Proc. A. Razmadze Math. Inst. 164(2014), 100–103

N. Danelia and V. Kokilashvili

APPROXIMATION OF PERIODIC FUNCTIONS IN GRAND VARIABLE EXPONENT LEBESGUE SPACES

The goal of this talk is to discuss some approximation problems for 2π-periodic functions in new function spaces, introduced and studied recently by V. Kokilashvili and A. Meskhi [1]. These spaces unified two non-standard Banach function spaces, in particular, grand and variable exponent Lebesgue spaces. It is worth mentioning that the grand variable exponent Lebesgue spaces are non-reflexive, non-separable and non-rearrangement invariant.

Let $\mathbb{T} = [-\pi, \pi]$ and let $s(x)$ be continuous, 2π-periodic function defined on $\mathbb{R}$. We suppose that $s(x)$ satisfies the log-Hölder continuity condition i.e. there exists a positive constant A such that for all $x, y \in \mathbb{R}$, $|x - y| < \frac{1}{2}$, the inequality

$$|s(x) - s(y)| \leq \frac{A}{-\log|x - y|}$$

holds.

In the sequel we denote the class of 2π-periodic functions satisfying the log-Hölder continuity condition by $\mathcal{P}^{\log}$. Further, we say that $s \in \mathcal{P}$ if

$$1 < s_- \leq s_+ < \infty,$$

where

$$s_- = \inf_{\mathbb{T}} |s(x)|, \quad s_+ = \sup_{\mathbb{T}} |s(x)|.$$

Definition. Let $p \in \mathcal{P}$ and $\theta > 0$. By $L^{p(\cdot),\theta}(\mathbb{T})$ we denote the class of those 2π-periodic measurable functions for which

$$\|f\|_{p(\cdot),\theta} = \sup_{0<\varepsilon<p_- - 1} \varepsilon^{\frac{\theta}{p_- - \varepsilon}} \|f\|_{p(\cdot)-\varepsilon} < \infty$$

where

$$\|f\|_{s(\cdot)} = \inf_{\lambda>0} \left\{ \lambda : \int_{\mathbb{T}} \left| \frac{f(x)}{\lambda} \right|^{s(x)} dx \leq 1 \right\}.$$

2010 *Mathematics Subject Classification:* 26A33, 41A10, 41A25.

Key words and phrases. Grand variable exponent Lebesgue spaces, trigonometric approximation, moduli of smoothness, best approximation.

Let $f \in L^{p(\cdot),\theta}$ and let

$$A_h f(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} f(t)dt, \quad x \in \mathbb{T}.$$

For $r > 0$ we set

$$\sigma_h^r f(x) := (I - A_h)^r f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} (A_k)^r f(x).$$

For $f \in L^{p(\cdot),\theta}(\mathbb{T})$ and $r > 0$ the fractional moduli of smoothness is defined as

$$\Omega_r(f,\delta)_{p(\cdot),\theta} = \sup_{0<h_i,t\le\delta} \left\| \prod_{i=1}^{[r]} (I - A_{h_i}) \sigma_i^{\{r_i\}} \right\|_{p(\cdot),\theta}$$

where

$$\prod_{i=1}^{0} (I - A_{h_i}) \sigma_t^r f := \sigma_t^r$$

for $0 < r < 1$.

The closure of the space $L^{p(\cdot)}(\mathbb{T})$ by the norm of $L^{p(\cdot),\theta}(\mathbb{T})$, $\theta > 0$, does not coincide with the latter space. Let us denote this closure by $\widetilde{L}^{p(\cdot),\theta}(\mathbb{T})$. This subspace of $L^{p(\cdot),\theta}$ is a set of functions for which

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p_- - \varepsilon}} \|f\|_{p(\cdot)-\varepsilon} = 0.$$

For $f \in \widetilde{L}^{p(\cdot),\theta}$ by $E_n(f)_{p(\cdot),\theta}$ we denote the best trigonometric approximation:

$$E_n(f)_{p(\cdot),\theta} = \inf \|f - T\|_{p(\cdot),\theta}$$

where the infimum is taken over all trigonometric polynomials T of order not greater than n. For $f \in \widetilde{L}^{p(\cdot),\theta}$ we have

$$\lim_{n\to\infty} E_n(f)_{p(\cdot),\theta} = 0.$$

We announce that the following statements are valid.

Theorem 1. *Let $p \in \mathcal{P} \cap \mathcal{P}^{\log}$, $\theta > 0$ and $r > 0$. Then for $f \in \widetilde{L}^{p(\cdot),\theta}(\mathbb{T})$ the following inequality holds*

$$\Omega_r(f, \frac{1}{n})_{p(\cdot),\theta} \le \frac{c}{n^{2r}} \sum_{\nu=0}^{n} (\nu+1)^{2r-1} E_\nu(f)_{p(\cdot),\theta} \tag{1}$$

with a constant $c > 0$ independent of f and n.

Theorem 2. *$p \in \mathcal{P} \cap \mathcal{P}^{\log}$, $\theta > 0$. If for $f \in \widetilde{L}^{p(\cdot),\theta}$ and some natural k the series*

$$\sum_{\nu=1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot),\theta} \tag{2}$$

converges, then the function $f^{(k-1)}$ is absolutely continuous, $f^{(k)} \in \widetilde{L}^{p(\cdot),\theta}$ and the inequality

$$E_n(f^{(k)})_{p(\cdot),\theta} \le c\bigg(n^k E_n(f)_{p(\cdot),\theta} + \sum_{k=n+1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot),\theta} \bigg) \tag{3}$$

holds with a constant c independent of f.

Theorem 3. *Let $p \in \mathcal{P} \cap \mathcal{P}^{\log}$, $\theta > 0$. Then under the conditions of Theorem 2 we have*

$$\Omega_r(f^{(k)}, \frac{1}{n})_{p(\cdot),\theta} \le \bigg(\frac{c}{n^{2r}} \sum_{\nu=0}^{n} (\nu+1)^{2r+k-1} E_\nu(f)_{p(\cdot),\theta} +$$
$$+ \sum_{\nu=n+1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot),\theta}.$$

The proofs of above-mentioned results are based on the boundedness of conjugate operator in $L^{p(\cdot),\theta}(\mathcal{P})$, Bernstein type inequality (see [3], Proposition 3.1) and the following

Lemma. Let $p \in \mathcal{P} \cap \mathcal{P}^{\log}$ and $\theta > 0$. Then for $f \in L^{p(\cdot),\theta}(\mathbb{T})$ with the condition $f^{(k)} \in L^{p(\cdot),\theta}(\mathbb{T})$ the inequality

$$\Omega_r(f, \delta) \le c\delta^{2r} \|f^{(k)}\|_{p(\cdot),\theta}$$

holds with a constant $c > 0$ independent of f and δ.

For the analogous results in variable exponent Lebesgue spaces we refer e.g. to [4].

Acknowledgement

The work was supported by the grant of Shote Rustaveli Science Foundation, contract $D-13/23$.

References

1. V. Kokilashvili and A. Meskhi, Integral operators in grand variable exponent Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* **162** (2013), 144–150.
2. V. Kokilashvili and A. Meskhi, Maximal and Calderon-Zygmund operatores in grand variable exponent Lebesgue spaces. *Georgian Math. J. (to appear).*
3. V. Kokilashvili and A. Meskhi, On weighted Bernstein type inequality in grand variable exponent Lebesgue spaces *(submitted).*

4. R. Akgün and V. Kokilashvili, Some notes on trigonometric approximation (α, ψ)-differentiable functions in weighted variable exponent Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* **161** (2013), 15–23.

Authors' addresses:

N. Danelia
Department of Exact and Natural Sciences,
I. Javakhishvili Tbilisi State University
2, University st., Tbilisi 0186, Georgia

V. Kokilashvili
A. Razmadze Mathematical Institute,
I. Javakhishvili Tbilisi State University
6, Tamarashvili st., Tbilisi 0177, Georgia