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Mathematics

On Trigonometric Approximation by Angle of Multivariable Functions in Weighted Variable Exponent Mixed-Norm Lebesgue Spaces

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ABSTRACT. The paper we present direct and inverse inequalities of trigonometric approximation by angle in variable exponent weighted mixed-norm Lebesgue spaces. *© 2019 Bull. Georg. Natl. Acad. Sci.*

Key words: approximation by angle, mixed-norm, variable exponent, Lebesgue spaces

Investigation of trigonometric approximation by angle of multivariable functions in classical $L^p\left(1<p<\infty\right)$ spaces was initiated by M. K. Potapov. His several papers deal with this problem [1-3]. Recently these results were extended to the weighted Lebesgue spaces with Muckenhoupt weights [4,5]. To the aforementioned problem in weighted mixed-norm Lebesgue spaces is devoted [6]. The present paper trigonometric approximation by angle is considered in more general setting, in the framework of weighted mixed-norm spaces.

Let $T=\left\{e^{i\varphi}:\ 0<\varphi\le 2\pi\right\}$ and $p:T\to\left(1,+\infty\right)$ be continuous on T, satisfying following two conditions:

$$1<p_-=\min_T p\left(x\right)\le\max_T p\left(x\right)=p_+<\infty,$$

there exists a constant $c>0$, such that for arbitrary $x,y\in T$

$$\left|p\left(x\right)-p\left(y\right)\right|\le\frac{c}{-\log\left|x-y\right|}.$$

The set of exponents satisfying conditions i) and ii) is denoted by $p^{\log}$.

Let w be weight function i. e. positive almost everywhere, integrable on T.

Definition 1. Weighted variable exponent Lebesgue space $L_w^{p(\cdot)}\left(T\right)$ is the set of all measurable functions $\varphi:T\to R^1$ for which the norm

$$\|\varphi\|_{p(\cdot),w(\cdot)} = \left\{\lambda > 0 \ \int_T \left|\frac{\varphi(x)w(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

is finite.

For the study of these spaces we refer the reader to the monographs [7-9].

Definition 2 [10]. For given variable exponent $p \in P^{\log}$ a weight $w \in A_{p(\cdot)}$, if

$$\sup |I|^{-1} \|w_I\|_{p(\cdot)} \cdot \|w^{-1}\chi_I\|_{p'(\cdot)} < \infty , \quad p' = \frac{p(\cdot)}{p(\cdot)-1},$$

where the supremum is taken over all intervals $I \subset R$.

From this definition it is clear that $w \in A_{p(\cdot)}$, then $w^{-1} \in A_{p'(\cdot)}$, $p' = \frac{p(\cdot)}{p(\cdot)-1}$.

Theorem A [11,12]**.** Let $p \in P^{\log}$ and $w \in A_{p(\cdot)}$. Then the Hardy-Littlewood maximal operator is bounded in $L^{p(\cdot)}_{w(\cdot)}$. Conversely, from the boundedness of maximal operator it follows that $p_- > 1$ and $w \in A_{p(\cdot)}$.

Definition 3 [12]**.** A pair $(p(\cdot), w(\cdot))$ is called M-pair, if the Hardy-Littlewood operator is bounded simultaneously in $L^{p(\cdot)}_{w}$ and $L^{p'(\cdot)}_{w^{-1}}$.

From Theorem 1 we infer that if $p \in P^{\log}$, then for arbitrary $w \in A_{p(\cdot)}$, the pair $(p(\cdot), w(\cdot))$ is M-pair.

Now we define weighted mixed-norm Lebesgue space with variable exponent. Later on we suppose that pairs $(p(\cdot), w(\cdot))$ $(i = 1,2)$ are M-pairs. We set $P = (p_1(\cdot), p_2(\cdot))$, $W = (w_1(\cdot), w_2(\cdot))$

Definition 4. The collection of all measurable functions $f : T^2 \to R^1$ is called as $L^P_W(T^2)$ space if the norm

$$\|f\|_{L^P_W} = \left\| \|f(x,\cdot)\|_{L^{p_2(\cdot)}_{w_2(\cdot)}} \right\|_{L^{p_1(\cdot)}_{w_1(\cdot)}}$$

is finite.

The space L^P_W is Banach function space. In the sequel by J we denote the rectangles with the sides parallel to the coordinate axis.

Theorem 1. Let $(p_i(\cdot), w_i(\cdot))$ be M-pairs. Then the strong maximal function

$$M^s f(x,y) = \sup_{J \ni (x,y)} \frac{1}{|J|} \int_J |f(t,s)| dtds$$

is bounded in L^P_W.

For unweighted case see [13].

For 2π-periodic (with respect to each variables) function from $L_W^P\left(T^2\right)$ via the Steklov means introduce the notion of moduli of smoothness:

$$\Omega\left(f;\delta_1,\delta_2\right)=\sup_{\substack{0<h\le\delta_1\\0<k\le\delta_2}}\left\|\frac{1}{hk}\int_{x-h}^{x+h}\int_{y-k}^{y+k}f\left(t,s\right)dtds-f\left(x,y\right)\right\|_{P,W}.$$

Define the constructive characteristics for $f\in L_W^P\left(T^2\right)$. By $P_{m,0}$ (respectively $P_{0,n}$) denote the set of all trigonometric polynomials of two variables of order not greater than m (not greater than n) with respect to x (with respect to y). By P_{mn} denote the set of all trigonometric polynomials of order not greater than m with respect to the variable x and of order not greater than n with respect to the variable y.

Partial best approximation orders are defined as:

$$E_{m,0}\left(f\right)_{P,W}=\inf\left\{\left\|f-T\right\|_{P,W}:\ T\in P_{m,0}\right\},$$

$$E_{0,n}\left(f\right)_{P,W}=\inf\left\{\left\|f-G\right\|_{P,W}:\ G\in P_{0,n}\right\}.$$

One of the important notions in our consideration is the best approximation by angle

$$E_{m,n}\left(f\right)_{P,W}=\inf\left\{\left\|f-T-G\right\|_{P,W}:\ T\in P_{m,0},\ \ G\in P_{0,n}\right\}.$$

The following statements are true:

Theorem 2. Let $\left(p\left(\cdot\right),w\left(\cdot\right)\right)$ $\left(i=1,2\right)$ be M-pairs. Then, for arbitrary $f\in L_W^P$ the following inequalities

$$E_{m,n}\left(f\right)_{P,W}\le c_1\Omega\left(f,\frac{1}{m},\frac{1}{n}\right)_{P,W}$$

and

$$\Omega\left(f,\frac{1}{m},\frac{1}{n}\right)_{P,W}\le\frac{c^2}{n^2}\sum_{i=0}^{m}\sum_{j=0}^{n}\left(i+1\right)\left(j+1\right)E_{i,j}\left(f\right)_{P,W}$$

hold.

Let us present some results, which play crucial role in the proof of Theorem 2.

In the sequel the derivatives are considered in Weil's sense.

Theorem 3 (Bernstein type inequalities). Let $\left(p_i\left(\cdot\right),w_i\left(\cdot\right)\right)$ be M-pairs. Assume $T_1\in P_{m,0}$, $T_2\in P_{0,n}$ and $T_3\in P_{m,n}$. Then

$$\left\|\frac{\partial^\alpha}{\partial x^\alpha}T_1\right\|_{P,W}\le c_1 m\left\|T_1\right\|_{P,W},\quad \left\|\frac{\partial^\beta}{\partial y^\beta}T_2\right\|_{P,W}\le c_2 n\left\|T_2\right\|_{P,W}$$

and

$$\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial y^\beta}T_3\right\|_{P,W}\le c_3 mn\left\|T_1\right\|_{P,W},$$

where the constants c_1, c_2 and c_3 $\left(i=1,2,3\right)$ are independent of polynomials.

Definition 5. Let $f \in L_W^P$, $(p(\cdot), w(\cdot))$ be M-pairs. The mixed K-functional is defined as

$$K(f,\delta,\varepsilon,p_1(\cdot),p_2(\cdot),w_1(\cdot),w_2(\cdot)) =$$

$$\inf_{h_1,h_2,h}\left\{\|f-h_1-h_2-h\|_{P,W}+\delta^2\left\|\frac{\partial^2 h_1}{\partial x^2}\right\|_{P,W}+\varepsilon^2\left\|\frac{\partial^2 h_2}{\partial y^2}\right\|_{P,W}+\delta^2\varepsilon^2\left\|\frac{\partial^4 h}{\partial x^2\partial y^2}\right\|_{P,W}\right\},$$

where the infimum is taken over the functions h_1, h_2 and h with conditions

$$\frac{\partial^2 h_1}{\partial x^2} \in L_W^P, \qquad \frac{\partial^2 h_2}{\partial y^2} \in L_W^P, \qquad \frac{\partial^4 h}{\partial x^2\partial y^2} \in L_W^P.$$

Theorem 4. Let $f \in L_W^P$, $(p_i(\cdot), w_i(\cdot))$ $(i=1,2)$ be M-pairs. Then the following two-sided estimate holds

$$b_1\Omega(f,\delta_1,\delta_2)_{P,W} \le K(f,\delta_1,\delta_2,p_1(\cdot),p_2(\cdot),w_1(\cdot),w_2(\cdot))_{P,W} \le b_2\Omega(f,\delta_1,\delta_2)_{P,W},$$

where the constants b_1 and b_2 are independent of f,δ_1 and δ_2.

Remark 1. Analogous results hold for the functions of n-variables, $n>2$ and mixed moduli of smoothness of higher and fractional orders.

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