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SOME APPROXIMATION RESULTS IN SUBSPACE OF WEIGHTED GRAND LEBESGUE SPACES

Let $\mathbb{T}=[-\pi,\pi]$, $1<p<\infty$ and $\theta>0$. The weighted grand Lebesgue space of $L_w^{p),\theta}(\mathbb{T})$ is defined as a set of measurable functions for which the norm

$$\|f\|_{L^{p),\theta,w}}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^\theta}{2\pi}\int\limits_{\mathbb{T}}|f(x)|^{p-\varepsilon}w(x)dx\right)^{\frac{1}{p-\varepsilon}}<\infty.$$

Here w is a 2π-periodic weight function, i.e. an almost everywhere positive integrable function.

$L_w^{p),\theta}(\mathbb{T})$ is the Banach function space, non-reflexive, non-separable and non-rearrangement. It is easy to check that the following continuous embeddings hold

$$L_w^p\hookrightarrow L_w^{p),\theta}\hookrightarrow L_w^{p-\varepsilon},\quad 0<\varepsilon<p-1.$$

Grand Lebesgue spaces on the bounded subsets of $\mathbb{R}^n$ were introduced by T. Iwaniec and C. Sbordone [1]. The closure of L_w^p $(1<p<\infty)$ by the norm of the grand Lebesgue spaces does not coincide with the latter space. Let us denote this closure by $\widetilde{L}_w^{p),\theta}$. It is known that this subspace of $L_w^{p),\theta}$ is a set of functions for which

$$\lim_{\varepsilon\to 0}\varepsilon^\theta\int\limits_{\mathbb{T}}|f(x)|^{p-\varepsilon}w(x)dx=0.$$

A weight function w is said to be of the Muckenhoupt class A_p $(1<p<\infty)$ if

$$\sup\left(\frac{1}{|I|}\int\limits_I w(x)dx\right)\left(\frac{1}{|I|}\int\limits_I w^{1-p'}(x)dx\right)^{p-1}<\infty,$$

where the supremum is taken over all intervals with length less than 2π, $p'=\frac{p}{p-1}$.

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We set $f\in L_w^{p),\theta}$ and

$$\mathcal{A}_h f(x):=\frac{1}{h}\int\limits_{x-h/2}^{x+h/2} f(t)\,dt,\quad x\in\boldsymbol{T}.$$

If $1<p<\infty$, $\theta>0$ and $w\in A_p$, then $\mathcal{A}_h$ is bounded in $L_w^{p),\theta}$. This follows, for example, from the boundedness of the Hardy-Littlewood maximal operator in weighted grand Lebesgue spaces due to A. Fiorenza, B. Gupta and P. Jain [2]. Consequently if $x,h\in\boldsymbol{T}$, $0\le r$, then we define, via Binomial expansion, that

$$\sigma_h^r f(x):=(I-\mathcal{A}_h)^r f(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(r+1)}{\Gamma(k+1)\,\Gamma(r-k+1)}(\mathcal{A}_h)^k$$

where $f\in L_w^{p),\theta}$, Γ is gamma function and I is the identity operator.

For $0\le r$ we define the *fractional moduli of smoothness* for $w\in A_p$ and $f\in L_w^{p),\theta}$ as

$$\Omega_r(f,\delta)_{p),\theta,w}:=\sup_{0<h_i,t\le\delta}\left\|\prod_{i=1}^{[r]}(I-\mathcal{A}_{h_i})\,\sigma_t^{\{r\}}f\right\|_{p),\theta,w},\quad \delta\ge 0,$$

where $\Omega_0(f,\delta)_{p),\theta,w}:=\|f\|_{p),\theta,w}$ and $\prod\limits_{i=1}^{0}(I-\mathcal{A}_{h_i})\,\sigma_t^r f:=\sigma_t^r f$ for $0<r<1$; $[r]$ denotes the integer part of the real number r and $\{r\}:=r-[r]$.

Let the Fourier series of a function $f\in L_w^{p),\theta}$ be written as

$$f(x)\backsim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}(a_k(f)\cos kx+b_k(f)\sin kx)=\sum_{k=0}^{\infty}A_k(x,f).$$

We will say that a function $f\in L_w^{p),\theta}$, $1<p<\infty$, $\theta>0$ and $w\in A_p$, has a (α,φ)-derivative f_α^φ if, for a given sequence $\varphi(k)$, $k=1,2,\dots$, and a number $\alpha\in\mathbb{R}$, the transformed trigonometric series

$$\sum_{k=1}^{\infty}\frac{1}{\varphi(k)}\left(a_k(f)\cos k\left(x+\frac{\alpha\pi}{2k}\right)+b_k(f)\sin k\left(x+\frac{\alpha\pi}{2k}\right)\right)$$

is the Fourier series of function f_α^φ.

We emphasize that the notion of generalized, so called (α,φ) derivatives was introduced and studied by A. I. Stepanets (see e.g. [3]) in view of approximation problems of periodic functions in classical Lebesgue spaces.

Let H be the set of some functions $\varphi(t)$ convex downwards for any $t\ge 1$ and satisfying the condition $\lim_{t\to\infty}\varphi(t)=0$.

We associate every function $\varphi\in H$ with a pair of functions $\eta(\tau)=\varphi^{-1}(\varphi(\tau)/2)$ and $\mu(\tau)=\tau/(\eta(\tau)-\tau)$. We set $H_0:=\{\varphi\in H:0<\mu(\tau)\le M\}$.

The following theorem states the Bernstein type inequality.

Theorem 1. *Let $1 < p < \infty$, $\theta > 0$ and $w \in A_p$. Suppose that $\varphi(k)$ is a nonincreasing sequence of non-negative numbers such that $\varphi(k) \to 0$ as $k \to \infty$ and $\frac{1}{\varphi(k)k^r}$ be nondecreasing. Then the following inequality*

$$\| (T_n)_r^\varphi \|_{p),\theta,w} \le \frac{c}{\varphi(n)} \Omega_{r/2}(T_n, 1/n)_{p),\theta,w}.$$

holds with a constant independent of T_n.

For $f \in \widetilde{L}_w^{p),\theta}$ by $E_n(f)_{p),\theta,w}$ we denote the best approximation by trigonometric polynomials

$$E_n(f)_{p),\theta,w} = \inf \|f - T\|_{p),\theta,w},$$

where the infimum is taken over all trigonometric polynomials T of order not greater than n. For $f \in \widetilde{L}^{p),\theta}{}_{,w}$, $w \in A_p$, $1 < p < \infty$ and $\theta > 0$ we have

$$\lim_{n\to\infty} E_n(f)_{p),\theta,w} = 0.$$

The following simultaneous approximation theorem is valid.

Theorem 2. *Let $1 < p < \infty$, $\theta > 0$ and $w \in A_p$. Suppose that $\alpha \in [0, \infty)$ and $f_\alpha^\varphi \in \widetilde{L}_w^{p),\theta}$. If $\varphi \in H_0$, then there exists a $T \in \mathcal{T}_n$, $n = 1, 2, 3, \dots$ and a constant $c > 0$ depending only on α and p such that*

$$\|f_\alpha^\varphi - T_\alpha^\varphi\|_{p),\theta,w} \le cE_n\left(f_\alpha^\varphi\right)_{p),\theta,\omega}$$

holds.

Theorem 3. *Let $1 < p < \infty$, $\theta > 0$, $r > 0$ and $w \in A_p$. Let T_n be the best approximating trigonometric polynomial for $f \in \widetilde{L}_w^{p),\theta}(\mathbb{T})$. Then there exists two positive constants c_1 and c_2, independent of f and n, such that the following chain of inequalities holds*

$$c_1\Omega_r\left(f, \frac{1}{n}\right)_{p),\theta,w} \le \|f - T_n\|_{p),\theta,w} + n^{-2r}\|T_n^{(2r)}\|_{p),\theta,w} \le$$
$$\le c_2\Omega_r\left(f, \frac{1}{n}\right)_{p),\theta,w}.$$

We claim that the following inverse theorem for (α, φ) differentiable functions in weighted grand Lebesgue spaces is true.

Theorem 4. *Let $1 < p < \infty$, $\theta > 0$. Suppose that $\alpha \in \mathbb{R}$ and $f \in \widetilde{L}_w^{p),\theta}$. If $\varphi \in H_0$, $r \in (0, \infty)$ and*

$$\sum_{\nu=1}^{\infty} \left(\nu\varphi\left(\nu\right)\right)^{-1} E_\nu\left(f\right)_{p),\theta,w} < \infty,$$

then $f_{\alpha}^{\varphi} \in \widetilde{L}^{p),\theta}{}_{,w}$ *and the following inequality*

$$\Omega_r\Big(f_{\alpha}^{\varphi}, \frac{1}{n}\Big)_{p)\theta,w} \le c\bigg(\frac{1}{n^{2r}}\sum_{\nu=1}^{n}\nu^{2r-1}(\varphi(\nu))^{-1}E_{\nu}(f)_{p),\theta,w}+ \\ + \sum_{\nu=n+1}^{\infty}(\nu\varphi(\nu))^{-1}E_{\nu}(f)_{p),\theta,w}\bigg)$$

holds with a constant $c > 0$ *independent of* f *and* $n \in \mathbb{N}$.

Proofs of above-mentioned results are based on the following statements.

Theorem 5. *Let* $1 < p < \infty$, $\theta > 0$ *and* $w \in A_p$. *Then the conjugate operator* $f \longrightarrow \widetilde{f}$ *is bounded in* $L^{p),\theta}{}_{,w}(\mathbb{T})$.

Theorem 6 (Marcinkiewicz type multiplier theorem)**.** *Let* $1 < p < \infty$, $\theta > 0$ *and* $w \in A_p$. *Suppose that* $\{\lambda_n\}_{n=1}^{\infty}$ *is a sequence of numbers satisfying the following conditions: there exists a positive number* $M > 0$, *such that*

$$|\lambda_n| \le M \quad \textit{and} \quad \sum_{k=2^n}^{2^{n+1}}|\lambda_{k+1} - \lambda_k| \le M$$

for arbitrary $n \in \mathbb{N}$.

Suppose $f \in L_w^{p),\theta}$ *and*

$$f(x) \backsim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx)$$

then the trigonometric series

$$f(x) \backsim \frac{\lambda_0 a_0(f)}{2} + \sum_{k=1}^{\infty}\lambda_k(a_k(f)\cos kx + b_k(f)\sin kx)$$

is the Fourier trigonometric series of some function $F \in L_w^{p),\theta}$ *and the following inequality holds*

$$\|F\|_{p),\theta,w} \le cM\|f\|_{p),\theta,w}$$

with a constant $c > 0$ *independent of* f.

It should be noted that the statements analogous to the Theorems 1, 2 and 4 in weighted variable exponent Lebesgue spaces where obtained in the paper [4]. For a particular case of Theorem 4 we refer to the paper [5].

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