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ANGULAR TRIGONOMETRIC APPROXIMATION IN THE FRAMEWORK OF NEW SCALE OF FUNCTION SPACES

V. KOKILASHVILI[1] AND TS. TSANAVA[1,2]

Abstract. In this note we announce our results on approximation by angle of functions of two variables in weighted Lebesgue spaces with mixed norms. The related problems of multidimensional Fourier Analysis are explored as well.

The investigation of trigonometric approximation by angle in classical Lebesgue spaces L^p $(1 < p < \infty)$ was initiated by M. K. Potapov. To this problem are devoted the series of his papers (see, for example, [6–8] and the survay paper [9]). Recently these results were extended to the L^p $(1 < p < \infty)$ spaces with Muckenhoupt weights [1, 2]. The main goal of present paper is to study the similar problem in weighted Lebesgue spaces with mixed norms.

1. In the sequel we denote by $\mathbb{T}^2$ the torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ is the circle $\{e^{i\varphi},\ \varphi \in [0, 2\pi]\}$. The function $v : \mathbb{T} \longrightarrow \mathbb{R}^1$ is called a weight if v is a measurable on $\mathbb{T}$, positive almost everywhere and integrable. For a Borel measure $e \subset \mathbb{T}$ we define the absolute continuous measure

$$ve = \int\limits_e v(x)dx.$$

A weight function v is said to be of Muckenhoupt class $A_p(\mathbb{T})$ if

$$\sup \bigg(\frac{1}{|I|}\int\limits_I v(x)dx\bigg)\bigg(\frac{1}{|I|}\int\limits_I v^{1-p'}(x)dx\bigg)^{p-1} < \infty, \quad p' = \frac{p}{p-1},$$

where the supremum is taken over all intervals $I \subset \mathbb{T}$.

In the sequel we consider the set of the measurable functions $f(x, y) : \mathbb{T}^2 \longrightarrow \mathbb{R}^1$ such that $f(x, y)$ is 2π-periodic with respect to each variable x and y.

Definition 1. Let $1 < p_i < \infty$ $(i = 1, 2)$, v and w be the weight functions defined on $\mathbb{T}$.

By $L^{p_1}_v\ (L^{p_2}_w)\ (\mathbb{T}^2)$, denote the set of measurable functions $f : \mathbb{T}^2 \longrightarrow \mathbb{R}$ for which the norm

$$\|f\|_{L^{p_1}_v(L^{p_2}_w)(\mathbb{T}^2)} = \Big\|\|f(x,\cdot)\|_{L^{p_2}_w}\Big\|_{L^{p_1}_v}$$

is finite.

The space $L^{p_1}_v(L^{p_2}_w)$ is a Banach function space. Introduction and the study of properties of this space was initiated in [3].

In the sequel always we assume that $v \in A_{p_1}(\mathbb{T})$ and $w \in A_{p_2}(\mathbb{T})$.

To avoid an inconvience of notation in this section we set $X := L^{p_1}_v(L^{p_2}_w)$.

The definition of the fractional modulus of smoothness in our case is similar as in [2], only the norm of Lebesgue spaces is changed by the norm in $L^{p_1}_v(L^{p_2}_w)(\mathbb{T}^2)$.

The mixed modulus of smoothness in $L^{p_1}_v(L^{p_2}_w)$ is denoted by $\Omega_r(f, \delta_1, \delta_2)$. We have

$$\Omega_r(f, \delta_1, \delta_2)_X \le c\|f\|_X$$

with a constant c nondepending on f, δ_1 and δ_2.

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By $P_{m,0}$ (respectively, $P_{0,n}$) is denoted the set of all trigonometric polynomial of degree m (at most n) with respect to the variable x (variable y). Also $P_{m,n}$ is defined as the set of all trigonometric polynomial of degree at most m with respect to variable x and of degree at most n with respect to variable y.

The best partial trigonometric approximation orders are defined as

$$E_{m,0}(f)_X = \inf\{\|f-T\|_X : T \in P_{m,0}\},$$

Analogously,

$$E_{0,m}(f)_X = \inf\{\|f-G\|_X : G \in P_{0,n}\}.$$

Then the best angular approximation order is defined by the equality

$$E_{m,n}(f)_X = \inf\{\|f-T-G\|_X : \mathbb{T} \in P_{m,0},\ G \in P_{0,n}\}.$$

The following assertions are true:

Theorem 1. *Let $1 < p_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Let $r > 0$. For $f \in X$ the following inequality*

$$E_{m,n}(f)_X \le c_1 \Omega_r\left(f, \frac{1}{m}, \frac{1}{n}\right)_X$$

holds with a constant c_1 nondepending on f, m and n.

Theorem 2. *Let $1 < p_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$, $f \in L_v^{p_1}(L_w^{p_2})(\mathbb{T}^2)$ and $r > 0$. Then*

$$\Omega_r\left(f, \frac{1}{m}, \frac{1}{n}\right)_X \le \frac{c}{m^{2r}n^{2r}} \sum_{i=0}^{m}\sum_{j=0}^{n} (i+1)^{2r-1}(j+1)^{2r-1}E_{ij}(f)_X.$$

In what follows we discuss some tools, contributing to the proving of aforementioned assertions.

Let $\sigma_{mn}^{\alpha,\beta}(f,x,y)$ ($\alpha > 0$, $\beta > 0$) be the Cesáro means of double Fourier trigonometric series of $f \in L_v^{p_1}(L_w^{p_2})(\mathbb{T}^2)$.

Theorem 3. *Let $1 < p_i, s_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Then*

$$\|\sigma_{mn}^{\alpha,\beta}(f)\|_X \le c\|f\|_X$$

and

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \|\sigma_{mn}^{\alpha,\beta}(f) - f\|_X = 0.$$

For the partial sums of double Fourier trigonometric series we have

$$\|S_{mn}(f)\|_X \le c\|f\|_X$$

with a constant c nondepending on $m, n \in \mathbb{N}$ and $f \in X$.

Further for $f \in X$

$$\lim_{n\to\infty} \|S_{n,n} - f\| = 0.$$

In the sequel under derivatives we assume the derivatives in Weyl's sense.

Theorem 4 (Bernstein type inequalities)**.** *Let $1 < p_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Assume that $\alpha, \beta > 0$.*

Let $T_1 \in P_{m,0}$, $T_2 \in P_{0,n}$ and $T_3 \in P_{mn}$. Then for α, β order Weyl's derivatives we have

$$\left\|\frac{\partial^\alpha}{\partial x^\alpha}T_1\right\|_X \le c_1 m^\alpha \|T_1\|_X$$

$$\left\|\frac{\partial^\beta}{\partial y^\beta}T_2\right\|_X \le c_2 n^\beta \|T_2\|_X$$

and

$$\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}T_3\right\|_X \le c_3 m^\alpha n^\beta \|T_3\|_X,$$

where the constants c_1, c_2 and c_3 are nondepending on m, n and polynomial.

For one and two weighted Bernstein inequalities in Lebesgue spaces we refer to [5], Chapter 6.

Definition 2. Let $f \in L_v^{p_1}(L_w^{p_2})(\mathbb{T}^2)$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Assume that $r > 0$. The mixed K-functional is defined as

$$K(f,\delta,\varepsilon,p_1,p_2,v,w,2r):$$

$$= \inf_{h_1 h_2, h} \left\{ \|f - h_1 - h_2 - h\|_X + \delta^{2r} \left\| \frac{\partial^{2r} h_1}{\partial x^{2r}} \right\|_X + \varepsilon^{2r} \left\| \frac{\partial^{2r} h_2}{\partial y^{2r}} \right\|_X + \delta^{2r}\varepsilon^{2r} \left\| \frac{\partial^{4r} h}{\partial x^{2r} \partial y^{2r}} \right\|_X \right\},$$

where the infimum is taken from all h_1, h_2, h such that $h_1 \in W_X^{2r,0}$, $h_2 \in W_X^{0,2r}$, $h \in W_X^{4r}$.

Here the following notations are used:

$$W_X^{2r,0} = \left\{ h_1 : \frac{\partial^{2r} h_1}{\partial x^{2r}} \in X \right\},$$

$$W_X^{0,2r} = \left\{ h_2 : \frac{\partial^{2r} h_2}{\partial y^{2r}} \in X \right\}$$

and

$$W_X^{4r} = \left\{ h : \frac{\partial^{4r} h}{\partial x^{2r} \partial y^{2r}} \in X \right\}.$$

The following statement is true

Theorem 5. *Let* $f \in L_v^{p_1}(L_w^{p_2})$, $1 < p_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. *Then the following equivalence*

$$\Omega_r(f,\delta_1,\delta_2)_X \approx K(f,\delta_1,\delta_2,p_1,p_2,v,w,2r)_X$$

holds with equivalence constants nondepending on f, δ_1 *and* δ_2.

It should be noted that mixed K-functionals were explored in [4] and [10]. This notion turn out to be very usefule in approximation and interpolation theory.

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[1]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

[2]Department of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi, Georgia
Email address: vakhtang.kokilashvili@tsu.ge
Email address: ts_tsanava75@yahoo.com