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CONTI–OPIAL TYPE EXISTENCE AND UNIQUENESS THEOREMS FOR NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS *

I. KIGURADZE † AND B. PŮŽA ‡

Dedicated to Lina Fazylovna Rakhmatullina and
Nikolaĭ Viktorovich Azbelev
on the occasion of their jubilees

Abstract. Conti–Opial type conditions are found for the solvability as well as for the unique solvability of the nonlinear singular boundary value problem

$$x^{(n)}(t) = (t-a)^{-\alpha}(b-t)^{-\beta} f(x)(t), \quad h_i(x) = 0 \ \ (i = 1, \dots, n).$$

Here α and $\beta \in [0, n-1]$, f is the operator (h_i $(i = 1, \dots, n)$ are the operators) acting from some subspace of the space of $(n-1)$-times continuously differentiable on the interval $]a, b[$ m-dimensional vector functions into the space of integrable on $[a, b]$ m-dimensional vector functions (into the space R^m).

Key Words. nonlinear singular boundary value problem, Conti–Opial type existence and uniqueness theorems

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§ 1. Formulation of the Main Results.

1.1. Statement of the problem and the main notation. Consider the functional differential equation

$$x^{(n)}(t) = (t-a)^{-\alpha}(b-t)^{-\beta} f(x)(t) \tag{1.1}$$

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† A. Razmadze Mathematical Institute of Georgian Academy of Sciences, 380093 Tbilisi, Georgia, E-mail: kig@rmi.acnet.ge

‡ Masaryk University, Faculty of Science, Department of Mathematical Analysis, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic, E-mail: puza@math.muni.cz

with the boundary conditions

$$h_i(x) = 0 \quad (i = 1, \dots, n), \tag{1.2}$$

where f is the operator (h_i $(i = 1, \dots, n)$ are the operators) acting from some subspace of the space of $(n-1)$-times continuously differentiable vector functions $x :]a, b[\to R^m$ into the space of integrable vector functions $y : [a, b] \to R^m$ (into the space R^m).

Problem (1.1), (1.2) is singular in the sense that for an arbitrary x the right-hand side of equation (1.1) may have nonintegrable singularities at the points a and b.

A survey of the literature on regular and singular boundary value problems of the type (1.1), (1.2) can be found respectively in [1] and [6]. In [6], a sufficient condition for singular problem (1.1), (1.2) to have the Fredholm property is found in the linear case, while in the nonlinear case the so-called principle of a priori boundedness is proved (see Theorem 1.0 given below).

In this paper, based on the above-mentioned principle, Conti–Opial type theorems are proved which are analogs of the Fredholm theorem for the nonlinear singular problem (1.1), (1.2). Using the same theorems, effective sufficient conditions for problem (1.1), (1.2) to be solvalble and uniquely solvable are derived when the boundary conditions (1.2) have the form

$$\begin{aligned} &\lim_{t \to a} x^{(i-1)}(t) = c_{1i} \quad (i = 1, \dots, n_1), \\ &\lim_{t \to b} x^{(i-1)}(t) = c_{2i} \quad (i = 1, \dots, n_2), \end{aligned} \tag{1.3}$$

where $c_{ki} \in R^m$ $(i = 1, \dots, n_k)$, $n_k \in \{1, \dots, n-1\}$ $(k = 1, 2)$ and

$$n_1 + n_2 = n.$$

Throughout the paper the following notations are used.

$R =]-\infty, +\infty[\,$, $R_+ = [0, +\infty[\,$.

R^m is the space of m-dimensional column vectors $x = (x_i)_{i=1}^m$ with the components $x_i \in R$ $(i = 1, \dots, m)$ and the norm

$$\|x\| = \sum_{i=1}^m |x_i|.$$

$R^m_+ = \{x = (x_i)_{i=1}^m : \; x_i \in R_+ \; (i = 1, \dots, m)\}$.

$R^{m \times m}$ is the space of $m \times m$ matrices $X = (x_{ik})_{i,k=1}^m$ with the components $x_{ik} \in R$ $(i, k = 1, \dots, m)$ and the norm

$$\|X\| = \sum_{i,k=1}^m |x_{ik}|.$$

If $x = (x_i)_{i=1}^m \in R^m$ and $X = (x_{ik})_{i,k=1}^m \in R^{m\times m}$, then

$$|x| = (|x_i|)_{i=1}^m \text{ and } |X| = (|x_{ik}|)_{i,k=1}^m.$$

$R_+^{m\times m} = \{X = (x_{ik})_{i,k=1}^m : x_{ik} \in R_+ \ (i,k = 1,\dots,m)\}$.

$r(X)$ is the spectral radius of the matrix $X \in R^{m\times m}$.

Inequalities between matrices and vectors are understood componentwise, i.e., for $x = (x_i)_{i=1}^m$, $y = (y_i)_{i=1}^m$, $X = (x_{ik})_{i,k=1}^m$ and $Y = (y_{ik})_{i,k=1}^m$ we have

$$x \le y \iff x_i \le y_i \ \ (i = 1,\dots,m)$$

and

$$X \le Y \iff x_{ik} \le y_{ik} \ \ (i,k = 1,\dots,m).$$

$C_{\alpha,\beta}^{n-1}(]a,b[\,;R^m)$ is the Banach space of $(n-1)$-times continuously differentiable vector functions $x :]a,b[\, \to R^m$ having limits *

$$\lim_{t\to a}(t-a)^{\alpha_i}x^{(i-1)}(t), \quad \lim_{t\to b}(b-t)^{\beta_i}x^{(i-1)}(t) \ \ (i = 1,\dots,n), \tag{1.4}$$

where

$$\alpha_i = \frac{\alpha+i-n+|\alpha+i-n|}{2}, \quad \beta_i = \frac{\beta+i-n+|\beta+i-n|}{2} \quad (i = 1,\dots,n). \tag{1.5}$$

The norm of an arbitrary element x of this space is defined by the equality

$$\|x\|_{C_{\alpha,\beta}^{n-1}} = \sup\left\{\sum_{k=1}^n (t-a)^{\alpha_i}(b-t)^{\beta_i}\|x^{(i-1)}(t)\| : \ a < t < b\right\}.$$

$\widetilde{C}_{\alpha,\beta}^{n-1}(]a,b[\,;R^m)$ is the space of $x \in C_{\alpha,\beta}^{n-1}(]a,b[\,;R^m)$ for which $x^{(n-1)}$ is locally absolutely continuous on $]a,b[$, i.e., absolutely continuous on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small positive ε.

$L([a,b];R^m)$ and $L([a,b];R^{m\times m})$ are respectively the Banach spaces of integrable vector functions $y : [a,b] \to R^m$ and integrable matrix functions $Y : [a,b] \to R^{m\times m}$ with the norms

$$\|y\|_L = \int_a^b \|y(t)\|\,dt, \quad \|Y\|_L = \int_a^b \|Y(t)\|\,dt.$$

* A vector function is said to be continuously differentiable, integrable, nondecreasing, etc., if its components are such.

$L([a,b];R^m_+) = \{y \in L([a,b];R^m) : \ y(t) \in R^m_+ \text{ for } t \in [a,b]\}$.

$L([a,b];R^{m\times m}_+) = \{Y \in L([a,b];R^{m\times m}) : \ Y(t) \in R^{m\times m}_+ \text{ for } t \in [a,b]\}$.

Each $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ is supposed to be defined on $[a,b]$ so that $x(a)$ $((x(b))$ is the right (left) limit of this vector function at the point a (at the point b).

In the sequel it will always be assumed that m and n are any natural numbers, $-\infty < a < b < +\infty$,

$$\alpha \in [0,n-1], \quad \beta \in [0,n-1], \tag{1.6}$$

whereas $f : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ and $h_i : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$ $(i = 1,\dots,n)$ are continuous operators.

By **a solution of the functional differential equation** (1.1) is understood a vector function $x \in \widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ satisfying (1.1) almost everywhere on $]a,b[$. A solution of (1.1) satisfying (1.2) is called **a solution of problem** (1.1), (1.2).

1.2. A priori boundedness principle. Following [6], we introduce

Definition 1.1. *The pair $(p,(\ell_i)^n_{i=1})$ of continuous operators $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ and $(\ell_i)^n_{i=1} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^{mn}$ is said to be* **consistent** *if:*

(i) *the operators $p(x,\cdot) : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ and $\ell_i(x,\cdot) : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$ are linear for any fixed $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ and $i \in \{1,\dots,n\}$;*

(ii) *for any x and $y \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ and for almost all $t \in [a,b]$ we have inequalities*

$$\|p(x,y)(t)\| \le \delta(t,\|x\|_{C^{n-1}_{\alpha,\beta}})\|y\|_{C^{n-1}_{\alpha,\beta}}, \quad \sum_{i=1}^n \|\ell_i(x,y)\| \le \delta_0(\|x\|_{C^{n-1}_{\alpha,\beta}})\|y\|_{C^{n-1}_{\alpha,\beta}},$$

where $\delta_0 : R_+ \to R_+$ is nondecreasing, $\delta(\cdot,\rho) \in L([a,b];R_+)$ for every $\rho \in R_+$, and $\delta(t,\cdot) : R_+ \to R_+$ is nondecreasing for almost all $t \in]a,b[$;

(iii) *there exist a positive number γ such that for any $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$, $q \in L([a,b];R^m)$ and $c_i \in R^m$ $(i = 1,\dots,n)$, an arbitrary solution y of the boundary value problem*

$$\begin{gathered} y^{(n)}(t) = (t-a)^{-\alpha}(b-t)^{-\beta}(p(x,y)(t) + q(t)), \\ \ell_i(x,y) = c_i \ \ (i = 1,\dots,n) \end{gathered} \tag{1.7}$$

admits the estimate

$$\|y\|_{C^{n-1}_{\alpha,\beta}} \le \gamma\Big(\sum_{i=1}^n \|c_i\| + \|q\|_L\Big). \tag{1.8}$$

In the paper [6] the following theorem is proved.

THEOREM 1.0. *Let the conditions*

$$\sup\Big\{\|f(x)(\cdot)\|: \ \|x\|_{C^{n-1}_{\alpha,\beta}} \le \rho\Big\} \le L([a,b];R_+), \tag{1.9}$$

$$\sup\Big\{\|h_i(x)(\cdot)\|: \ \|x\|_{C^{n-1}_{\alpha,\beta}} \le \rho\Big\} < +\infty \ \ (i=1,\dots,n) \tag{1.10}$$

hold for every $\rho \in R_+$. Moreover, let there exist a positive number ρ_0 and a consistent pair $(p,(\ell_i)_{i=1}^n)$ of continuous operators $p: C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ and $(\ell_i)_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^{mn}$ such that for any $\lambda \in]0,1[$ an arbitrary solution of the problem

$$x^{(n)}(t) = (t-a)^{-\alpha}(b-t)^{-\beta}((1-\lambda)p(x,x)(t) + \lambda f(x)(t)), \tag{1.11}$$

$$\ell_i(x,x) = \lambda(\ell_{ii}(x,x) - h_i(x)) \ \ (i=1,\dots,n) \tag{1.12}$$

admits the estimate

$$\|x\|_{C^{n-1}_{\alpha,\beta}} \le \rho_0. \tag{1.13}$$

Then problem (1.1), (1.2) *is solvable.*

1.3. Conti–Opial type theorems. Along with problem (1.1), (1.2) we will have to consider the vector differential inequality

$$|(t-a)^{\alpha}(b-t)^{\beta}y^{(n)}(t) - p_0(y)(t)| \le q_0(y)(t), \tag{1.14}$$

with the boundary conditions

$$|\ell_{0i}(y)| \le h_{0i}(y) \ \ (i=1,\dots,n), \tag{1.15}$$

where $p_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ and $\ell_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ $(i=1,\dots,n)$ are linear operators; $q_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b]\,;R^m_+)$ and $h_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m_+$ are positively homogeneous operators. A solution of problem (1.14), (1.15) will also be sought in the class $\widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$.

To formulate the main results of this paper we need

DEFINITION 1.2. *An operator $q_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ (an operator $h_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$) is called positively homogeneous if the equality*

$$q_0(\lambda x)(t) = \lambda q_0(x)(t) \quad (\,h_0(\lambda x) = \lambda h_0(x)\,)$$

is fulfilled for all $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$, $\lambda \in R_+$ *and almost all* $t \in]a,b[$.

DEFINITION 1.3. *A positively homogeneous operator* $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ *(a positively homogeneous operator* $\ell : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$*) is called strongly bounded (bounded) if there exists a function* $\zeta \in L([a,b];R_+)$ *(a positive number* ζ_0*) such that the inequality*

$$\|p(x)(t)\| \le \zeta(t)\|x\|_{C^{n-1}_{\alpha,\beta}} \quad \left(\|\ell(x)\| \le \zeta_0\|x\|_{C^{n-1}_{\alpha,\beta}}\right)$$

holds for all $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and almost all* $t \in]a,b[$.

DEFINITION 1.4. *Let* $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^{m_0}) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L_{\alpha,\beta}(]a,b[\,;R^m)$ *be continuous operators. We say that the pair* $(p_0,(\ell_{0i})_{i=1}^n)$ *of operators* $p_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L_{\alpha,\beta}(]a,b[\,;R^m)$, $(\ell_{0i})_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^{mn}$ *belongs to the set* $\mathcal{E}^{m_0}_{p;\ell_1,\dots,\ell_n}$ *if there exists a sequence* $x_k \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^{m_0})$ $(k=1,2,\dots)$ *such that the equalities*

$$\lim_{k\to\infty}\int_a^t p(x_k,y)(s)\,ds = \int_a^t p_0(y)(s)\,ds,$$
$$\lim_{k\to+\infty} \ell_i(x_k,y) = \ell_{0i}(y) \quad (i=1,\dots,n)$$

hold for every $y \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and* $t \in [a,b]$.

DEFINITION 1.5. *Let* $q_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m_+)$ *and* $h_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m_+$ $(i=1,\dots,n)$ *be any positively homogeneous operators. We say that the pair* $(p,(\ell_i)_{i=1}^n)$ *of continuous operators* $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^{m_0}) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$, $(\ell_i)_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^{m_0}) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$ *belong to the Opial class* $O^{m_0;\alpha,\beta}_{q_0;h_{01},\dots,h_{0m}}$ *if:*

(i) *the operators* $p(x,\cdot) : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$ *and* $\ell_i(x,\cdot) : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$ $(i=1,\dots,n)$ *are linear for any fixed* $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^{m_0})$;

(ii) *there exist* $\zeta \in L([a,b];R_+)$ *and* $\zeta_0 \in R_+$ *such that the inequalities*

$$\|p(x,y)(t)\| \le \zeta(t)\|y\|_{C^{n-1}_{\alpha,\beta}}, \quad \|\ell_i(x,y)\| \le \zeta_0\|y\|_{C^{n-1}_{\alpha,\beta}} \quad (i=1,\dots,n)$$

hold for any x *and* $y \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and for almost all* $t \in [a,b]$;

(iii) *for every* $(p_0,(\ell_{0i})_{i=1}^n) \in \mathcal{E}^{m_0}_{p;\ell_1,\dots,\ell_n}$ *problem* (1.14), (1.15) *has only a trivial solution.*

THEOREM 1.1. *Let there exist a positively homogeneous, continuous, strongly bounded operator* $q_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m_+)$, *positively homogeneous, continuous, bounded operators* $h_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m_+$ $(i=1,\dots,n)$, *and a pair of operators*

$$(p,(\ell_i)_{i=1}^n) \in O^{m;\alpha,\beta}_{q_0;h_{01},\dots,h_{0m}} \tag{1.16}$$

such that for any $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and almost all* $t \in [a,b]$ *the inequalities*

$$|f(x)(t) - p(x,x)(t)| \le q_0(x)(t) + q(t, \|x\|_{C^{n-1}_{\alpha,\beta}}), \tag{1.17}$$

$$|h_i(x) - \ell_i(x,x)| \le h_{0i}(x) + h(\|x\|_{C^{n-1}_{\alpha,\beta}}) \quad (i = 1,\dots,n) \tag{1.18}$$

hold, where $q : [a,b] \times R_+ \to R^m_+$ *is an integrable in the first argument and nondecreasing in the second argument vector function, and* $h : R_+ \to R^m_+$ *is a nondecreasing vector function. Let, moreover,*

$$\lim_{\rho \to +\infty} \left(\frac{\|h(\rho)\|}{\rho} + \frac{1}{\rho} \int_a^b \|q(s,\rho)\|\, ds \right) = 0. \tag{1.19}$$

Then problem (1.1), (1.2) *is solvable.*

In the case $n = 1$ and $\alpha = \beta = 0$, from this theorem follows Theorem 1.1 in [5] which itself is a generalization of the Conti theorem ([2], Theorem 2) and the Opial theorem ([7], Theorem 1).

COROLLARY 1.1. *Let there exist a strongly bounded linear operator* $p_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$, *bounded linear operators* $\ell_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$ $(i = 1,\dots,n)$, *positively homogeneous continuous, strongly bounded operator* $q_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m_+)$ *and positively homogeneous, continuous, bounded operators* $h_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m_+$ $(i = 1,\dots,n)$ *such that for any* $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and almost all* $t \in [a,b]$ *the inequalities*

$$|f(x)(t) - p_0(x)(t)| \le q_0(x)(t) + q(t, \|x\|_{C^{n-1}_{\alpha,\beta}}),$$
$$|h_i(x) - \ell_{0i}(x)| \le h_{0i}(x) + h(\|x\|_{C^{n-1}_{\alpha,\beta}})$$

hold, where $q : [a,b] \times R_+ \to R^m_+$ *is an integrable in the first argument and nondecreasing in the second argument vector function, and* $h : R_+ \to R^m_+$ *is a nondecreasing vector function. If, moreover, problem* (1.14), (1.15) *has only a trivial solution and condition* (1.19) *is fulfilled, then problem* (1.1) (1.2) *is solvable.*

THEOREM 1.2. *Let there exist a positively homogeneous, continuous, strongly bounded operator* $q_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m_+)$, *positively homogeneous, continuous, bounded operators* $h_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m_+$ $(i = 1,\dots,n)$, *and a pair of operators*

$$(p, (\ell_i)_{i=1}^n) \in O^{2m;\alpha,\beta}_{q_0;h_{01},\dots,h_{0m}} \tag{1.20}$$

such that for any $x, \overline{x} \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and for almost all* $t \in [a,b]$ *the inequalities*

$$|f(x)(t) - f(\overline{x})(t) - p(x,\overline{x},x-\overline{x})(t)| \le q_0(x-\overline{x})(t), \tag{1.21}$$

$$|h_i(x) - h_i(\overline{x}) - \ell_i(x,\overline{x},x-\overline{x})| \le h_{0i}(x-\overline{x}) \quad (i=1,\dots,n) \tag{1.22}$$

hold. Then problem (1.1), (1.2) *is uniquely solvable.*

COROLLARY 1.2. *Let there exist a strongly bounded linear operator* $p_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m)$, *bounded linear operators* $\ell_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$ $(i=1,\dots,n)$, *positively homogeneous continuous, strongly bounded operator* $q_0 : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to L([a,b];R^m_+)$ *and positively homogeneous, continuous, bounded operators* $h_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m_+$ $(i=1,\dots,n)$ *such that for any* $x, \overline{x} \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and for almost all* $t \in [a,b]$ *the inequalities*

$$|f(x)(t) - f(\overline{x})(t) - p_0(x-\overline{x})(t)| \le q_0(x-\overline{x})(t),$$
$$|h_i(x) - h_i(\overline{x}) - \ell_{0i}(x-\overline{x})| \le h_{0i}(x-\overline{x}) \quad (i=1,\dots,n)$$

hold. If, moreover, problem (1.14), (1.15) *has only a trivial solution, then problem* (1.1), (1.2) *is uniquely solvable.*

1.4. Theorems on the solvability and unique solvability of problem (1.1), (1.3). In this subsection, besides of the above-introduced, we will use also the following notation:

$$n_0 = \min\{n_1, n_2\}, \quad u_{\alpha,\beta}(t) = (t-a)^{n-1-\alpha}(b-t)^{n-1-\beta}.$$

For an arbitrary linear operator $p : C([a,b];R^m) \to L([a,b];R^m)$, by $p(u_{\alpha,\beta}E_m)$ it will be understood a matrix function satisfying the equality

$$p(u_{\alpha,\beta}c)(t) = p(u_{\alpha,\beta}E_m)(t)c \ \text{ for } \ t \in [a,b], \quad c \in R^m. \tag{1.23}$$

DEFINITION 1.6. *A linear operator* $p : C([a,b];R^m) \to L([a,b];R^m)$ *is called* **positive** *if*

$$p(x) \in L([a,b];R^m_+) \ \text{ for } \ x \in C([a,b];R^m_+).$$

THEOREM 1.3. *Let* $\alpha \in [n_2-1, n_2]$, $\beta \in [n_1-1, n_1]$ *and let there exist a positive linear operator* $p : C([a,b];R^m) \to L([a,b];R^m)$ *such that for any* $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ *and for almost all* $t \in [a,b]$ *the inequality*

$$|f(x)(t)| \le p(|x|)(t) + q(t, \|x\|_{C^{n-1}_{\alpha,\beta}})$$

holds, where $q : [a,b] \times R_+ \to R_+^m$ *is an integrable in the first argument and nondecreasing in the second argument vector function such that*

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \|q(s,\rho)\| \, ds = 0.$$

If, moreover,

$$r\Big(\int_a^b p(u_{\alpha,\beta} E_m)(s)\, ds \Big) < n_0 (n_1 - 1)!(n_2 - 1)!(b-a)^{n-1}, \tag{1.24}$$

then problem (1.1), (1.3) *is solvable.*

THEOREM 1.4. *Let* $\alpha \in [n_2 - 1, n_2]$, $\beta \in [n_1 - 1, n_1]$ *and let there exist a positive linear operator* $p : C([a,b]; R^m) \to L([a,b]; R^m)$ *such that for any* $x, y \in C_{\alpha,\beta}^{n-1}(]a,b[\,; R^m)$ *and for almost all* $t \in [a,b]$ *the inequality*

$$|f(x)(t) - f(y)(t)| \le p(|x-y|)(t)$$

holds. If, moreover, condition (1.24) *is fulfilled, then problem* (1.1), (1.3) *is uniquely solvable.*

As an example, we consider the differential equation with a deviating argument

$$u^{(n)}(t) = (t-a)^{-\alpha}(b-t)^{-\beta} f_0(t, u(\tau(t))), \tag{1.25}$$

where $f_0 : [a,b] \times R^m \to R^m$ is a vector function from the Carathéodory class, and $\tau : [a,b] \to [a,b]$ is a measurable function.

Theorems 1.4 and 1.5 imply, respectively, the following statements.

COROLLARY 1.3. *Let* $\alpha \in [n_2 - 1, n_2]$, $\beta \in [n_1 - 1, n_1]$ *and let there exist a matrix and a vector functions* $\mathcal{P} \in L([a,b]; R_+^{m\times m})$ *and* $q \in L([a,b]; R_+^m)$ *such that the inequality*

$$|f_0(t,x)| \le \mathcal{P}(t)|x| + q(t)$$

holds in $[a,b] \times R^m$. *If, moreover,*

$$r\Big(\int_a^b (\tau(s) - a)^{n-1-\alpha} (b - \tau(s))^{n-1-\beta} \mathcal{P}(s)\, ds \Big) < \\ < n_0 (n_1 - 1)!(n_2 - 1)!(b-a)^{n-1}, \tag{1.26}$$

then problem (1.25), (1.3) *is solvable.*

COROLLARY 1.4. *Let* $\alpha \in [n_2 - 1, n_2]$, $\beta \in [n_1 - 1, n_1]$ *and let there exist a matrix function* $\mathcal{P} \in L([a,b]; R_+^{m\times m})$ *such that for any* $x, y \in R^m$ *and* $t \in]a,b[$, *the inequality*

$$|f_0(t,x) - f_0(t,y)| \le \mathcal{P}(t)|x-y|$$

holds. If, moreover, condition (1.26) *is fulfilled, then problem* (1.25), (1.3) *is uniquely solvable.*

§ 2. Auxiliary Propositions.

2.1. Lemma on sequences of elements from the set $\mathcal{E}^{m_0}_{p;\ell_1,\dots,\ell_n}$.

LEMMA 2.1. *Let conditions* (i) *and* (ii) *of Definition* 1.5 *be fulfilled and let*

$$(p_k, (\ell_{ki})_{i=1}^n) \in \mathcal{E}^{m_0}_{p;\ell_1,\dots,\ell_n} \quad (k = 1, 2, \dots). \tag{2.1}$$

Then there exist $(p_0, (\ell_{0i})_{i=1}^n) \in \mathcal{E}^{m_0}_{p;\ell_1,\dots,\ell_n}$ and a subsequence $(p_{kk}, (\ell_{kki})_{i=1}^n)$ $(k = 1, 2, \dots)$ of the sequence $(p_k, (\ell_{ki})_{i=1}^n)$ $(k = 1, 2, \dots)$ such that for any $y \in C^{n-1}_{\alpha,\beta}(]a, b[\,; R^m)$, we have

$$\lim_{k\to+\infty} \int_a^t (p_{kk}(y)(\xi) - p_0(y)(\xi))\, d\xi = 0 \ \ \textit{uniformly on} \ \ [a, b], \tag{2.2}$$

$$\lim_{k\to+\infty} \ell_{kki}(y) = \ell_{0i}(y) \quad (i = 1, \dots, n). \tag{2.3}$$

Proof. By virtue of condition (ii) of Definition 1.5 and condition (2.1), there exist $\zeta \in L([a, b]; R_+)$ and $\zeta_0 \in R_+$ such that for any $y \in C^{n-1}_{\alpha,\beta}(]a, b[\,; R^m)$ the inequalities

$$\|p_k(y)(t)\| \le \zeta(t) \|y\|_{C^{n-1}_{\alpha,\beta}} \quad (k = 1, 2, \dots), \tag{2.4}$$

$$\sum_{i=1}^n \|\ell_{ki}(y)\| \le \zeta_0 \|y\|_{C^{n-1}_{\alpha,\beta}} \quad (i = 1, \dots, n; \ \ k = 1, 2, \dots) \tag{2.5}$$

are fulfilled.

Let

$$z_k(y)(t) = \frac{1}{(n-1)!} \int_a^t (t - \xi)^{n-1} p_k(y)(\xi)\, d\xi \quad (k = 1, 2, \dots).$$

In view of conditions (2.4) and (2.5), $z_k : C^{n-1}_{\alpha,\beta}(]a, b[\,; R^m) \to C^{n-1}([a, b]; R^m)$ and $\ell_{ki} : C^{n-1}_{\alpha,\beta}(]a, b[\,; R^m) \to R^m$ $(i = 1, \dots, n)$ are linear bounded operators for every natural k.

Let $\{y_1, y_2, \dots\}$ be a set, everywhere dense in $C^{n-1}_{\alpha,\beta}(]a, b[\,; R^m)$. According to the Arzellá–Ascoli lemma and conditions (2.4) and (2.5), from the sequence $((z_k, (\ell_{ki})_{i=1}^n))_{k=1}^{+\infty}$ we can choose a subsequence $((z_{1k}, (\ell_{1ki})_{i=1}^n))_{k=1}^{+\infty}$ such that the sequences $(z_{1k}(y_1))_{k=1}^{+\infty}$ and $(\ell_{1ki}(y_1))_{k=1}^{+\infty}$ $(i = 1, \dots, n)$ would be converging by the norms of the spaces $C^{n-1}([a, b]; R^m)$ and R^m, respectively.

Similarly, from the sequence $((z_{1k}, (\ell_{1ki})_{i=1}^n))_{k=1}^{+\infty}$ we can choose a subsequence $((z_{2k}, (\ell_{2ki})_{i=1}^n))_{k=1}^{+\infty}$ such that the sequences $(z_{2k}(y_2))_{k=1}^{+\infty}$ and $(\ell_{2ki}(y_2))_{k=1}^{+\infty}$ $(i = 1, \dots, n)$ would be converging. If we continue this process endlessly, then we get the system of sequences $((z_{jk}, (\ell_{jki})_{i=1}^n))_{k=1}^{+\infty}$ $(j = 1, 2, \dots)$ such that for any natural j_1 and $j_2 \geq j_1$, $((z_{j_2k}, (\ell_{j_2ki})_{i=1}^n))_{k=1}^{+\infty}$ is a subsequence of the sequence $((z_{j_1k}, (\ell_{j_1ki})_{i=1}^n))_{k=1}^{+\infty}$, and sequences $(z_{j_2k}(y_{j_2}))_{k=1}^{+\infty}$ and $(\ell_{j_2ki}(y_{j_2}))_{k=1}^{+\infty}$ $(i = 1, \dots, n)$ are converging. Consider the sequence $((z_{kk}, (\ell_{kki})_{i=1}^n))_{k=1}^{+\infty}$. It is obvious that for every natural j sequences $(z_{kk}(y_j))_{k=1}^{+\infty}$ and $(\ell_{kki}(y_j))_{k=1}^{+\infty}$ $(i = 1, \dots, n)$ are converging. Thus, by virtue of the Banach–Steinhaus theorem ([4], Chapter VII, § 1, Theorem 3), there exist linear bounded operators $z_0 : C_{\alpha,\beta}^{n-1}(]a, b[\,; R^m) \to C^{n-1}([a, b]; R^m)$ and $\ell_{0i} : C_{\alpha,\beta}^{n-1}(]a, b[\,; R^m) \to R^m$ $(i = 1, \dots, n)$ such that for every $y \in C_{\alpha,\beta}^{n-1}(]a, b[\,; R^m)$ equalities (2.3) and

$$\lim_{k \to +\infty} \|z_{kk}(y) - z_0(y)\|_{C_{\alpha,\beta}^{n-1}} = 0 \tag{2.6}$$

are satisfied. On the basis of the last equality and condition (2.4), we find

$$z_{kk}^{(n-1)}(y)(a) = 0, \quad \|z_{kk}^{(n-1)}(y)(t) - z_{kk}^{(n-1)}(y)(s)\| = \left\| \int_s^t p_{kk}(y)(\xi)\, d\xi \right\| \leq$$
$$\leq \|y\|_{C_{\alpha,\beta}^{n-1}} \int_s^t \zeta(\xi)\, d\xi \ \text{ for } \ a < s < t < b$$

and

$$z_0^{(n-1)}(a) = 0, \quad \|z_0^{(n-1)}(y)(t) - z_0^{(n-1)}(y)(s)\| \leq \|y\|_{C_{\alpha,\beta}^{n-1}} \int_s^t \zeta(\xi)\, d\xi$$
$$\text{for } \ a \leq s < t \leq b.$$

Hence it is clear that $z_0^{(n-1)}(y) : [a, b] \to R^m$ is absolutely continuous and

$$z_0^{(n-1)}(y)(t) = \int_a^t p_0(y)(s)\, ds, \tag{2.7}$$

where $p_0(y)(t) = z_0^{(n)}(y)(t)$ and

$$\|p_0(y)(t)\| \leq \zeta(t)\|y\|_{C_{\alpha,\beta}^{n-1}} \ \text{ for almost all } \ t \in [a, b].$$

(2.6) and (2.7) result in (2.2). On the other hand, due to conditions (2.1)–(2.3) and Definition 1.4, it is evident that $(p_0, (\ell_{0_i})_{i=1}^n) \in \mathcal{E}_{p;\ell_1,\dots,\ell_n}^{m_0}$. □

2.2. Lemma on an a priori estimate.

LEMMA 2.2. *Let $q_0 : C_{\alpha,\beta}^{n-1}(]a, b[\,; R^m) \to L([a, b]; R_+^m)$ be a positively homogeneous, continuous, strongly bounded operator, $h_{0i} : C_{\alpha,\beta}^{n-1}(]a, b[\,; R^m) \to$*

R_+^m $(i = 1,\dots,n)$ be positively homogeneous, continuous, bounded operators, and $(p,(\ell_i)_{i=1}^n)$ be a pair of operators satisfying condition (1.16). *Then there exists a positive number ρ^* such that no matter how are $(p_0,(\ell_{0i})_{i=1}^n) \in \mathcal{E}^m_{p;\ell_1,\dots,\ell_n}$, $q_1 \in L([a,b];R_+^m)$ and $h_1 \in R_+^m$, an arbitrary solution x of the problem*

$$(2.8)\qquad |(t-a)^\alpha(b-t)^\beta x^{(n)}(t) - p_0(x)(t)| \le q_0(x)(t) + q_1(t),$$

$$(2.9)\qquad |\ell_{0i}(x)| \le h_{0i}(x) + h_1 \quad (i = 1,\dots,n)$$

admits the estimate

$$(2.10)\qquad \|x\|_{C^{n-1}_{\alpha,\beta}} \le \rho^*(\|h_1\| + \|q_1\|_L).$$

Proof. We assume the contrary that the lemma is not true. Then for any natural k, there exist

$$(p_k,(\ell_{ki})_{i=1}^n) \in \mathcal{E}^m_{p;\ell_1,\dots,\ell_n},\quad q_{1k} \in L([a,b];R_+^m),\quad h_{1k} \in R_+^m$$

and a solution x_k of the problem

$$(2.11)\qquad |(t-a)^\alpha(b-t)^\beta x_k^{(n)}(t) - p_k(x_k)(t)| \le q_0(x_k)(t) + q_{1k}(t),$$

$$(2.12)\qquad |\ell_{ki}(x_k)| \le h_{0i}(x_k) + h_{1k} \quad (i = 1,\dots,n)$$

such that

$$(2.13)\qquad \|x_k\|_{C^{n-1}_{\alpha,\beta}} > k(\|h_{1k}\| + \|q_{1k}\|_L).$$

In view of condition (ii) of Definition 1.5, condition (2.1) and the strongly boundedness of the operator q_0, there exist $\zeta \in L([a,b];R_+)$ and $\zeta_0 \in R_+$ such that for any $y \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$, along with (2.4) and (2.5), the inequality

$$(2.14)\qquad \|q_0(y)(t)\| \le \zeta(t)\|y\|_{C^{n-1}_{\alpha,\beta}}$$

is fulfilled.

By Lemma 2.1, without loss of generality we can assume that for any $y \in C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ the equalities

$$(2.15)\qquad \lim_{k\to+\infty}\int_a^t p_k(y)(\xi)\,d\xi = \int_a^t p_0(y)(\xi)\,d\xi \ \text{ uniformly on } [a,b],$$

$$(2.16)\qquad \lim_{k\to+\infty}\ell_{ki}(y) = \ell_{0i}(y) \quad (i = 1,\dots,n)$$

are satisfied, where

$$(p_0, (\ell_{0i})_{i=1}^n) \in \mathcal{E}^m_{p;\ell_1,\dots,\ell_n}. \tag{2.17}$$

Suppose

$$y_k(t) = \|x_k\|^{-1}_{C^{n-1}_{\alpha,\beta}} x_k(t),$$

$$\widetilde{q}_{1k}(t) = \|x_k\|^{-1}_{C^{n-1}_{\alpha,\beta}} q_{1k}(t), \quad \widetilde{h}_{1k} = \|x_k\|^{-1}_{C^{n-1}_{\alpha,\beta}} h_{1k}.$$

Then

$$\|y_k\|_{C^{n-1}_{\alpha,\beta}} = 1 \ \ (k = 1, 2, \dots). \tag{2.18}$$

On the other hand, in view of inequalities (2.4) and (2.11)–(2.14), we have

$$|(t-a)^\alpha (b-t)^\beta y_k^{(n)}(t) - p_k(y_k)(t)| \le q_0(y_k)(t) + \widetilde{q}_{1k}(t), \tag{2.19}$$

$$|\ell_{ki}(y_k)| \le h_{0i}(y_k) + \widetilde{h}_{1k} \ \ (i = 1, \dots, n), \tag{2.20}$$

$$\|\widetilde{q}_{1k}\|_L < \frac{1}{k}, \tag{2.21}$$

$$\|\widetilde{h}_{1k}\| < \frac{1}{k}, \tag{2.22}$$

and

$$\|y_k^{(n)}(t)\| \le (t-a)^{-\alpha}(b-t)^{-\beta}(2\zeta(t) + \|q_{1k}(t)\|). \tag{2.23}$$

By Lemma 2.1 from [6] and conditions (1.6), (2.18), (2.21) and (2.23), without loss of generality the sequence $(y_k)_{k=1}^{+\infty}$ can be assumed to be converging by the norm of the space $C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$ to some vector function $y \in \widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[\,;R^m)$.

(2.18)–(2.20) imply

$$\|y\|_{C^{n-1}_{\alpha,\beta}} = 1, \tag{2.24}$$

$$\left| y_k^{(n-1)}(t) - y_k^{(n-1)}(s) - \int_s^t (\xi-a)^{-\alpha}(b-\xi)^{-\beta} p_k(y)(\xi)\, d\xi \right| \le \tag{2.25}$$

$$\le \int_s^t (\xi-a)^{-\alpha}(b-\xi)^{-\beta}(q_0(y_k)(\xi) + |p_k(y-y_k)(\xi)| + \widetilde{q}_{1k}(\xi))\, d\xi$$

$$\text{for } a < s < t < b,$$

$$|\ell_{ki}(y)| \le h_{0i}(y_k) + |\ell_{ki}(y-y_k)| + \widetilde{h}_{1k} \ \ (i = 1, \dots, n). \tag{2.26}$$

On the other hand, due to (2.15), it is evident that

$$\lim_{k\to+\infty}\int_s^t(\xi-a)^{-\alpha}(b-\xi)^{-\beta}p_k(y)(\xi)\,d\xi=\int_s^t(\xi-a)^{-\alpha}(b-\xi)^{-\beta}p_0(y)(\xi)\,d\xi \quad \text{for } a<s<t<b.$$

If together with this condition we take into account conditions (2.4) and (2.21) and pass to the limit in inequality (2.25) as $k\to+\infty$, then we get

$$\left|y^{(n-1)}(t)-y^{(n-1)}(s)-\int_s^t(\xi-a)^{-\alpha}(b-\xi)^{-\beta}p_0(y)(\xi)\,d\xi\right|\le$$
$$\le\int_s^t(\xi-a)^{-\alpha}(b-\xi)^{-\beta}q_0(y)(\xi)\,d\xi \quad \text{for } a<s<t<b.$$

Hence it immediately follows that y is a solution of the differential inequality (1.14).

If now we pass to the limit in inequalities (2.26) as $k\to+\infty$, then in view of conditions (2.5) and (2.22) we get inequality (1.15). Therefore y is a solution of problem (1.14), (1.15). Thus $y(t)\equiv 0$, since by conditions (1.16) and (2.17) the above-mentioned problem has only a trivial solution. But this contradicts (2.23). The contradiction obtained proves the lemma. □

2.3. Lemma on the estimate of the Green function of two-point boundary value problem. We consider the boundary value problem

$$u^{(n)}(t)=0, \tag{2.27}$$
$$u^{(i-1)}(a)=0 \quad (i=1,\dots,n_1), \qquad u^{(i-1)}(b)=0 \quad (i=1,\dots,n_2), \tag{2.28}$$

where $n\ge 2$, $n_1\in\{1,\dots,n-1\}$ and $n_2=n-n_1$. Suppose

$$n_0=\min\{n_1,n_2\}, \quad u_{\alpha,\beta}(t)=(t-a)^{n-1-\alpha}(b-t)^{n-1-\beta}.$$

LEMMA 2.3. *Let $\alpha\in[n_2-1,n_2]$ and $\beta\in[n_1-1,n_1]$. Then the Green function g of problem* (2.27), (2.28) *admits the estimate*

$$0\le(-1)^{n_2}g(t,s)\le \tag{2.29}$$
$$\le\frac{(b-a)^{1-n}}{n_0(n_1-1)!(n_2-1)!}u_{\alpha,\beta}(t)(s-a)^{\alpha}(b-s)^{\beta} \quad \textit{for } a\le s,t\le b.$$

Proof. By V. V. Ostroumov [8] (see also Lemma 9.6 in the monograph [3] by U. Elias) is shown that

$$0\le(-1)^{n_2}g(t,s)\le \tag{2.30}$$
$$\le\frac{(b-a)^{1-n}}{n_0(n_1-1)!(n_2-1)!}(t-a)^{n_1-1}(b-t)^{n_2-1}(s-a)^{n_2-1}(b-s)^{n_1-1}g_0(t,s),$$

where

$$g_0(t,s) = \min\{(t-a)(b-s),(s-a)(b-t)\}.$$

Let $\alpha_1 = \alpha - n_2 + 1$, $\beta_1 = \beta - n_1 + 1$. Then for $s \le t$ we have

$$g_0(t,s) = (s-a)(b-t) \le (t-a)^{1-\alpha_1}(b-t)^{1-\beta_1}(s-a)^{\alpha_1}(b-s)^{\beta_1}.$$

If $t \le s$, then

$$g_0(t,s) = (t-a)(b-s) \le (t-a)^{1-\alpha_1}(b-t)^{1-\beta_1}(s-a)^{\alpha_1}(b-s)^{\beta_1}.$$

Thus estimate (2.30) results in estimate (2.29). □

§ 3. Proof of the Main Results.

Proof of Theorem 1.1. First we note that conditions (1.16)–(1.18) yield conditions (1.9) and (1.10). On the other hand, by Lemma 2.2 there exists a positive number γ such that for any $x \in C^{n-1}_{\alpha,\beta}(]a,b[;R^m)$, $q \in L([a,b];R^m)$ and $c_i \in R^m$ $(i=1,\dots,n)$, an arbitrary solution y of the boundary value problem (1.7) admits estimate (1.8). Therefore the pair $(p,(\ell_i)_{i=1}^n)$ is consistent.

By virtue of Theorem 1.0, it is sufficient to show the existence of a positive number ρ_0 such that for any $\lambda \in]0,1[$ an arbitrary solution of problem (1.11), (1.12) would admit estimate (1.13).

Let ρ^* be the number appearing in Lemma 2.2. Due to condition (1.19), there exists $\rho_0 > 0$ such that

$$\rho^*\left(\|h(\rho)\| + \int_a^b \|q(s,\rho)\|\,ds\right) < \rho \quad \text{for} \quad \rho > \rho_0. \tag{3.1}$$

Let x be a solution of problem (1.11), (1.12) for some $\lambda \in]0,1[$. Set

$$\rho_x = \|x\|_{C^{n-1}_{\alpha,\beta}}. \tag{3.2}$$

Then, in view of conditions (1.17) and (1.18), we have

$$|(t-a)^\alpha(b-t)^\beta x^{(n)}(t) - p(x,x)(t)| = \lambda|f(x)(t) - p(x,x)(t)| \le$$
$$\le q_0(x)(t) + q(t,\rho_x)$$

and

$$|\ell_i(x,x)| = \lambda|h_i(x) - \ell_i(x,x)| \le h_{0i}(x) + h(\rho_x) \quad (i=1,\dots,n).$$

Therefore, x is a solution of problem (2.8), (2.9), where

$$p_0(y)(t) = p(x,y)(t), \quad \ell_{0i}(y) = \ell_i(x,y) \text{ for any } y \in C^{n-1}_{\alpha,\beta}(]a,b[;R^m),$$
$$q_1(t) = q(t,\rho_x), \quad h_1 = h(\rho_x),$$

and, moreover, $(p_0, (\ell_{0i})_{i=1}^n) \in \mathcal{E}^m_{p;\ell_1,\dots,\ell_n}$. By Lemma 2.2, x admits estimate (2.10), i.e.,

$$\rho_x \le \rho^*\Big(\|h(\rho_x)\| + \int_a^b \|q(t,\rho_x)\|\,dt\Big).$$

Hence, by virtue of condition (3.1) and notation (3.2), we get (1.13). □

Proof of Theorem 1.2. Suppose

$$\overline{p}(x,y)(t) = p(x,0,y)(t), \quad \overline{\ell}_i(x,y) = \ell_i(x,0,y) \ \ (i=1,\dots,n),$$
$$q(t) = |f(0)(t)|, \quad h = \max\{|h_1(0)|,\dots,|h_n(0)|\}.$$

Then conditions (1.20)–(1.22) yield the conditions

$$(\overline{p}, (\overline{\ell}_i)_{i=1}^n) \in O^{m,\alpha,\beta}_{q_0;h_{01},\dots,h_{0n}},$$
$$|f(x)(t) - \overline{p}(x,x)(t)| \le q_0(x)(t) + q(t)$$

and

$$|h_i(x) - \overline{\ell}_i(x,x)| \le h_{0i}(x) + h \ \ (i=1,\dots,n).$$

These conditions, by Theorem 1.1, guarantee the solvability of problem (1.1), (1.2). It remains to show that this problem has at most one solution.

Let x and $\overline{x}$ be arbitrary solutions of problem (1.1), (1.2). Suppose

$$y(t) = x(t) - \overline{x}(t),$$
$$p_0(y)(t) = p(x,\overline{x},y)(t), \quad \ell_{0i}(y) = \ell_i(x,\overline{x},y) \ \ (i=1,\dots,n).$$

Then, due to conditions (1.21), (1.22), the vector function y is a solution of problem (1.14), (1.15). On the other hand, by (1.20) this problem has only a trivial solution. Thus $y(t) \equiv 0$, and consequently, $x(t) \equiv \overline{x}(t)$. □

Corollary 1.1 (Corollary 1.2) follows from Theorem 1.1 (Theorem 1.2) in the case where $p(x,y)(t) \equiv p_0(y)(t)$, $\ell_i(x,y) \equiv \ell_{0i}(y)$ $(i=1,\dots,n)$ $(p(x,\overline{x},y)(t) \equiv p_0(y)(t)$, $\ell_i(x,\overline{x},y) \equiv \ell_{0i}(y)$ $(i=1,\dots,n))$.

Proof of Theorems 1.3 *and* 1.4. In view of conditions $\alpha \in [n_2-1,n_2]$, $\beta \in [n_1-1,n_1]$ and equalities (1.5), the existence of limits (1.4) implies the existence of the limits

$$\ell_{0i}(x) \overset{\text{def}}{=} \lim_{t\to a} x^{(i-1)}(t) \ (i=1,\dots,n_1), \quad \ell_{0n_1+i}(x) \overset{\text{def}}{=} \lim_{t\to b} x^{(i-1)}(t) \ (i=1,\dots,n_2).$$

Moreover, it is evident that $\ell_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;R^m) \to R^m$ $(i=1,\dots,n)$ are linear bounded operators.

By Theorems 1.1 and 1.2, to prove Theorems 1.3 and 1.4 it is sufficient to show that if $p_0(y)(t) \equiv 0$, $q_0(y)(t) \equiv p(|y|)(t)$ and $h_{0i}(y) \equiv 0$ $(i=1,\dots,n)$, then problem (1.14), (1.15) has only a trivial solution. However, in the case under consideration this problem has the form

$$(t-a)^\alpha(b-t)^\beta|y^{(n)}(t)| \le p(|y|)(t), \tag{3.3}$$

$$\lim_{t\to a} y^{(i-1)}(t) = 0 \ (i=1,\dots,n_1), \quad \lim_{t\to b} y^{(i-1)}(t) = 0 \ (i=1,\dots,n_2). \tag{3.4}$$

Let $y \in \widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[;R^m)$ be a solution of problem (3.3), (3.4). Then

$$y(t) = \int_a^b g(t,s)y^{(n)}(s)\,ds, \tag{3.5}$$

where g is the Green function of problem (2.27), (2.28), admitting, by Lemma 2.3, estimate (2.29).

According to inequalities (2.29) and (3.3), from (3.5) we find

$$(u_{\alpha,\beta}(t))^{-1}|y(t)| \le \frac{(b-a)^{1-n}}{n_0(n_1-1)!(n_2-1)!}\int_a^b p(|y|)(s)\,ds \ \text{ for } \ a\le t\le b. \tag{3.6}$$

Let $y(t) = (y_j(t))_{j=1}^m$. Suppose

$$\rho_j = \sup\{|y_j(t)|/u_{\alpha,\beta}(t) : \ a<t<b\} \ \ (j=1,\dots,m), \quad \rho = (\rho_j)_{j=1}^m.$$

Then

$$|y(t)| \le u_{\alpha,\beta}(t)\rho \ \text{ for } \ a\le t\le b.$$

If together with this estimate we take into account the positiveness of the operator p and identity (1.23), then we get

$$p(|y|)(s) \le p(u_{\alpha,\beta}\,\rho)(s) \le p(u_{\alpha,\beta}E_m)(s)\rho \ \text{ for almost all } \ s\in[a,b].$$

On account of this inequality, from (3.6) we obtain

$$(u_{\alpha,\beta}(t))^{-1}|y(t)| \le A\rho \ \text{ for } \ a\le t\le b,$$

where

$$A = \frac{(b-a)^{1-n}}{n_0(n_1-1)!(n_2-1)!}\int_a^b p(u_{\alpha,\beta}E_m)(s)\,ds.$$

Hence, in view of the definition of ρ, it is obvious that

$$(E_m - A)\rho \leq 0. \tag{3.7}$$

On the other, due to (1.24) we have

$$r(A) < 1.$$

According to this inequality and the nonnegativeness of the matrix A, the matrix $(E_m - A)^{-1}$ is nonnegative. If now we multiply both sides of inequality (3.7) by $(E_m - A)^{-1}$, then we obtain $\rho \leq 0$, i. e., $\rho = 0$ and $y(t) \equiv 0$. □

Proof of Corollaries 1.3 *and* 1.4. Equation (1.25) is obtained from equation (1.1) in the case, where

$$f(x)(t) \equiv f_0(t, x(\tau(t))).$$

Suppose

$$p(x)(t) = \mathcal{P}(t)\, x(\tau(t)).$$

Then

$$p(u_{\alpha,\beta}E_m)(t) = u_{\alpha,\beta}(\tau(t))\mathcal{P}(t) = (\tau(t) - a)^{n-1-\alpha}(b - \tau(t))^{n-1-\beta}\mathcal{P}(t).$$

Thus inequality (1.26) implies inequality (1.24). If now we apply Theorem 1.3 (Theorem 1.4), then the validity of Corollary 1.3 (Corollary 1.4) becomes evident. □

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