

$$I_2' \leqslant (1-\delta)\, \varphi^\gamma (1/\delta)\, \delta^{-1} \sup_{u<h} \|\mathscr{A}_u f - f\|^\gamma_{p(\cdot),\omega} \to 0 \quad \text{as } h \to 0.$$

Thus, by (2.6),

$$\|f - \mathscr{A}_h f\|^{k,\varphi}_{p(\cdot),\gamma} \to 0 \quad \text{as } h \to 0.$$

□

Proof of Theorem 1.12. By [16, Theorem 1.1], we have

$$\Omega_r \Big(f_\alpha^\psi, \frac{1}{n}\Big)_{p(\cdot),\omega} \geqslant \frac{c}{n^{2r}} \left(\sum_{\nu=1}^n \frac{\nu^{2\beta r} E_\nu^\beta (f_\alpha^\psi)_{p(\cdot),\omega}}{\nu} \right)^{1/\beta} =: L.$$

By [19, Theorem 1.1], we have

$$L \geqslant \frac{c}{n^{2r}} \left(\sum_{\nu=1}^n \frac{\nu^{2\beta r} E_\nu^\beta (f)_{p(\cdot),\omega}}{\nu \psi^\beta (\nu)} \right)^{1/\beta}.$$

Theorem 1.12 is proved. □

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