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On Nonlinear Boundary Value Problems for Two-Dimensional Differential Systems

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1. STATEMENT OF THE MAIN RESULTS

1.1. Statement of the Problems

We study the boundary value problem

$$\frac{du_i}{dt} = f_i\,(t, u_1, u_2) \qquad (i = 1, 2), \tag{1.1}$$

$$\varphi_i\,(u_1(a), u_2(a), u_1(b), u_2(b)) = 0 \qquad (i = 1, 2), \tag{1.2}$$

where the $f_i : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are functions satisfying the local Carathéodory conditions and the $\varphi_i : \mathbb{R}^4 \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions satisfying one of the following two inequalities in $\mathbb{R}^4$:

$$(\varphi_1\,(x_1, x_2, x_3, x_4) - x_1)\,x_2 - (\varphi_2\,(x_1, x_2, x_3, x_4) - x_3)\,x_4 \le \gamma, \tag{1.3}$$

$$(\varphi_1\,(x_1, x_2, x_3, x_4) - x_1)\,x_2 - (\varphi_2\,(x_1, x_2, x_3, x_4) - x_4)\,x_3 \le \gamma. \tag{1.4}$$

Here $\gamma = \text{const} \ge 0$.

We separately consider the case in which $f_i\,(t, x_1, x_2) \equiv f_i\,(t, x_{3-i})$ $(i = 1, 2)$ and either

$$\varphi_1\,(x_1, x_2, x_3, x_4) = x_1 - \mu x_4 + \psi_1\,(x_2)\,,$$
$$\varphi_2\,(x_1, x_2, x_3, x_4) \equiv x_3 - \mu x_2 - \psi_2\,(x_4)$$

or

$$\varphi_1\,(x_1, x_2, x_3, x_4) = x_1 - \mu x_3 + \psi_1\,(x_2)\,,$$
$$\varphi_2\,(x_1, x_2, x_3, x_4) \equiv x_4 - \mu x_2 - \psi_2\,(x_3)\,,$$

that is, the case in which system (1.1) has the form

$$\frac{du_1}{dt} = f_1\,(t, u_2)\,, \qquad \frac{du_2}{dt} = f_2\,(t, u_1)\,, \tag{1.5}$$

and the boundary conditions (1.2) have one of the following two forms:

$$u_1(a) = \mu u_2(b) - \psi_1\,(u_2(a))\,, \qquad u_1(b) = \mu u_2(a) + \psi_2\,(u_2(b))\,, \tag{1.2$_1$}$$

$$u_1(a) = \mu u_1(b) - \psi_1\,(u_2(a))\,, \qquad u_2(b) = \mu u_2(a) + \psi_2\,(u_1(b))\,, \tag{1.2$_2$}$$

where μ is an arbitrary real number and the $\psi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions such that

$$x\psi_1(x) + y\psi_2(y) \le \gamma \quad \text{for} \quad (x, y) \in \mathbb{R}^2. \tag{1.6}$$

This class of boundary conditions includes, for example, well-known two-point, periodic, and antiperiodic boundary conditions

$$u_1(a) = 0, \qquad u_1(b) = 0, \tag{1.2$_3$}$$

$$u_1(a) = 0, \qquad u_2(b) = 0, \tag{1.2$_4$}$$

$$u_1(a) = u_1(b), \qquad u_2(a) = u_2(b), \tag{1.2$_5$}$$

$$u_1(a) = -u_1(b), \qquad u_2(a) = -u_2(b). \tag{1.2$_6$}$$

There is a vast literature (e.g., see [1–17] and the bibliography therein) about boundary value problems of the form (1.1), (1.2) [in particular, problems (1.1), (1.2_k), $k = 1, \dots, 6$]. Nevertheless, the case in which the right-hand sides of system (1.1) are rapidly growing functions of the phase variables has been much less studied. The results given below deal with this case.

1.2. Problem (1.1), (1.2)

Throughout the paper, we use the following notation: $\mathbb{R}_+ = [0, +\infty[$; $M([a,b] \times \mathbb{R}_+)$ is the set of functions $\omega : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ integrable with respect to the first argument, continuous and nondecreasing with respect to the second argument, and such that $\omega(t, 0) \equiv 0$;

$$D_0(x_0) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 x_3 > 0, \ \ |x_1| \geq x_0, \ \ |x_3| \geq x_0\};$$

$$D(x_0) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ |x_1| + |x_4| \geq x_0, \ x_1 x_3 > 0\} \\ \cup \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : |x_1| + |x_4| \geq x_0, \ x_2 x_4 > 0\}.$$

If $u_{i0} : [a,b] \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions, then by $U_r(u_{10}, u_{20})$ we denote the set of continuous vector functions $(u_1, u_2) : [a,b] \to \mathbb{R}$ such that $|u_1(t) - u_{10}(t)| + |u_2(t) - u_{20}(t)| < r$ for $a \leq t \leq b$.

Along with problem (1.1), (1.2), we consider the perturbed problem

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) + \eta_i(t, u_1, u_2) \qquad (i = 1, 2), \tag{1.7}$$

$$\varphi_i(u_1(a), u_2(a), u_1(b), u_2(b)) + \zeta_i(u_1(a), u_2(a), u_1(b), u_2(b)) = 0 \qquad (i = 1, 2). \tag{1.8}$$

Just as in [4], we introduce the following definition.

Definition 1.1. Problem (1.1), (1.2) is said to be *well posed* if it has a unique solution (u_{10}, u_{20}) and if, for arbitrary numbers $r > 0$ and $\varepsilon \in\,]0, r[$ and an arbitrary function $\omega \in M([a,b] \times \mathbb{R}_+)$, there exists a positive number δ such that, for arbitrary Carathéodory functions $\eta_i : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ and arbitrary continuous functions $\zeta_i : \mathbb{R}^4 \to \mathbb{R}$ $(i = 1, 2)$ satisfying the conditions

$$\sum_{k=1}^{2} \left| \int_a^t \eta_k(s, x_1, x_2)\, ds \right| \leq \delta,$$

$$\sum_{k=1}^{2} |\eta_k(t, x_1, x_2) - \eta_k(t, y_1, y_2)| \leq \omega(t, |x_1 - y_1| + |x_2 - y_2|) \quad \text{for} \quad a \leq t \leq b,$$

$$\sum_{k=1}^{2} |x_k - u_{k0}(t)| \leq r, \qquad \sum_{k=1}^{2} |y_k - u_{k0}(t)| \leq r,$$

$$\sum_{k=1}^{2} |\zeta_k(x_1, x_2, x_3, x_4)| \leq \delta \quad \text{for} \quad \sum_{k=1}^{2} (|x_k - u_{k0}(a)| + |x_{2+k} - u_{k0}(b)|) \leq r,$$

problem (1.1), (1.2) has at least one solution $(u_1, u_2) \in U_r(u_{10}, u_{20})$ and the inequality

$$\sum_{k=1}^{2} |u_k(t) - u_{k0}(t)| < \varepsilon, \qquad a \leq t \leq b,$$

holds for every such solution.

Theorem 1.1. *Let functions* φ_1 *and* φ_2 *satisfy condition* (1.3) *or condition* (1.4), *where* $\gamma = \text{const} \geq 0$. *Moreover, suppose that there exist functions* $h_i \in M([a,b] \times \mathbb{R}_+)$ $(i = 1, 2)$, *integrable functions* $h_{0i}:[a,b]\to\mathbb{R}_+$ $(i=1,2)$ *and* $h:[a,b]\to\mathbb{R}_+$, *and constants* $\ell\ge 0$, $\delta>0$, *and* $x_0>0$ *such that*

$$f_i\left(t,x_1,x_2\right)x_{3-i}\ge h_i\left(t,\left|x_{3-i}\right|\right)-h_{0i}(t)\quad \textit{for}\quad a\le t\le b,\quad (x_1,x_2)\in\mathbb{R}^2\quad (i=1,2), \tag{1.9}$$

$$\left|f_i\left(t,x_1,x_2\right)\right|\le \ell h_{3-i}\left(t,\left|x_i\right|\right)+h(t)\quad \textit{for}\quad a<t<b,\quad x_i\in\mathbb{R},\quad \left|x_{3-i}\right|\le\delta\quad (i=1,2), \tag{1.10}$$

and at least one of the following three conditions is satisfied:

$$\int\limits_a^b h_i\left(s,x_0\right)ds>\gamma+\int\limits_a^b\left(h_{01}(s)+h_{02}(s)\right)ds,\qquad i=1,2; \tag{1.11}$$

$$\begin{gathered}\int\limits_a^b h_1\left(s,x_0\right)ds>\gamma+\int\limits_a^b\left(h_{01}(s)+h_{02}(s)\right)ds\qquad \textit{and}\\ \sum_{k=1}^2\left|\varphi_k\left(x_1,x_2,x_3,x_4\right)\right|>0\quad \textit{for}\quad \left(x_1,x_2,x_3,x_4\right)\in D_0\left(x_0\right);\end{gathered} \tag{1.12}$$

$$\sum_{k=1}^2\left|\varphi_k\left(x_1,x_2,x_3,x_4\right)\right|>0\quad \textit{for}\quad \left(x_1,x_2,x_3,x_4\right)\in D\left(x_0\right). \tag{1.13}$$

Then problem (1.1), (1.2) *is solvable.*

We consider problems (1.1), (1.2_1) and (1.1), (1.2_2) for the case in which conditions (1.12) and (1.13) are replaced by the conditions

$$\int\limits_a^b h_1\left(s,x_0\right)ds>\gamma+\int\limits_a^b\left(h_{01}(s)+h_{02}(s)\right)ds,\quad \mu=0,\quad |\psi_1(x)|\le x_0\quad \text{for}\quad x\in\mathbb{R}, \tag{1.12_1}$$

$$\int\limits_a^b h_1\left(s,x_0\right)ds>\gamma+\int\limits_a^b\left(h_{01}(s)+h_{02}(s)\right)ds,\quad \mu\le 0,\quad |\psi_1(x)|\le x_0\quad \text{for}\quad x\in\mathbb{R}, \tag{1.12_2}$$

$$\mu\le 0,\qquad |\psi_1(x)|+|\psi_2(x)|\le x_0\quad \text{for}\quad x\in\mathbb{R}. \tag{$1.13'$}$$

Theorem 1.1 readily implies the following assertion.

Corollary 1.1. *Suppose that there exist functions* $h_i\in M\left([a,b]\times\mathbb{R}_+\right)$ $(i=1,2)$, *integrable functions* $h_{0i}:[a,b]\to\mathbb{R}_+$ $(i=1,2)$ *and* $h:[a,b]\to\mathbb{R}_+$, *and constants* $\ell\ge 0$, $\gamma\ge 0$, $\delta>0$, *and* $x_0>0$ *such that conditions* (1.6), (1.9), *and* (1.10), *as well as one of conditions* (1.11) *and* (1.12_1) [*respectively, one of conditions* (1.11), (1.12_2), *and* $(1.13')$], *are satisfied. Then problem* (1.1), (1.2_1) [*respectively, problem* (1.1), (1.2_2)] *is solvable.*

Example 1.1. Let m be a positive integer, let $p_0\in\mathbb{R}$, and let $p:[a,b]\to\,]0,+\infty[$ be an integrable function. Then the differential system

$$\frac{du_1}{dt}=\frac{1}{\left(1+|u_2|\right)^2}u_2+p_0\frac{1}{1+|u_2|},\qquad \frac{du_2}{dt}=p(t)u_1^{2m-1} \tag{1.14}$$

satisfies conditions (1.9) and (1.10), where $h_1(t,x)=(1+x)^{-2}x^2$, $h_2(t,x)=p(t)x^{2m}$, $h_{10}(t)=|p_0|$, $h_{20}(t)=0$, $\ell=0$, $\delta=1$, and $h(t)=1+|p_0|+p(t)$. Therefore, to satisfy inequality (1.11) with $\gamma=0$ for some sufficiently large x_0, it is necessary and sufficient that $|p_0|<1$. This, together with Corollary 1.1, implies that if $|p_0|<1$, then problems (1.14), (1.2_3) and (1.14), (1.2_5) are solvable. On the other hand, if $|p_0|\ge 1$, then the first component of an arbitrary solution of system (1.14) is an increasing function and hence problems (1.14), (1.2_3) and (1.14), (1.2_5) have no solutions.

This example shows that the conditions imposed in Theorem 1.1 and the corollary on the functions h_i and h_{i0} $(i=1,2)$ are in a sense optimal.

Theorem 1.2. *If the assumptions of Theorem* 1.1 *are satisfied, then the unique solvability of problem* (1.1), (1.2) *guarantees its well-posedness.*

1.3. Problems (1.5), (1.2_1) *and* (1.5), (1.2_2)

If $f_i(t,x_1,x_2) \equiv f_i(t,x_{3-i})$ $(i=1,2)$, then condition (1.9) acquires the form

$$f_i(t,x)x \ge h_i(t,|x|) - h_{0i}(t) \quad \text{for} \quad a \le t \le b, \quad x \in \mathbb{R} \quad (i=1,2). \tag{1.15}$$

As to condition (1.10), it is necessarily satisfied for $\ell = 0$, $\delta = 1$, and

$$h(t) = \max\{|f_1(t,x)| + |f_2(t,x)| : \ |x| \le 1\}.$$

Therefore, Corollary 1.1 and Theorem 1.2 imply the following assertion.

Theorem 1.3. *Suppose that there exist functions* $h_i \in M([a,b]\times\mathbb{R}_+)$ $(i=1,2)$, *integrable functions* $h_{0i} : [a,b] \to \mathbb{R}_+$ $(i=1,2)$ *and* $h : [a,b] \to \mathbb{R}_+$, *and constants* $\gamma \ge 0$ *and* $x_0 > 0$ *such that, along with conditions* (1.6) *and* (1.15), *one of conditions* (1.11) *and* (1.12_1) [*respectively, one of conditions* (1.11), (1.12_2), *and* $(1.13')$] *is satisfied. Moreover, let* f_i $(i=1,2)$ *be increasing functions of the second argument, and let* ψ_i $(i=1,2)$ *be nonincreasing functions of the second argument. Then problem* (1.5), (1.2_1) [*respectively, problem* (1.5), (1.2_2)] *is well posed.*

The Emden–Fowler system

$$\frac{du_i}{dt} = \sum_{k=1}^{m_i} p_{ik}(t)\,|u_{3-i}|^{\lambda_{ik}} \operatorname{sgn} u_{3-i} + q_i(t) \qquad (i=1,2), \tag{1.16}$$

where $\lambda_{ik} = \text{const} > 0$ and p_{ik} and $q_i : [a,b] \to \mathbb{R}$ $(i=1,2;\ k=1,\ldots,m_i)$ are integrable functions, is a special case of system (1.5).

Corollary 1.2. *Let*

$$p_{ik}(t) \ge 0, \qquad \sum_{j=1}^{m_i} p_{ij}(t) > 0 \quad \textit{for} \quad a \le t \le b \quad (i=1,2; \quad k=1,\ldots,m_i), \tag{1.17}$$

$$\int_a^b \left[\sum_{k=1}^{m_i} p_{ik}(s)\right]^{-1/\lambda_i} |q_i(s)|^{1+1/\lambda_i}\, ds < +\infty \qquad (i=1,2), \tag{1.18}$$

where $\lambda_i = \min\{\lambda_{ik} : \ k=1,\ldots,m_i\}$. *Furthermore, let* ψ_1 *and* ψ_2 *be nondecreasing functions satisfying condition* (1.6) *with* $\gamma = \text{const} \ge 0$. *Then problems* (1.16), (1.2_1) *and* (1.16), (1.2_2) *are well posed.*

In particular, it follows from Corollary 1.2 that if conditions (1.17) and (1.18) are satisfied, then problems (1.16), (1.2_k) $(k=3,4,5,6)$ are well posed.

2. AUXILIARY ASSERTIONS

2.1. Lemmas on A Priori Estimates

Suppose that δ is a positive constant, $\ell \ge 0$,

$$\nu_\delta(x) = \begin{cases} 0 & \text{if} \quad x > \delta \\ 1 & \text{if} \quad 0 \le x \le \delta, \end{cases}$$

$\tilde{h}_i \in M([a,b] \times \mathbb{R}_+)$ $(i = 1,2)$, and $h_{0i} : [a,b] \to \mathbb{R}_+$ $(i = 1,2)$ and $\tilde{h} : [a,b] \to \mathbb{R}_+$ are integrable functions. Consider the system of differential inequalities

$$u_i'(t)u_{3-i}(t) \geq \tilde{h}_i(t, |u_{3-i}(t)|) - h_{0i}(t) \qquad (i = 1,2), \tag{2.1}$$

$$|u_i'(t)|\,\nu_\delta(|u_{3-i}(t)|) \leq \ell \tilde{h}_{3-i}(t, |u_i(t)|) + \tilde{h}(t) \qquad (i = 1,2). \tag{2.2}$$

A *solution* of this system is understood as a vector function (u_1, u_2) with absolutely continuous components $u_i : [a,b] \to \mathbb{R}$ $(i = 1,2)$ that satisfies the differential inequalities (2.1) and (2.2) almost everywhere on $[a,b]$.

Lemma 2.1. *Let γ and x_0 be some nonnegative constants. Then an arbitrary solution (u_1, u_2) of system* (2.1), (2.2) *satisfying the conditions*

$$u_1(b)u_2(b) - u_1(a)u_2(a) \leq \gamma, \tag{2.3}$$

$$\min\{|u_i(t)| : \ a \leq t \leq b\} \leq x_0 \qquad (i = 1,2), \tag{2.4}$$

admits the estimates

$$|u_i(t)| \leq \varrho \quad \textit{for} \quad a \leq t \leq b \quad (i = 1,2), \tag{2.5}$$

where

$$\varrho = x_0 + \left(\ell + \frac{1}{\delta}\right)\gamma + \left(\ell + \frac{2}{\delta}\right)\int_a^b (h_{01}(s) + h_{02}(s))\,ds + \int_a^b \tilde{h}(s)ds. \tag{2.6}$$

Proof. By inequalities (2.1) and (2.3), we have

$$u_i'(t)u_{3-i}(t) = |u_i'(t)u_{3-i}(t) + h_{0i}(t)| - h_{0i}(t) \geq |u_i'(t)u_{3-i}(t)| - 2h_{0i}(t) \qquad (i = 1,2),$$

$$\int_a^b [u_1'(s)u_2(s) + u_1(s)u_2'(s)]\,ds = u_1(b)u_2(b) - u_1(a)u_2(a) \leq \gamma.$$

Therefore,

$$\int_a^b \left[\tilde{h}_1(s, |u_2(s)|) + \tilde{h}_2(s, |u_2(s)|)\right] ds \leq \gamma + \int_a^b (h_{01}(s) + h_{02}(s))\,ds, \tag{2.7}$$

$$\int_a^b [|u_1'(s)u_2(s)| + |u_1(s)u_2'(s)|]\,ds \leq \gamma + 2\int_a^b (h_{01}(s) + h_{02}(s))\,ds. \tag{2.8}$$

Let $I_k = \{t \in [a,b] : \ |u_{3-k}(t)| \leq \delta\}$ $(k = 1,2)$. Then, with regard to condition (2.4), we obtain

$$\begin{aligned} |u_i(t)| &\leq x_0 + \int_a^b |u_i'(s)|\,ds \\ &\leq x_0 + \int_{I_i} |u_i'(s)|\,ds + \frac{1}{\delta}\int_{[a,b]\setminus I_i} |u_i'(s)u_{3-i}(s)|\,ds \qquad \text{for} \qquad a \leq t \leq b \qquad (i = 1,2). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{I_i} |u_i'(s)|\,ds &\leq \int_{I_i} \left[\ell \tilde{h}_{3-i}(s, |u_i(s)|) + \tilde{h}(s)\right] ds \\ &\leq \int_a^b \left[\ell \tilde{h}_{3-i}(s, |u_i(s)|) + \tilde{h}(s)\right] ds \qquad (i = 1,2) \end{aligned}$$

by (2.2). If we now use conditions (2.7) and (2.8), then it becomes clear that the estimate (2.5) is valid, where ϱ is the number given by (2.6). The proof of the lemma is complete.

Lemma 2.2. *Suppose that there exist numbers $\gamma \geq 0$ and $x_0 > 0$ such that*

$$\int_a^b \tilde{h}_1(s, x_0)\, ds > \gamma + \int_a^b (h_{01}(s) + h_{02}(s))\, ds. \tag{2.9}$$

Then an arbitrary solution (u_1, u_2) of system (2.1), (2.2) *satisfying condition* (2.3) *and the inequality*

$$\min\{|u_1(t)| : \ a \leq t \leq b\} \leq x_0 \tag{2.10}$$

admits the estimate (2.5), *where ϱ is the number given by* (2.6).

Proof. Let (u_1, u_2) be an arbitrary solution of system (2.1), (2.2) satisfying conditions (2.3) and (2.10). Then, as shown above, inequality (2.7) is valid. Now we set $\mu_0 = \min\{|u_2(t)| : \ a \leq t \leq b\}$ and obtain

$$\int_a^b \tilde{h}_1(s, \mu_0)\, ds \leq \gamma + 2\int_a^b (h_{01}(s) + h_{02}(s))\, ds$$

from (2.7). This, together with (2.9), implies that $\mu_0 \leq x_0$. Consequently, inequality (2.4) is valid. Now an application of Lemma 2.1 makes Lemma 2.2 obvious.

The following assertion can be proved in a similar way.

Lemma 2.3. *Suppose that there exist numbers $\gamma \geq 0$ and $x_0 > 0$ such that*

$$\int_a^b \tilde{h}_i(s, x_0)\, ds > \gamma + \int_a^b (h_{01}(s) + h_{02}(s))\, ds \qquad (i = 1, 2). \tag{2.11}$$

Then an arbitrary solution (u_1, u_2) of system (2.1), (2.2) *satisfying condition* (2.3) *admits the estimate* (2.5), *where ϱ is the number given by* (2.6).

2.2. Lemmas on the Solvability and Well-Posedness of Problem (1.1), (1.2)

For an arbitrary positive number r, we set

$$\chi_r(x) = \begin{cases} 1 & \text{if } \ 0 \leq x \leq r \\ 2 - x/r & \text{if } \ r < x < 2r \\ 0 & \text{if } \ x \geq 2r. \end{cases} \tag{2.12}$$

Along with (1.1), (1.2), we consider the auxiliary linear and nonlinear problems

$$u_i' = \sum_{k=1}^{2} p_{ik}(t) u_k \qquad (i = 1, 2), \tag{2.13}$$

$$\sum_{k=1}^{2} (\alpha_{ik} u_k(a) + \beta_{ik} u_k(b)) = 0 \qquad (i = 1, 2), \tag{2.14}$$

$$u_i' = \sum_{k=1}^{2} p_{ik}(t) u_k + q_{i\varrho}(t, u_1, u_2) \qquad (i = 1, 2), \tag{2.15}$$

$$\sum_{k=1}^{2} (\alpha_{ik} u_k(a) + \beta_{ik} u_k(b)) = \Delta_{i\varrho}(u_1(a), u_2(a), u_1(b), u_2(b)) \qquad (i = 1, 2), \tag{2.16}$$

where

$$q_{i\varrho}(t,x_1,x_2)=\chi_{2\varrho}(|x_1|+|x_2|)\left[f_i(t,x_1,x_2)-\sum_{k=1}^{2}p_{ik}(t)x_k\right]\qquad (i=1,2), \tag{2.17}$$

$$\Delta_{i\varrho}(x_1,x_2,x_3,x_4)=\chi_{4\varrho}\left(\sum_{k=1}^{4}|x_k|\right)\left[\sum_{k=1}^{2}(\alpha_{ik}x_k+\beta_{ik}x_{k+2})-\varphi_i(x_1,x_2,x_3,x_4)\right]\quad (i=1,2). \tag{2.18}$$

Lemma 2.4. *Suppose that there exist integrable functions* $p_{ik}:[a,b]\to\mathbb{R}$ $(i,k=1,2)$ *and constants* $\alpha_{ik}\in\mathbb{R}$, $\beta_{ik}\in\mathbb{R}$ $(i,k=1,2)$, *and* $\varrho\in\,]0,+\infty[$ *such that problem* (2.13), (2.14) *has only the trivial solution and an arbitrary solution* (u_1,u_2) *of problem* (2.15), (2.16) *admits the estimate* (2.5). *Then problem* (2.15), (2.16) *is solvable, and each solution is simultaneously a solution of problem* (1.1), (1.2).

Proof. By virtue of notation (2.12), (2.17), and (2.18), it is clear that there exists an integrable function $q_\varrho^*:[a,b]\to\,]0,+\infty[$ and a positive constant Δ_ϱ^* such that the inequalities

$$|q_{i\varrho}(t,x_1,x_2)|\le q_\varrho^*(t),\qquad |\Delta_{i\varrho}(x_1,x_2,x_3,x_4)|\le\Delta_\varrho^*\qquad (i=1,2) \tag{2.19}$$

are valid on $[a,b]\times\mathbb{R}^2$ and in $\mathbb{R}^4$, respectively.

It follows from the Conti theorem [13] (see also [4, Corollary 2.1]) that condition (2.19) and the unique solvability of the homogeneous problem (2.13), (2.14) guarantee the solvability of problem (2.15), (2.16).

Let (u_1,u_2) be an arbitrary solution of problem (2.15), (2.16). Then, by one of the assumptions of the lemma, the estimate (2.5) is valid, and consequently,

$$\chi_{2\varrho}(|u_1(t)|+|u_2(t)|)\equiv 1,\qquad \chi_{4\varrho}(|u_1(a)|+|u_2(a)|+|u_1(b)|+|u_2(b)|)=1.$$

By using relations (2.17) and (2.18), we find that (u_1,u_2) is a solution of problem (1.1), (1.2). The proof of the lemma is complete.

Lemma 2.5. *Suppose that there exist integrable functions* $p_{ik}:[a,b]\to\mathbb{R}$ $(i,k=1,2)$ *and constants* $\alpha_{ik}\in\mathbb{R}$, $\beta_{ik}\in\mathbb{R}$ $(i,k=1,2)$, *and* $\varrho_0>0$ *such that problem* (2.13), (2.14) *has only the trivial solution and, for each* $\varrho\ge\varrho_0$, *an arbitrary solution* (u_1,u_2) *of problem* (2.15), (2.16) *admits the estimates* (2.5). *Then the unique solvability of problem* (1.1), (1.2) *implies its well-posedness.*

Proof. Problem (1.1), (1.2) is solvable, since all assumptions of Lemma 2.4 are valid. Our aim is to show that if this problem has a unique solution (u_{10},u_{20}), then it is well posed.

By Lemma 2.4, for each $\varrho\ge\varrho_0$, the vector function (u_{10},u_{20}) is also the unique solution of problem (2.15), (2.16). By Definition 3.1 in [4], this means that (u_{10},u_{20}) is a strongly isolated solution of problem (1.1), (1.2) in arbitrary radius. Now it follows by Theorem 3.1 in [4] that problem (1.1), (1.2) is well posed. The proof of the lemma is complete.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. We carry out the proof only for the case in which condition (1.3) is satisfied; the case of condition (1.4) can be treated in a similar way. Furthermore, without loss of generality, we assume that $x_0>1$.

First, suppose that the functions h_i $(i=1,2)$ satisfy conditions (1.11) and set

$$p_i(t)=1+h_i(t,x_0)\qquad (i=1,2),\qquad \tilde h(t)=(p_1(t)+p_2(t))\,\delta+h(t), \tag{3.1}$$

$$q_{i\varrho}(t,x_1,x_2)=\chi_{2\varrho}(|x_1|+|x_2|)\,[f_i(t,x_1,x_2)-p_i(t)x_{3-i}]\qquad (i=1,2), \tag{3.2}$$

$$\Delta_{i\varrho}(x_1,x_2,x_3,x_4)=\chi_{4\varrho}\left(\sum_{k=1}^{4}|x_k|\right)[x_{2i-1}-\varphi_i(x_1,x_2,x_3,x_4)]\qquad (i=1,2), \tag{3.3}$$

where ϱ and χ_r are the number and the function given by (2.6) and (2.12), respectively. Then the linear homogeneous problem

$$\frac{du_i}{dt} = p_i(t)u_{3-i} \qquad (i = 1, 2), \qquad u_1(a) = 0, \qquad u_1(b) = 0,$$

has only the trivial solution, since $p_i(t) > 0$ for $a \le t \le b$ $(i = 1, 2)$. By Lemma 2.4, to prove the solvability of problem (1.1), (1.2), it suffices to show that an arbitrary solution (u_1, u_2) of the problem

$$\frac{du_i}{dt} = p_i(t)u_{3-i} + q_{i\varrho}(t, u_1, u_2) \qquad (i = 1, 2), \tag{3.4}$$

$$u_1(a) = \Delta_{1\varrho}(u_1(a), u_2(a), u_1(b), u_2(b)), \qquad u_1(b) = \Delta_{2\varrho}(u_1(a), u_2(a), u_1(b), u_2(b)), \tag{3.5}$$

admits the estimates (2.5).

Let $\lambda(t) = \chi_{2\varrho}(|u_1(t)| + |u_2(t)|)$, $\lambda_0 = \chi_{4\varrho}(|u_1(a)| + |u_2(a)| + |u_1(b)| + |u_2(b)|)$, and

$$\tilde{h}_i(t, x) = (1 - \lambda(t))h_i(t, x_0)\,x + \lambda(t)h_i(t, x) \qquad (i = 1, 2). \tag{3.6}$$

With regard to (3.2) and (3.3), from (3.4) and (3.5), we obtain

$$u_i'(t) = (1 - \lambda(t))p_i(t)u_{3-i}(t) + \lambda(t)f_i(t, u_1(t), u_2(t)) \qquad (i = 1, 2),$$
$$u_1(a) = \lambda_0(u_1(a) - \varphi_1(u_1(a), u_2(a), u_1(b), u_2(b))),$$
$$u_1(b) = \lambda_0(u_1(b) - \varphi_2(u_1(a), u_2(a), u_1(b), u_2(b))).$$

This, together with conditions (1.3), (1.9), (1.10), (3.1), and (3.6), shows that the vector function (u_1, u_2) is a solution of the system of differential inequalities (2.1), (2.2) supplemented by condition (2.3). On the other hand, by (3.6), inequalities (1.11) lead to (2.11), since $x_0 > 1$. Therefore, all assumptions of Lemma 2.3 are satisfied, which guarantees the validity of the estimates (2.5). This completes the proof of the solvability of problem (1.1), (1.2).

Let us proceed to the case in which condition (1.12) or (1.13) is satisfied. Without loss of generality, we assume that $x_0 > \delta$. Let $\tilde{h}(t) = h(t)$, and let ϱ be the number given by (2.6). We set $\zeta_\varrho(x) = 0$ for $|x| \le \varrho$, $\zeta_\varrho(x) = (|x| - \varrho)\operatorname{sgn} x$ for $|x| > \varrho$, and

$$\tilde{f}_i(t, x_1, x_2) = f_i(t, x_1, x_2) + \zeta_\varrho(x_{3-i}), \qquad \tilde{h}_i(t, x) = h_i(t, x) + \zeta_\varrho(x)x \qquad (i = 1, 2) \tag{3.7}$$

and consider the differential system

$$\frac{du_i}{dt} = \tilde{f}_i(t, u_1, u_2) \qquad (i = 1, 2). \tag{3.8}$$

By (1.9) and (1.10), from (3.7), we obtain

$$\tilde{f}_i(t, x_1, x_2)\,x_{3-i} \ge \tilde{h}_i(t, |x_{3-i}|) - h_{0i}(t) \quad \text{for} \quad a \le t \le b, \quad (x_1, x_2) \in \mathbb{R}^2 \quad (i = 1, 2), \tag{3.9}$$

$$\left|\tilde{f}_i(t, x_1, x_2)\right| \le \ell\tilde{h}_{3-i}(t, |x_i|) + \tilde{h}(t) \quad \text{for} \quad a \le t \le b, \quad x_i \in \mathbb{R}, \quad |x_{3-i}| \le \delta \quad (i = 1, 2). \tag{3.10}$$

On the other hand, obviously,

$$\int_a^b \tilde{h}_i(s, \tilde{x}_0)\,ds > \gamma + \int_a^b (h_{01}(s) + h_{02}(s))\,ds \quad (i = 1, 2), \tag{3.11}$$

where $\tilde{x}_0 = \varrho + 1 + (b - a)^{-1}\int_a^b (h_{01}(s) + h_{02}(s))\,ds$ $(i = 1, 2)$. However, as was proved above, conditions (1.3) and (3.9)–(3.11) ensure the solvability of problem (3.8), (1.2).

Let (u_1, u_2) be an arbitrary solution of problem (3.8), (1.2). Then, by conditions (3.9) and (3.10), the vector function (u_1, u_2) is also a solution of the system of differential inequalities (2.1), (2.2) supplemented by condition (2.3). On the other hand, it follows from conditions (1.2) and (1.12) [respectively, (1.2) and (1.13)] that inequalities (2.9) and (2.10) [respectively, (2.4)] are valid. Consequently, all assumptions of Lemma 2.2 (respectively, Lemma 2.1) are valid, which guarantees the validity of the estimates (2.5). These estimates, together with notation (3.7), imply that (u_1, u_2) is a solution of problem (1.1), (1.2). The proof of the theorem is complete.

Theorem 1.2 can be proved by analogy with Theorem 1.1. The only difference is that Lemma 2.5 is used instead of Lemma 2.4.

Proof of Theorem 1.3. As was mentioned above (see Subsection 1.3), if the assumptions of Theorem 1.3 are valid for problem (1.5), (1.2_1) or problem (1.5), (1.2_2), then the assumptions of Theorem 1.1 are also valid. Thus, by Theorem 1.2, to prove Theorem 1.3, it suffices to show that if the f_i $(i = 1, 2)$ are increasing functions of the second argument and the ψ_i $(i = 1, 2)$ are nonincreasing functions of the second argument, then problem (1.5), (1.2_1), as well as problem (1.5), (1.2_2), has at most one solution.

Suppose the contrary: problem (1.5), (1.2_1) [respectively, problem (1.5), (1.2_2)] has two distinct solutions (u_1, u_2) and (v_1, v_2). We set $w_i(t) = u_i(t) - v_i(t)$ $(i = 1, 2)$. Then

$$(w_1(t)w_2(t))' = \Delta(t), \qquad w_1(b)w_2(b) - w_1(a)w_2(a) = \Delta_0, \tag{3.12}$$

where $\Delta(t) = (f_1(t, v_2(t) + w_2(t)) - f_1(t, v_2(t)))\, w_2(t) + (f_2(t, v_1(t) + w_1(t)) - f_2(t, v_1(t)))\, w_1(t)$ and

$$\Delta_0 = (\psi_1(v_2(a) + w_2(a)) - \psi_1(v_2(a)))\, w_2(a) + (\psi_2(v_2(b) + w_2(b)) - \psi_2(v_2(b)))\, w_2(b)$$
$$\Big(\Delta_0 = (\psi_1(v_2(a) + w_2(a)) - \psi_1(v_2(a)))\, w_2(a) + (\psi_2(v_1(b) + w_1(b)) - \psi_2(v_1(b)))\, w_1(b)\Big).$$

Moreover, $\Delta(t) \ge 0$ for $a \le t \le b$, $\int_a^b \Delta(t)dt > 0$, and $\Delta_0 \le 0$. Therefore, from (3.12), we obtain $0 \ge \Delta_0 = \int_a^b \Delta(t)dt > 0$. The contradiction thus obtained completes the proof of the theorem.

Proof of Corollary 1.2. System (1.16) is obtained from system (1.5) if

$$f_i(t, x) = \sum_{k=1}^{m_i} p_{ik}(t)|x|^{\lambda_{ik}} \operatorname{sgn} x + q_i(t) \qquad (i = 1, 2). \tag{3.13}$$

This, together with condition (1.17), implies that f_1 and f_2 are increasing functions of the second argument. On the other hand, if we set $p_{0i}(t) = \sum_{k=1}^{m_i} p_{ik}(t)$ $(i = 1, 2)$, then, by the Young inequality and conditions (1.17) and (3.13), we obtain the inequalities

$$\begin{aligned} f_i(t, x)x &\ge p_{0i}(t)|x|^{\lambda_i+1} - |q_i(t)||x| \\ &\ge \frac{1}{2} p_{0i}(t)|x|^{\lambda_i+1} - (2p_{0i}(t))^{-1/\lambda_i} |q_i(t)|^{1+1/\lambda_i} \quad \text{for} \quad a \le t \le b, \quad |x| \ge 1 \quad (i = 1, 2). \end{aligned}$$

Consequently, inequalities (1.15) are valid, where

$$h_i(t, x) = (1/2)p_{0i}(t)|x|^{\lambda_i+1}, \qquad h_{0i}(t) = |q_i(t)| + (2p_{0i}(t))^{-1/\lambda_i} |q_i(t)|^{1/\lambda_i} \qquad (i = 1, 2);$$

moreover, by condition (1.18), the h_{0i} $(i = 1, 2)$ are integrable functions. On the other hand, since the p_{0i} $(i = 1, 2)$ are positive, it follows that inequalities (1.11) are valid for some sufficiently large x_0. If we now use Theorem 1.3, then the validity of Corollary 1.2 becomes obvious.

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