FUNCTIONAL DIFFERENTIAL EQUATIONS

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THE NEUMANN PROBLEM FOR THE SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS AT RESONANCE *

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Dedicated to the blessed memory
of my friend, Professor M. Drakhlin

Abstract. The optimal sufficient conditions of solvability and unique solvability of the boundary value problem

$$u'' = f(t,u) + f_0(t); \quad u'(a) = c_1, \ \ u'(b) = c_2$$

are established.

Key Words. Second order nonlinear differential equation, the Neumann problem at resonance, the existence and uniqueness theorems

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§ 1. Statement of the Main Results

Let $-\infty < a < b < +\infty$, $c_i \in R$ $(i = 1,2)$, $f_0 : [a,b] \to R$ be the Lebesgue integrable function, and $f : [a,b] \times R \to R$ be a function, satisfying the local Carathéodory conditions. We consider the differential equation

$$u'' = f(t,u) + f_0(t) \tag{1.1}$$

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with the Neumann boundary conditions

$$u'(a) = c_1, \quad u'(b) = c_2. \tag{1.2}$$

In the case, where

$$\liminf_{|x|\to+\infty} \Big|\frac{f(t,x)}{x}\Big| = 0$$

or

$$\limsup_{|x|\to+\infty} \Big|\frac{f(t,x)}{x}\Big| = +\infty,$$

$(1.1), (1.2)$ is the problem at resonance since the corresponding linear homogeneous problem

$$u'' = 0; \quad u'(a) = 0, \quad u'(b) = 0$$

has an infinite set of nontrivial solutions.

In the present paper, we establish new and unimprovable, in a certain sense, sufficient conditions which guarantee, respectively, the solvability and unique solvability of the problem $(1.1), (1.2)$. These results cover the resonance case and differ substantially from the well-known theorems of the existence and uniqueness of a solution of the problem $(1.1), (1.2)$ (see, e.g., [1]–[6]).

In the sequel, it will be assumed that

$$f(t,0) = 0 \quad \text{for} \quad a \le t \le b. \tag{1.3}$$

For arbitrary functions $p_i : [a,b] \to R$ $(i = 1,2)$, the writing $p_1(t) \not\equiv p_2(t)$ means that they differ from each other on a set of positive measure.

THEOREM 1.1. *Let there exist a nonnegative number r and an integrable function $g : [a,b] \to R$ such that*

$$f(t,x) \operatorname{sgn} x \ge g(t) \quad \text{for} \quad a < t < b, \quad |x| > r \tag{1.4}$$

and

$$\Big| \int_a^b f_0(s)\, ds + c_1 - c_2 \Big| \le \int_a^b g(s)\, ds. \tag{1.5}$$

Then the problem $(1.1), (1.2)$ has at least one solution.

THEOREM 1.2. *Let*

$$f(t,x) - f(t,y) > 0 \quad \text{for} \quad a < t < b, \quad x > y \tag{1.6}$$

and for some $r \geq 0$ *the inequality* (1.5) *be fulfilled, where*

$$g(t) = \min\{f(t,r), |f(t,-r)|\}. \tag{1.7}$$

Then the problem (1.1), (1.2) *has one and only one solution.*

To formulate the next theorem, we need the following definition.

DEFINITION 1.1. *We say that the integrable function* $p : [a,b] \to [0,+\infty[$ *belongs to the set* $\mathcal{U}_N([a,b])$ *if for an arbitrary integrable function* $p_0 : [a,b] \to R$, *satisfying the inequality*

$$0 \leq p_0(t) \leq p(t) \quad \text{for} \quad a \leq t \leq b, \tag{1.8}$$

the boundary value problem

$$u'' + p_0(t)u = 0; \quad u'(a) = 0, \quad u'(b) = 0 \tag{1.9}$$

has no nontrivial solution with alternating signs.

THEOREM 1.3. *Let there exist a nonnegative number* r, *the function*

$$p \in \mathcal{U}_N([a,b]) \tag{1.10}$$

and integrable functions $q : [a,b] \to [0,+\infty[$ *and* $g : [a,b] \to R$ *such that*

$$-p(t)|x| - q(t) \leq f(t,x)\operatorname{sgn} x \leq -g(t) \quad \text{for} \quad a < t < b, \quad |x| > r \tag{1.11}$$

and the inequality (1.5) *is fulfilled. Then the problem* (1.1), (1.2) *has at least one solution.*

COROLLARY 1.1. *Let there exist a nonnegative number* r *and integrable functions* $g : [a,b] \to R$, $p : [a,b] \to [0,+\infty[$, *and* $q : [a,b] \to [0,+\infty[$ *such that the inequalities* (1.5) *and* (1.11) *are fulfilled. Let, moreover, either*

$$\int_a^b p(s)\,ds \leq \frac{4}{b-a}, \tag{1.12}$$

or

$$p(t) \leq \frac{\pi^2}{(b-a)^2}, \quad p(t) \not\equiv \frac{\pi^2}{(b-a)^2}. \tag{1.13}$$

Then the problem (1.1), (1.2) *has at least one solution.*

THEOREM 1.4. *Let*

$$-p(t)(x-y) \le f(t,x) - f(t,y) < 0 \ \ for \ \ a<t<b, \ \ x>y, \tag{1.14}$$

where the function p *satisfies the condition* (1.10). *Let, moreover, for some* $r \ge 0$ *the inequality* (1.5) *be fulfilled, where*

$$g(t) = \min\{|f(t,r)|, f(t,-r)\}. \tag{1.15}$$

Then the problem (1.1), (1.2) *has one and only one solution.*

COROLLARY 1.2. *Let the condition* (1.14) *be fulfilled, where* $p : [a,b] \to [0,+\infty[$ *is a function, satisfying either the inequality* (1.12), *or the inequalities* (1.13). *Let, moreover, for some* $r \ge 0$ *the inequality* (1.5) *be fulfilled, where* g *is the function, given by the equality* (1.15). *Then the problem* (1.1), (1.2) *has one and only one solution.*

In the above-given theorems and their corollaries the condition (1.5) is, in a certain sense, optimal. Moreover, the following propositions are valid.

THEOREM 1.5. *Let there exist numbers* $\sigma \in \{-1,1\}$, $r>0$, *and an integrable function* $g : [a,b] \to [0,+\infty[$ *such that*

$$|f(t,x)| \le g(t) \ \ for \ \ |x| \le r, \quad f(t,x) = \sigma g(t)\operatorname{sgn} x \ \ for \ \ |x|>r. \tag{1.16}$$

Then for the solvability of the problem (1.1), (1.2) *it is necessary and sufficient that the inequality* (1.5) *be fulfilled.*

THEOREM 1.6. *Let the condition* (1.6) (*the conditions* (1.10) *and* (1.14)) *be fulfilled, and*

$$f(t,-x) = -f(t,x) \ \ for \ \ a<t<b, \ \ x \in R. \tag{1.17}$$

Let, moreover, there exist an integrable function $f^* : [a,b] \to [0,+\infty[$ *such that*

$$\lim_{x\to+\infty} |f(t,x)| = f^*(x) \ \ for \ \ a<t<b. \tag{1.18}$$

Then for the unique solvability of the problem (1.1), (1.2) *it is necessary and sufficient that*

$$\Big|\int_a^b f_0(s)\,ds + c_1 - c_2\Big| < \int_a^b f^*(s)\,ds. \tag{1.19}$$

Let $p(t) > 0$ for $a < t < b$, and the condition (1.12) (the condition (1.13)) be fulfilled. Then by virtue of Corollary 1.2, the differential equation

$$u'' = -p(t)u + f_0(t) \tag{1.20}$$

under the boundary conditions (1.2) has a unique solution, no matter how $c_i \in R$ $(i = 1, 2)$ and the integrable function $f_0 : [a, b] \to R$ are. Suppose now that

$$\alpha = \frac{2}{b-a}, \quad \varepsilon > 0, \quad \beta = 1 + \frac{8}{\varepsilon},$$

$$p(t) = \alpha^2(1-\alpha(t-a))^{\beta-2}(\beta+1-(1-\alpha(t-a))^\beta) \text{ for } a \le t \le \frac{a+b}{2},$$

$$p(t) = p(b+a-t) \text{ for } \frac{a+b}{2} < t \le b,$$

$$u_0(t) = (1-\alpha(t-a))\exp\big(-\beta^{-1}(1-\alpha(t-a))^\beta\big) \text{ for } a \le t \le \frac{a+b}{2},$$

$$u_0(t) = -u_0(b+a-t) \text{ for } \frac{a+b}{2} < t \le b.$$

Then the homogeneous problem

$$u'' + p(t)u = 0; \quad u'(a) = u'(b) = 0$$

has a nontrivial solution $u(t) = u_0(t)$, while the inhomogeneous problem (1.20), (1.2) has no solution, if only

$$c_2 - c_1 \neq \int_a^b f_0(s)u_0(s)\,ds.$$

On the other hand, in this case we have

$$\int_a^b p(s)\,ds = 2\int_a^{\frac{b+a}{2}} p(s)\,ds < 2\alpha^2(\beta+1)\int_a^{\frac{a+b}{2}} (1-\alpha(s-a))^{\beta-2}\,ds = \frac{4+\varepsilon}{b-a}.$$

Remark 1.1. *The above-constructed example shows that the inequality* (1.12) *in Corollaries* 1.1 *and* 1.2 *cannot be replaced by the inequality*

$$\int_a^b p(s)\,ds < \frac{4+\varepsilon}{b-a}$$

no matter how small $\varepsilon > 0$ *is. In the same corollaries, the inequalities* (1.13) *cannot be replaced by the identity*

$$p(t) \equiv \frac{\pi^2}{(b-a)^2},$$

since in this case the problem (1.20), (1.2) *has no solution, if only*

$$c_2 - c_1 \neq \int_a^b f_0(s) \cos \frac{\pi(s-a)}{b-a}\, ds.$$

Finally, as an example let us consider the following differential equations

$$u'' = \sigma f_1(t) \frac{|u|^\alpha}{1+|u|^\beta} \operatorname{sgn} u + f_0(t) \tag{1.21}$$

and

$$u'' = \sigma f_1(t) \frac{|u|^\alpha |\sin u|}{1+|u|^\beta} \operatorname{sgn} u + f_0(t), \tag{1.22}$$

where $\alpha > 0$, $\beta \geq 0$, $\sigma \in \{-1, 1\}$, and $f_i : [a,b] \to R$ $(i = 1,2)$ are integrable functions.

From Theorem 1.6 we have

Corollary 1.3. *Let*

$$\alpha = \beta, \quad f_1(t) > 0 \;\; \textit{for} \;\; a < t < b,$$

and either $\sigma = 1$, *or*

$$\sigma = -1, \;\; \alpha \geq 1, \quad \alpha \int_a^b f_1(s)\, ds \leq \frac{4}{b-a}\,.$$

Then for the unique solvability of the problem (1.21), (1.2) *it is necessary and sufficient that*

$$\Big| \int_a^b f_0(s)\, ds + c_1 - c_2 \Big| < \int_a^b f_1(s)\, ds.$$

Corollary 1.4. *Let*

$$\alpha > \beta, \quad f_1(t) > 0 \;\; \textit{for} \;\; a < t < b$$

and either $\sigma = 1$, *or*

$$\sigma = -1, \quad 1 \le \alpha \le \beta + 1, \quad \alpha \int\limits_a^b f_1(s)\,ds \le \frac{4}{b-a}\,.$$

Then the problem (1.21), (1.2) *has one and only one solution.*

From Theorem 1.1 and Corollary 1.1 it follows

COROLLARY 1.5. *Let*

$$f_1(t) \ge 0 \;\; \text{for} \;\; a < t < b$$

and

$$\int\limits_a^b f_0(s)\,ds = c_2 - c_1. \tag{1.23}$$

Let, moreover, either $\sigma = 1$, *or* $\sigma = -1$ *and* $\alpha < \beta + 1$, *or*

$$\sigma = -1, \quad \alpha = \beta + 1, \quad \alpha \int\limits_a^b f_1(s)\,ds \le \frac{4}{b-a}\,.$$

Then the problem (1.22), (1.2) *has at least one solution.*

Note that if $f_1(t) \equiv 0$, then the problem (1.22), (1.2) is solvable if and only if the equality (1.23) is fulfilled.

§ 2. Auxiliary Propositions.

By C (by C^1) we denote the space of continuous (continuously differentiable) functions $u : [a,b] \to R$ with the norm

$$\|u\|_C = \max\{|u(s)| : \; a \le s \le b\} \quad (\,\|u\|_{C^1} = \|u\|_C + \|u'\|_C\,),$$

and by L we denote the space of Lebesgue integrable functions $v : [a,b] \to R$ with the norm

$$\|v\|_L = \int\limits_a^b |v(s)|\,ds.$$

For an arbitrary $u \in C$, we set

$$\mu(u) = \min\{|u(s)| : \; a \le s \le b\}.$$

2.1. Lemmas on a priori estimates. Along with (1.1), we have to consider the differential inequalities

$$u''(t)\operatorname{sgn} u(t) \geq -q(t) \tag{2.1}$$

and

$$-p(t)|u(t)| - q(t) \leq u''(t)\operatorname{sgn} u(t) \leq q(t), \tag{2.2}$$

where p and $q : [a,b] \to [0,+\infty[$ are the Lebesgue integrable functions.

The function $u : [a,b] \to R$ is said to be a solution of the differential inequality (2.1) (differential inequality (2.2)) if it is absolutely continuous together with u' and satisfies that inequality almost everywhere on $[a,b]$.

LEMMA 2.1. *An arbitrary solution u of the differential inequality* (2.1) *admits the estimate*

$$\|u\|_C \leq (2+b-a)\Big(\mu(u) + |u'(a)| + |u'(b)| + \|q\|_L\Big). \tag{2.3}$$

Proof. According to (2.1), almost everywhere on $[a,b]$ the inequality

$$-u''(t)u(t) \leq q(t)|u(t)|$$

is fulfilled.

Integrating both parts of the above inequality from a to b, we obtain

$$u'(a)u(a) - u'(b)u(b) + \rho^2 \leq \int\limits_a^b q(t)|u(t)|\,dt, \tag{2.4}$$

where

$$\rho = \Big(\int\limits_a^b u'^2(t)\,dt\Big)^{1/2}.$$

On the other hand, it is clear that

$$\|u\|_C \leq \mu(u) + \int\limits_a^b |u'(s)|\,ds \leq \mu(u) + (b-a)^{1/2}\rho. \tag{2.5}$$

Therefore,

$$|u'(b)u(b) - u(a)u'(a)| \leq (|u'(a)| + |u'(b)|)\Big(\mu(u) + (b-a)^{1/2}\rho\Big),$$

$$\int\limits_a^b q(t)|u(t)|\,dt \leq \Big(\mu(u) + (b-a)^{1/2}\rho\Big)\|q\|_L.$$

Taking into account the above estimates, the inequality (2.4) yields

$$\rho^2 \leq \Big(|u'(a)|+|u'(b)|+\|q\|_L\Big)\mu(u)+\Big(|u'(a)|+|u'(b)|+\|q\|_L\Big)(b-a)^{1/2}\rho.$$

However,

$$\Big(|u'(a)|+|u'(b)|+\|q\|_L\Big)(b-a)^{1/2}\rho \leq \frac{b-a}{2}\,\Big(|u'(a)|+|u'(b)|+\|q\|_L\Big)^2+\frac{\rho^2}{2}\,.$$

Thus from the above inequality it follows that

$$\rho^2 \leq 2\Big(|u'(a)| + |u'(b)| + \|q\|_L\Big)\mu(u) + (b-a)\Big(|u'(a)| + |u'(b)| + \|q\|_L\Big)^2 \leq$$
$$\leq \Big[(b-a)^{-\frac{1}{2}}\mu(u) + (b-a)^{1/2}\Big(|u'(a)| + |u'(b)| + \|q\|_L\Big)\Big]^2$$

and, consequently,

$$\rho \leq (b-a)^{-\frac{1}{2}}\mu(u) + (b-a)^{1/2}\Big(|u'(a)| + |u'(b)| + \|q\|_L\Big).$$

According to this estimate, (2.5) results in the estimate (2.3). $\square$

Lemma 2.2. *If the condition* (1.10) *be fulfilled, then there exists a positive number ρ_0 such that for an arbitrary nonnegative function $q \in L$ every solution u of the differential inequality* (2.2) *admits the estimate*

$$\|u\|_{C^1} \leq \rho_0\Big(\mu(u) + |u'(a)| + |u'(b)| + \|q\|_L\Big). \tag{2.6}$$

Proof. Assume that the above lemma is not true. Then for an arbitrary natural number k there exist a nonnegative function $q_k \in L$ and a solution u_k of the differential inequality

$$0 \leq -u_k''(t)\operatorname{sgn} u_k(t) + q_k(t) \leq p(t)|u_k(t)| + 2q_k(t) \tag{2.7}$$

such that

$$\|u_k\|_{C^1} > k\Big(\mu(u_k) + |u_k'(a)| + |u_k'(b)| + \|q_k\|_L\Big). \tag{2.8}$$

Let

$$u_{0k}(t) = \frac{u_k(t)}{\|u_k\|_{C^1}}\,, \quad q_{0k}(t) = \frac{q_k(t)}{\|u_k\|_{C^1}}\,, \quad \delta_k(t) = p(t)|u_{0k}(t)| + 2q_{0k}(t),$$
$$\eta_k(t) = \begin{cases} 0 & \text{for } \ \delta_k(t) = 0 \\ (-u_{0k}''(t)\operatorname{sgn} u_{0k}(t) + q_{0k}(t))/\delta_k(t) & \text{for } \ \delta_k(t) = 0 \end{cases},$$
$$p_k(t) = \eta_k(t)p(t), \quad \mathcal{P}_k(t) = \int_a^t p_k(s)\,ds,$$
$$q_{1k}(t) = (1 - 2\eta_k(t))q_{0k}(t)\operatorname{sgn} u_{0k}(t).$$

Then by virtue of the inequalities (2.7) and (2.8) we have

$$0 \le \eta_k(t) \le 1 \ \text{ for almost all } \ t \in [a,b],$$

$$\|u_{0k}\|_{C^1} = 1, \tag{2.9}$$

$$\mu(u_{0k}) + |u'_{0k}(a)| + |u'_{0k}(b)| < \frac{1}{k}\,, \tag{2.10}$$

$$\|q_{1k}\|_L < \frac{1}{k}\,. \tag{2.11}$$

Moreover, it is clear that for every natural number k the function u_{0k} is a solution of the differential equation

$$u''_{0k}(t) + p_k(t)u_{0k}(t) = q_{1k}(t), \tag{2.12}$$

and the function $\mathcal{P}_k$ satisfies the conditions

$$\mathcal{P}_k(0) = 0, \quad 0 \le \mathcal{P}_k(t) - \mathcal{P}_k(\tau) \le \int_\tau^t p(s)\,ds \ \text{ for } \ a \le \tau \le t \le b. \tag{2.13}$$

By (2.9) and (2.11), from (2.13) we have

$$\|u''_{0k}\|_L \le \|p\|_L + 1. \tag{2.14}$$

On the other hand, by the Arzela–Ascoli lemma and the conditions (2.10), (2.13) and (2.14), without loss of generality we can assume that the sequences $(u_k)_{k=1}^{+\infty}$, $(u'_k)_{k=1}^{+\infty}$ and $(\mathcal{P}_k)_{k=1}^{+\infty}$ are uniformly convergent on $[a,b]$, i.e., there exist $u \in C^1$ and $\mathcal{P}_0 \in C$ such that

$$\lim_{k\to+\infty} \|u_{0k} - u\|_{C^1} = 0, \quad \lim_{k\to+\infty} \|\mathcal{P}_k - \mathcal{P}_0\|_C = 0. \tag{2.15}$$

Thus from (2.9), (2.10), and (2.13) we find

$$\|u\|_{C^1} = 1, \tag{2.16}$$

$$\mu(u) = 0, \quad u'(a) = 0, \ \ u'(b) = 0, \tag{2.17}$$

$$\mathcal{P}_0(a) = 0, \quad 0 \le \mathcal{P}_0(t) - \mathcal{P}_0(\tau) \le \int_\tau^t p(s)\,ds \ \text{ for } \ a \le \tau \le t \le b. \tag{2.18}$$

By the conditions (2.18), the function $\mathcal{P}_0$ is absolutely continuous and admits the representation

$$\mathcal{P}_0(t) = \int_a^t p_0(s)\,ds \ \text{ for } \ a \le t \le b, \tag{2.19}$$

where $p_0 \in L$ is a function, satisfying the inequality (1.8).

By virtue of Lemma 1.1 of [4] and the conditions (2.15) and (2.19), we have

$$\lim_{k\to+\infty} \int_a^t p_k(s)\, u_{0k}(s)\, ds = \int_0^t p_0(s) u(s)\, ds \ \text{ for } \ t \in [a,b].$$

If along with this we take into account the conditions (2.10) and (2.11), then from the representation

$$u'_{0k}(t) = u'_{0k}(a) - \int_a^t (p_k(s)u_{0k}(s) - q_{1k}(s))\, ds \ \text{ for } \ a \le t \le b$$

we find

$$u'(t) = -\int_a^t p_0(s)u(s)\, ds \ \text{ for } \ a \le t \le b.$$

Consequently, u is a solution of the equation

$$u'' + p_0(t)u = 0,$$

satisfying the conditions (2.16) and (2.17). Thus it is clear that u is a solution with alternating sings of the problem (1.9). But this contradicts the condition (1.10) since p_0 satisfies the inequality (1.8). The obtained contradiction proves the lemma. □

2.2. Lemmas on the set $\mathcal{U}_N([a,b])$.

LEMMA 2.3. *Let $p \in L$ be a nonnegative function, satisfying either the inequality* (1.12)*, or the inequalities* (1.13)*. Then the condition* (1.10) *is fulfilled.*

Proof. Let $p_0 \in L$ be an arbitrary function, satisfying the inequality (1.8). We have to show that the problem (1.9) has no solution with alternating sings.

Assume the contrary that the above-mentioned problem has a solution u with alternating signs. Then there exist numbers t_0, t_1 and t_2 such that $a \le t_1 < t_0 < t_2 \le b$,

$$u(t_0) = 0, \quad u'(t) < 0 \ \text{ for } \ t_1 < t < t_2, \quad u'(t_1) = u'(t_2) = 0, \tag{2.20}$$
$$\int_{t_1}^{t_0} p_0(s)\, ds > 0, \quad \int_{t_0}^{t_2} p_0(s)\, ds > 0.$$

If along with this we take into account the fact that p_0 is nonnegative, then it becomes clear that

$$\max\{|u(t)|:\ t_1 \le t \le t_0\} = u(t_1) = \int_{t_1}^{t_0} |u'(s)|\,ds < (t_0 - t_1)|u'(t_0)|,$$

$$\max\{|u(t)|:\ t_0 \le t \le t_1\} = |u(t_2)| = \int_{t_0}^{t_2} |u'(s)|\,ds < (t_2 - t_0)|u'(t_0)|.$$

Therefore,

$$|u'(t_0)| = \int_{t_1}^{t_0} p_0(s)|u(s)|\,ds \le |u(t_1)|\int_{t_1}^{t_0} p_0(s)\,ds < (t_0 - t_1)|u'(t_0)|\int_{t_1}^{t_0} p_0(s)\,ds,$$

$$|u'(t_0)| = \int_{t_0}^{t_2} p_0(s)|u(s)|\,ds < |u(t_2)|\int_{t_0}^{t_2} p_0(s)\,ds < (t_2 - t_0)|u'(t_0)|\int_{t_0}^{t_2} p_0(s)\,ds.$$

It follows from the above estimates that

$$\int_{t_1}^{t_0} p_0(s)\,ds > \frac{1}{t_0 - t_1}\,, \quad \int_{t_0}^{t_2} p_0(s)\,ds > \frac{1}{t_2 - t_0},$$

and

$$\int_{t_1}^{t_2} p(s)\,ds = \int_{t_1}^{t_0} p_0(s)\,ds + \int_{t_0}^{t_2} p_0(s)\,ds >$$

$$> \frac{1}{t_0 - t_1} + \frac{1}{t_2 - t_0} = \frac{t_2 - t_1}{(t_0 - t_1)(t_2 - t_0)} \ge \frac{4}{t_2 - t_1} \ge \frac{4}{b - a}\,.$$

Thus the inequality (1.12) is violated.

It remains to consider the case where the inequalities (1.13) are fulfilled. Obviously, either $t_1 - t_0 \le (b-a)/2$, or $t_2 - t_0 \le (b-a)/2$. For the definiteness, we assume that $t_1 - t_0 \le (b-a)/2$. Then by (1.13) we have

$$p(t) \le \frac{\pi^2}{4(t_1 - t_0)^2} \ \text{ for } \ t_1 \le t \le t_0, \quad p(t) \not\equiv \frac{\pi^2}{4(t_1 - t_0)^2}\,.$$

In view of the Sturm lemma and the conditions (2.20), it follows that the boundary value problem

$$v'' + \frac{\pi^2}{4(t_0 - t_1)^2}\,v = 0; \quad v'(t_1) = 0, \ \ v(t_0) = 0$$

has no positive on $]t_0, t_1[$ solution. But this is not the case because this problem has the solution $v(t) = \cos\frac{\pi(t-t_1)}{2(t_0-t_1)}$. The obtained contradiction proves the lemma. □

LEMMA 2.4. *If the condition* (1.10) *is fulfilled, then there exists* $\varepsilon > 0$ *such that an arbitrary function* $\overline{p} \in L$*, satisfying the inequalities*

$$p(t) \le \overline{p}(t) \text{ for } a < t < b, \quad \int_a^b (\overline{p}(s) - p(s))\,ds < \varepsilon, \tag{2.21}$$

satisfies the condition

$$\overline{p} \in \mathcal{U}_N([a,b]).$$

Proof. Let ρ_0 be the number appearing in Lemma 2.2,

$$\varepsilon = \frac{1}{\rho_0}, \tag{2.22}$$

and $p_0 \in L$ and $\overline{p} \in L$ be arbitrary functions, satisfying the inequalities (2.21) and

$$0 \le p_0(t) \le \overline{p}(t) \text{ for } a < t < b. \tag{2.23}$$

We have to show that the problem (1.9) has no nontrivial solution satisfying the equality

$$\mu(u) = 0. \tag{2.24}$$

Assume the contrary that there exists a nontrivial solution u of the problem (1.9), satisfying the equality (2.24). Then u is likewise a solution of the differential inequality (2.2), where

$$q(t) = (\overline{p}(t) - p(t))|u(t)|.$$

Hence, owing to Lemma 2.2, we find

$$0 < \|u\|_C \le \rho_0 \int_a^b (\overline{p}(s) - p(s))|u(s)|\,ds.$$

This inequality by virtue of (2.21) and (2.22) yields

$$\|u\|_C \le \|u\|_C \rho_0 \int_a^b (\overline{p}(s) - p(s))\,ds < \|u\|_C.$$

The obtained contradiction proves the lemma. □

2.3. Lemmas on the auxiliary boundary value problems. Along with (1.1), (1.2), let us consider the auxiliary boundary value problems

$$u'' = \ell(t)u; \quad u'(a) = 0, \ \ u'(b) = 0 \tag{2.25}$$

and

$$u'' = (1-\lambda)\ell(t)u + \lambda f(t,u) + \lambda f_0(t), \tag{2.26}$$

$$u'(a) = \lambda c_1, \ \ u'(b) = \lambda c_2, \tag{2.27}$$

where $\ell \in L$, and $\lambda \in]0,1[$ is a parameter.

Lemma 2.5. *Let either*

$$\ell(t) \geq 0 \ \textit{ for } \ a \leq t \leq b, \quad \ell(t) \not\equiv 0, \tag{2.28}$$

or

$$\ell(t) = -p(t) \leq 0 \ \textit{ for } \ a \leq t \leq b, \quad \ell(t) \not\equiv 0, \tag{2.29}$$

and the condition (1.10) *be fulfilled. Then the problem* (2.25) *has only a trivial solution.*

Proof. Let u be an arbitrary solution of the problem (2.25). Then since ℓ is of alternating signs, we have

$$\int_a^b |\ell(s)|u(s)\,ds = 0.$$

If we assume that $\mu(u) > 0$, the above equality yields

$$0 = \int_a^b |\ell(s)|\,|u(s)|\,ds \geq \mu(u)\int_a^b |\ell(s)|\,ds > 0 \ \text{ since } \ \ell(t) \not\equiv 0.$$

The obtained contradiction proves that $\mu(u) = 0$.

Assume now that the inequalities (2.28) are fulfilled. Then u is a solution of the differential inequality

$$u''(t)\operatorname{sgn} u(t) \geq 0,$$

satisfying the conditions (2.17), whence by Lemma 2.1 it follows that $u(t) \equiv 0$.

Consider now the case where the conditions (1.10) and (2.29) are fulfilled. Then u is a solution of the differential inequality

$$-p(t)|u(t)| \leq u''(t)\operatorname{sgn} u(t) \leq 0,$$

whence by Lemma 2.2 it follows that $u(t) \equiv 0$.

Thus we have proved that the problem (2.25) has only a trivial solution. □

From Corollary 2 of [7] we have the following

Lemma 2.6. *Let the problem* (2.25) *have only a trivial solution, and there exist a positive constant ρ such that for an arbitrary $\lambda \in]0,1[$, every solution u of the problem* (2.26), (2.27) *admits the estimate*

$$\|u\|_{C^1} \le \rho. \tag{2.30}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Lemma 2.7. *Let there exist numbers $\sigma \in \{-1,1\}$, $r \ge 0$, and an integrable function $g : [a,b] \to R$ such that along with* (1.5) *the inequalities*

$$\sigma\ell(t) > 0, \quad \sigma f(t,x)\operatorname{sgn} x \ge g(t) \ \text{ for } \ a < t < b, \ \ |x| \ge r \tag{2.31}$$

are fulfilled. Then for every $\lambda \in]0,1[$ an arbitrary solution u of the problem (2.26), (2.27) *admits the estimate*

$$\mu(u) \le r. \tag{2.32}$$

Proof. Assume the contrary that the lemma is not true. Then there exist $\lambda \in]0,1[$ and a solution u of the problem (2.26), (2.27) such that $\mu(u) > r$.

If we assume $\sigma_0 = \sigma \operatorname{sgn} u(a)$, then in view of (2.31), from (2.26) we find

$$\sigma_0(u''(t) - \lambda f_0(t)) \ge (1-\lambda)\mu(u)\ell(t) + \lambda g(t).$$

Integrating the above inequality from a to b, according to (2.27) we obtain

$$\sigma_0\lambda\Big(c_2 - c_1 - \int_a^b f_0(s)\,ds\Big) \ge$$

$$> (1-\lambda)\mu(u)\int_a^b \ell(s)\,ds + \lambda\int_a^b g(s)\,ds > \lambda\int_a^b g(s)\,ds.$$

But this inequality contradicts the inequality (1.5). The obtained contradiction proves the lemma. □

§ 3. Proof of the Main Results.

Proof of Theorem 1.1. Let

$$f^*(t,y) = \max\{|f(t,x)|:\ |x|\le y\} + |f_0(t)| \ \text{ for } \ y\ge 0, \tag{3.1}$$

$$q(t) = f^*(r,t) + |g(t)|, \tag{3.2}$$

$$\rho_0 = (2+b-a)\Big(r+|c_1|+|c_2|+\|q\|_L\Big), \tag{3.3}$$

$$\rho = \rho_0 + |c_1| + \int\limits_a^b (1+f^*(s,\rho_0))\,ds, \tag{3.4}$$

and

$$\ell(t)\equiv 1. \tag{3.5}$$

Then by Lemma 2.5, the problem (2.25) has only a trivial solution. In view of that fact and Lemma 2.6, to prove the theorem it suffices to state that for every $\lambda\in]0,1[$ an arbitrary solution u of the problem (2.26), (2.27) admits the estimate (2.30).

By virtue of Lemma 2.7, the conditions (1.4), (1.5), and (3.5) guarantee the validity of the estimate (2.32). On the other hand, according to the notation (3.1) and (3.2), the function u is a solution of the differential inequality (2.1), and

$$|u''(t)| \le 1 + f^*(t,\|u\|_C) \tag{3.6}$$

almost everywhere on $[a,b]$.

By Lemma 2.1 and the conditions (2.27) and (2.32), we have

$$\|u\|_C \le \rho_0,$$

where ρ_0 is a number given by the equality (3.3). Taking into account this estimate and notation (3.4), from (3.6) we obtain

$$\|u'\|_{C^1} \le |c_1| + \int\limits_a^b |u''(s)|\,ds \le |c_1| + \int\limits_a^b (1+f^*(s,\rho_0))\,ds = \rho-\rho_0.$$

Consequently, the estimate (2.30) is valid. □

Proof of Theorem 1.2. (1.3), (1.6), and (1.7) result in the inequality (1.4). Thus all the conditions of Theorem 1.1 are fulfilled which guarantee the solvability of the problem (1.1), (1.2).

It remains to prove that the problem (1.1), (1.2) has no more than one solution. Let u_1 and u_2 be arbitrary solutions of that problem, and $u(t) = u_2(t) - u_1(t)$. Assuming $\mu(u) > 0$, from (1.6) we have

$$u''(t)\sigma_0 > 0 \;\text{ almost everywhere on }\; [a,b],$$

where $\sigma_0 = \operatorname{sgn} u(0)$. But this contradicts the equalities $u'(a) = u'(b) = 0$. The obtained contradiction proves that $\mu(u) = 0$.

By the condition (1.6), the function u is a solution of the differential inequality

$$u''(t)\operatorname{sgn} u(t) \geq 0.$$

Hence, by Lemma 2.1 and the condition (2.17), it follows that $u(t) \equiv 0$, i.e., $u_1(t) \equiv u_2(t)$. □

Proof of Theorem 1.3. Due to Lemma 2.4 and the condition (1.11), without loss of generality we can assume that

$$p(t) > 0 \;\text{ for }\; a < t < b, \tag{3.7}$$

$$f(t,x)\operatorname{sgn} x \leq -g(t) \;\text{ for }\; a < t < b, \;\; |x| \geq r, \tag{3.8}$$

and

$$-p(t)|x| - q(t) \leq \big(f(t,x) + f_0(t)\big)\operatorname{sgn} x \leq q(t) \;\text{ for }\; a < t < b, \;\; x \in R. \tag{3.9}$$

By the condition (1.10) and Lemma 2.2, there exists a positive constant ρ_0 such that an arbitrary solution of the differential inequality (2.2) admits the estimate (2.6). We put

$$\rho = \rho_0\Big(r + |c_1| + |c_2| + \|q\|_L\Big). \tag{3.10}$$

Let

$$\ell(t) = -p(t). \tag{3.11}$$

Then by the conditions (1.10), (3.7) and Lemma 2.5, the problem (2.25) has only a trivial solution. Taking this fact and Lemma 2.6 into account, to prove Theorem 1.3 it suffices to state that for every $\lambda \in]0,1[$ an arbitrary solution u of the problem (2.26), (2.27) admits the estimate (2.30).

By Lemma 2.7, the conditions (1.5), (3.7), (3.8), and (3.11) guarantee the validity of the estimate (2.32). On the other hand, in view of the conditions (3.9) and (3.11), it follows from (2.26) that u is a solution of the differential inequality (2.2). Therefore this function admits, as is said above, the estimate (2.6). However, this estimate, due to the conditions (2.27), (2.32) and (3.10), results in the estimate (2.30). □

Proof of Theorem 1.4. From (1.3), (1.14), and (1.15) we have the inequality (1.11). However, by Theorem 1.3, this inequality together with the conditions (1.5) and (1.10) guarantees the solvability of the problem (1.1), (1.2). It remains to prove that this problem has no more than one solution.

Let u_1 and u_2 be arbitrary solutions of the problem (1.1), (1.2), and $u(t) = u_2(t) - u_1(t)$. If we assume that $\mu(u) > 0$, then in view of (1.14), almost everywhere on $[a, b]$ we have $\sigma_0 u''(t) < 0$, where $\sigma_0 = \operatorname{sgn} u(a)$. But this is impossible since $u'(a) = u'(b) = 0$. The obtained contradiction proves that $\mu(u) = 0$.

By the condition (1.14), the function u is a solution of the differential inequality

$$-p(t)|u(t)| \le u''(t) \operatorname{sgn} u(t) \le 0,$$

whence by Lemma 2.2 it follows that $u(t) \equiv 0$, i.e., $u_1(t) \equiv u_2(t)$. □

Corollary 1.1 (Corollary 1.2) follows from Theorem 1.3 (Theorem 1.4) and Lemma 2.3.

Proof of Theorem 1.5. First, let us prove the necessity. Let the problem (1.1), (1.2) have a solution u. Then

$$c_2 - c_1 = \int_a^b f(s, u(s))\, ds + \int_a^b f_0(s)\, ds. \tag{3.12}$$

Hence, due to (1.6), we get

$$\Big| \int_a^b f_0(s)\, ds + c_1 - c_2 \Big| \le \int_a^b |f(s, u(s))|\, ds \le \int_a^b g(s)\, ds,$$

i.e., (1.5) is fulfilled.

It remains to prove that if along with (1.16) the condition (1.5) is fulfilled, then the problem (1.1), (1.2) is solvable.

Owing to (1.16), in the case $\sigma = 1$ (in the case $\sigma = -1$), the function f satisfies the inequality (1.4) (the inequality (1.11), where $p(t) \equiv 0$, $q(t) \equiv g(t)$). However, by virtue of Theorem 1.1 (of Corollary 1.1), this inequality together with the condition (1.5) guarantees the solvability of the problem (1.1), (1.2). □

Proof of Theorem 1.6. First, let us prove the necessity. Let the problem (1.1), (1.2) have a solution u. Then the equality (3.12) is valid. On the other hand, by the conditions (1.6), (1.17), and (1.18) (by the conditions (1.14), (1.17), and (1.18)), we have

$$|f(t,x)| < f^*(t) \text{ for } a < t < b, \quad x \in R.$$

According to that estimate, from (3.12) it follows the inequality (1.19).

Now we prove the sufficiency. Let along with (1.16), (1.17), and (1.18) (along with (1.10), (1.14), (1.17), and (1.18)) the condition (1.19) be fulfilled. Then

$$\lim_{x \to +\infty} \int_a^b |f(s,x)|\, ds = \int_a^b f^*(t)\, dt.$$

Therefore there exists $r > 0$ such that the function $g(t) = |f(t,r)| = |f(t,-r)|$ satisfies the inequality (1.5). However, by Theorem 1.2 (by Theorem 1.4), the inequality (1.5) together with the condition (1.6) (together with the conditions (1.10) and (1.14)) guarantees the solvability of the problem (1.1), (1.2).

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