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Mathematics

On the Approximation of Periodic Functions in Variable Exponent Lorentz Spaces

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ABSTRACT. The paper presents results on approximation by trigonometric polynomials in Lorentz spaces with variable exponents. The inequalities are obtained, which establish the connection between the best approximation by trigonometric polynomials and the generalized modulus of smoothness so that the exponents of space metrics are different on both sides of the inequalities. The analogues of Jackson's and inverse inequalities are proved in variable exponent Lorentz spaces. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: variable exponent Lorentz spaces, generalized moduli of smoothness, best approximation. Bernstein and Nikol'ski inequalities

Approximation problems of periodic functions in classical Lorentz spaces are explored in [1, 2]. In [3] the Lorentz spaces with variable exponents are introduced and boundedness of maximal, singular and potential operators are established. In present paper we study trigonometric approximation problems in variable exponent Lorentz spaces.

Let $T=[-\pi;\pi]$ and let on the entire axis a continuous $2\pi-$ periodic function $p:T\to(1,+\infty)$ be such that

$$1<p_{-}\le p(x)\le p_{+}<+\infty$$

and

$$|p(t)-p(0)|\le\frac{c}{|t|}\qquad \text{when}\qquad |t|\le\frac{1}{2}.$$

We denote by Q the set of such variable exponents.

The decreasing rearrangement of the function f is denoted by f^*

$$f^*(t)=\sup\left\{\tau\ge 0:\left|\left\{x\in T:|f(x)|>\tau\right\}\right|>t\right\}.$$

Let the functions p and r belong to the class Q. $L^{p(.),r(.)}$ denotes the set of those $2\pi-$periodic measurable functions f, for which

$$\|f\|_{L^{p(\cdot),r(\cdot)}} = \left\| t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f^*(t) \right\|_{L^{r(\cdot)}(T)} < +\infty.$$

We introduce the structural and constructive characteristics of the function f:

$$\Omega(f,\delta)_{L^{p(\cdot),r(\cdot)}} = \sup_{0<h\le\delta} \left\| \frac{1}{2h} \int_{x-h}^{x+h} f(t) - f(x) \right\|_{L^{p(\cdot),r(\cdot)}}, \quad 0<\delta<\pi$$

the generalized moduli of smoothness and the best approximation by trigonometric polynomials

$$E_n(f)_{L^{p(\cdot),r(\cdot)}} = \inf_{T_k} \|f - T_k\|_{L^{p(\cdot),r(\cdot)}},$$

where the infimum is taken with respect to all trigonometric polynomials, whose order does not exceed n.

Now we are ready to give the main results of this paper.

Theorem 1. Let the functions p and q belong to the class Q, then

$$\frac{1}{p(x)} - \frac{1}{q(x)} \equiv s, \ s>0, \ x\in T,$$

and $p_+ < \frac{1}{s}$.

If the condition

$$\sum_{\nu=1}^{\infty} \nu^{s-1} E_\nu(f)_{L^{p(\cdot)}} < +\infty$$

is fulfilled, then $f \in L^{q(\cdot),p(\cdot)}(T)$ and the inequality

$$E_n(f)_{L^{q(\cdot),p(\cdot)}} \le c\left(n^s E_n(f)_{L^{p(\cdot)}} + \sum_{\nu=n+1}^{\infty} \nu^{s-1} E_\nu(f)_{L^{p(\cdot)}} \right)$$

holds, where the constant c does not depend on f and n.

Theorem 2. In the conditions of Theorem 1 the inequality

$$\Omega\left(f,\frac{1}{n}\right)_{L^{q(\cdot),p(\cdot)}} \le c\left(\frac{1}{n^2} \sum_{\nu=0}^{n} (\nu+1)^{s+1} E_\nu(f)_{L^{p(\cdot)}} + \sum_{\nu=n+1}^{\infty} \nu^{s-1} E_\nu(f)_{L^{p(\cdot)}} \right)$$

is fulfilled, where c does not depend on f and n.

Corollary 1. Let $f \in L^{p(\cdot)}(T)$ and $p \in Q$. If

$$E_n(f)_{L^{p(\cdot)}} = O\left(\frac{1}{n^{2+s}}\right), \ n\in N,$$

then

$$\Omega\left(f,\frac{1}{n}\right)_{q(\cdot),p(\cdot)} = O\left(\frac{\ln n}{n^2}\right).$$

The proofs of these theorems are based on Bernstein-Zygmund and Nikol'ski type inequalities for trigonometric polynomials and on realization type theorem in variable exponent Lorentz spaces.

Theorem 3. Let the functions p and r belong to the class Q. Then the inequality

$$\left\|T_n'\right\|_{L^{p(.),r(.)}} \le cn\left\|T_n\right\|_{L^{p(.),r(.)}}$$

holds for each trigonometric polynomial T_n.

Theorem 4. Let the functions p and q satisfy the conditions of Theorem 1. Then for each trigonometric polynomial T_n the inequality

$$\left\|T_n\right\|_{L^{q(.),p(.)}} \le cn^s \left\|T_n\right\|_{L^{p(.)}}$$

holds, where the constant c does not depend on the polynomial T_n.

We introduce K_2 functional for the function $f \in L^{p(.),r(.)}$. Namely,

$$K_2\left(f,t;L^{p(.),r(.)},W''_{p(.),r(.)}\right) = \inf\left\{\left\|f-g\right\|_{L^{p(.),r(.)}} + t^2\left\|g''\right\|_{L^{p(.),r(.)}}\right\},$$

the infimum is taken over all $g \in W''_{q(.),p(.)}$, where

$$W''_{p(.),r(.)} = \left\{g \in L^{p(.),r(.)} : g'' \in L^{q(.),p(.)}\right\}.$$

Theorem 5. Assume that the functions $p(.)$ and $r(.)$ belong to the class Q. Then the following chain inequality is fulfilled:

$$c_1\Omega(f,t)_{L^{p(.),r(.)}} \le K_2\left(f,t;L^{p(.),r(.)},W''_{p(.),r(.)}\right) \le c_2\Omega(f,t)_{L^{p(.),r(.)}}$$

A more general assertion than Theorem 2 can be proved, when $p(x) = q(x)$. In particular, the following assertion is valid.

Theorem 6. Assume that the variable exponents p and r belong to the class Q. Then the estimate

$$\Omega\left(f,\frac{1}{n}\right)_{L^{p(.),r(.)}} \le c\left(\frac{1}{n^2}\sum_{\nu=0}^{n}(\nu+1)E_\nu(f)_{L^{p(.),r(.)}}\right)$$

holds for each function $f \in L^{p(.),r(.)}$.

This assertion gives rise to:

Corollary 2. Let p and r belong to the class Q. If for some $\alpha > 0$

$$E_n(f)_{L^{p(.),r(.)}} = O\left(\frac{1}{n^\alpha}\right),\ n \in N,$$

then

$$\Omega(f,\delta)_{L^{p(.),r(.)}} = \begin{cases} O\left(\delta^\alpha\right) & for\ \ \alpha < 2 \\ \delta^2 \log\dfrac{1}{\delta} & for\ \ \alpha = 2 \\ \delta^2 & for\ \ \alpha > 2 \end{cases}$$

Theorem 7 (an analogue of Jackson's first inequality). Assume that $p, r \in Q$. Then for each function $f \in L^{p(.),r(.)}$ we have the inequality

$$E_n(f)_{L^{p(\cdot),r(\cdot)}} \le c\Omega\left(f,\frac{1}{n}\right)_{L^{p(\cdot),r(\cdot)}},$$

where the constant c does not depend on n and f.

The proof of this theorem, except some inequalities obtained above, is based on the following lemma.

Lemma 1. Assume that $p,r\in Q$. Then for each function $f\in W^{(\alpha)}_{p(\cdot),r(\cdot)}$, $\alpha>0$, the inequality

$$E_n(f)_{L^{p(\cdot),r(\cdot)}} \le \frac{c}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{L^{p(\cdot),r(\cdot)}}$$

holds, where the constant c does not depend on n and f.

By virtue of Theorem 7 and Lemma 1 we conclude that the following assertion is valid.

Theorem 8 (an analogue of Jackson's second inequality). Assume that $p,r\in Q$ and $f\in W^{(\alpha)}_{p(\cdot),q(\cdot)}$, $\alpha>0$. Then we have

$$E_n(f)_{L^{p(\cdot),r(\cdot)}} \le \frac{c}{(n+1)^{\alpha}}\Omega\left(f^{(\alpha)},\frac{1}{n}\right)_{L^{p(\cdot),r(\cdot)}},$$

where the constant c does not depend on n and f.

We introduce into consideration the generalized Lipschitz classes $Lip\left(\alpha,L^{p(\cdot),r(\cdot)}\right)$ with the condition

$$\Omega(f,\delta)_{L^{p(\cdot),r(\cdot)}} = O\left(\delta^{\alpha}\right) \quad 0<\alpha<2.$$

From Corollary 2 and Theorem 7 we obtain the following constructive characteristic for the classes $Lip\left(\alpha,L^{p(\cdot),r(\cdot)}\right)$, $0<\alpha<2$.

Corollary 3. In order that $f\in Lip\left(\alpha,L^{p(\cdot),r(\cdot)}\right)$, $0<\alpha<2$, it is necessary and sufficient that

$$E_n(f)_{L^{p(\cdot),r(\cdot)}} = O\left(\frac{1}{n^{\alpha}}\right).$$

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