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Periodic solutions of nonautonomous ordinary differential equations

I. Kiguradze · A. Lomtatidze

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Abstract For higher order ordinary differential equations, new sufficient conditions on the existence and uniqueness of periodic solutions are established. Results obtained cover the case when the right-hand side of the equation is not of a constant sign with respect to an independent variable.

Keywords Nonautonomous differential equation · Periodic solution · Existence · Uniqueness

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Introduction

In the present paper, for a higher order nonautonomous ordinary differential equation we investigate the problem on the existence of a periodic solution with a prescribed

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I. Kiguradze (✉)
A. Razmadze Mathematical Institute, 1, Aleksidze Str., 0193 Tbilisi, Georgia
e-mail: kig@rmi.acnet.ge

A. Lomtatidze
Department of Mathematics and Statistics, Faculty of Sciences,
Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
e-mail: bacho@math.muni.cz

period. In Sect. 1, the optimal, in a certain sense, conditions are found guaranteeing the existence of a unique ω-periodic solution of the linear differential equation

$$u^{(n)} = p(t)u + q(t)$$

with ω-periodic coefficients $p, q\colon \mathbb{R} \to \mathbb{R}$. In spite of previously known results (see [1, 10, 13, 17]), they also cover the case when the function p is not of a constant sign. On the base of the results of Sect. 1, the sufficient conditions of the existence and uniqueness of an ω-periodic solution of the nonlinear equation

$$u^{(n)} = f\left(t, u, u', \dots, u^{(n-1)}\right)$$

are established in Sect. 2. Here we suppose that the function $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is ω-periodic with respect to a time variable and satisfies the conditions

$$\begin{aligned} p_1(t)|x_1| - \delta\left(t, \sum_{k=1}^{n} |x_k|\right) &\le f(t, x_1, x_2, \dots, x_n) \operatorname{sgn} x_1 \\ &\le p_2(t)|x_1| + \delta\left(t, \sum_{k=1}^{n} |x_k|\right), \end{aligned}$$

where $\delta\colon \mathbb{R} \times [0, +\infty[\to [0, +\infty[$ is a sublinear function with respect to the second variable. Moreover, in spite of previously known results (see [2–7], [11–20] and the references therein), we do not restrict signs of the functions p_1 and p_2.

Throughout the paper, we assume that $n \ge 2$ and $\omega > 0$. We also use the following notation.

$$[x]_+ = \frac{1}{2}(|x| + x), \quad [x]_- = \frac{1}{2}(|x| - x).$$

ζ is the Riemann zeta-function, i.e.,

$$\zeta(x) = \sum_{k=1}^{+\infty} \frac{1}{k^x} \quad \text{for} \quad x > 1.$$

L_ω is the space of all ω-periodic real functions which are Lebesgue integrable on $[0, \omega]$.

L^2_ω is the space of all ω-periodic real functions which are square Lebesgue integrable on $[0, \omega]$.

C_ω, resp. AC_ω, is the space of continuous, resp. absolutely continuous, ω-periodic functions $u\colon \mathbb{R} \to \mathbb{R}$,

$$\|u\|_{C_\omega} = \max\{|u(t)|\colon t \in [0, \omega]\}.$$

AC^k_ω denotes the space of ω-periodic functions $u\colon \mathbb{R} \to \mathbb{R}$ which are continuous together with their first k derivatives and $u^{(k)} \in AC_\omega$.

Z_ω is the set of all nondecreasing in the second argument functions $\delta\colon \mathbb{R} \times [0, +\infty[\to [0, +\infty[$ such that $\delta(\cdot, \varrho) \in L_\omega$ for $\varrho \geq 0$ and

$$\lim_{\varrho \to +\infty} \frac{1}{\varrho} \int_0^\omega \delta(t, \varrho)dt = 0.$$

If $p \in L_\omega$ and $\int_0^\omega p(t)dt \neq 0$, then

$$\gamma_0(p) = \left(1 + \frac{\int_0^\omega |p(t)|dt}{\left|\int_0^\omega p(t)dt\right|}\right)^2, \quad \gamma(p) = \gamma_0(p) \int_0^\omega |p(t)|dt.$$

If $p_1, p_2 \in L_\omega$ and $\int_0^\omega p_2(t)dt \neq 0$, then

$$\eta_0(p_1, p_2) = \left(1 + \frac{\int_0^\omega p_0(t)dt}{\left|\int_0^\omega p_2(t)dt\right|}\right)^2, \quad \eta(p_1, p_2) = \eta_0(p_1, p_2) \int_0^\omega p_0(t)dt,$$

where

$$p_0(t) = \frac{1}{2}\left(|p_1(t)| + |p_2(t)| + ||p_1(t)| - |p_2(t)||\right).$$

If $u \in L_\omega$, then the number c_0 defined by the relation

$$c_0 = \frac{1}{\omega} \int_0^\omega u(t)dt$$

is called the mean value of the function u.

For any $x, y \in L_\omega$, the writing $x(t) \not\equiv y(t)$ means that the functions x and y differ from each other on a set of positive measure.

Under the ω-periodic solution of the above-mentioned equations we understand a function $u \in AC^{n-1}_\omega$ which satisfies them almost everywhere on $\mathbb{R}$.

1 Linear problem

In this section, we will consider the equation

$$u^{(n)} = p(t)u + q(t), \tag{1.1}$$

where $p, q \in L_\omega$.

The following lemma is well-known from the general theory of linear boundary value problems (see, e.g., [9, Theorem 1.1]).

Lemma 1.1 *Equation* (1.1) *has a unique ω-periodic solution iff the corresponding homogeneous equation*

$$u^{(n)} = p(t)u \tag{1.1_0}$$

has no nontrivial ω-periodic solution.

Except of this we will need the next three lemmas.

Lemma 1.2 *Let ℓ be a natural number,*

$$u \in AC_{\omega}^{\ell-1}, \qquad u^{(\ell)} \in L_{\omega}^{2}, \tag{1.2}$$

and c_0 be the mean value of the function u. Then

$$\int_0^{\omega} |u(t) - c_0|^2 dt \leq \left(\frac{\omega}{2\pi}\right)^{2\ell} \int_0^{\omega} \left|u^{(\ell)}(t)\right|^2 dt \tag{1.3}$$

and

$$\|u - c_0\|_{C_{\omega}}^2 \leq \frac{\zeta(2\ell)}{\pi} \left(\frac{\omega}{2\pi}\right)^{2\ell-1} \int_0^{\omega} \left|u^{(\ell)}(t)\right|^2 dt. \tag{1.4}$$

Moreover, the equality

$$\int_0^{\omega} |u(t) - c_0|^2 dt = \left(\frac{\omega}{2\pi}\right)^{2\ell} \int_0^{\omega} \left|u^{(\ell)}(t)\right|^2 dt \tag{1.5}$$

holds if and only if

$$u(t) \equiv c_0 + c \sin \frac{2\pi}{\omega} (t - t_0) \tag{1.6}$$

for some $c, t_0 \in \mathbb{R}$, while the equality

$$\|u - c_0\|_{C_{\omega}}^2 = \frac{\zeta(2\ell)}{\pi} \left(\frac{\omega}{2\pi}\right)^{2\ell-1} \int_0^{\omega} \left|u^{(\ell)}(t)\right|^2 dt \tag{1.7}$$

is satisfied if and only if

$$u(t) \equiv c_0 + c \sum_{k=1}^{+\infty} \frac{1}{k^{2\ell}} \cos \frac{2k\pi}{\omega} (t - t_0) \tag{1.8}$$

for some $c, t_0 \in \mathbb{R}$.

Proof On account of (1.2), it is clear that

$$u(t) = c_0 + \sum_{k=1}^{+\infty} h_k(t) \quad \text{for} \quad t \in \mathbb{R} \tag{1.9}$$

and

$$u^{(\ell)}(t) = \left(\frac{2\pi}{\omega}\right)^{\ell} \sum_{k=1}^{+\infty} k^{\ell} h_k\left(t + \frac{\ell\omega}{4k}\right) \quad \text{for} \quad t \in \mathbb{R},$$

where

$$h_k(t) = c_{1k} \sin \frac{2k\pi}{\omega} t + c_{2k} \cos \frac{2k\pi}{\omega} t.$$

Hence, by virtue of Parseval's equality, we get

$$\int_0^{\omega} |u(t) - c_0|^2 dt = \frac{\omega}{2} \sum_{k=1}^{+\infty} \left(c_{1k}^2 + c_{2k}^2\right) \tag{1.10}$$

and

$$\int_0^{\omega} \left|u^{(\ell)}(t)\right|^2 dt = \frac{\omega}{2} \left(\frac{2\pi}{\omega}\right)^{2\ell} \sum_{k=1}^{+\infty} k^{2\ell} \left(c_{1k}^2 + c_{2k}^2\right). \tag{1.11}$$

Inequality (1.3) now immediately follows from (1.10) and (1.11). Moreover, equality (1.5) holds if and only if

$$c_{1k} = 0 \quad \text{and} \quad c_{2k} = 0 \quad \text{for} \quad k = 2, 3, \dots,$$

i.e., when

$$u(t) \equiv c_0 + c_{11} \sin \frac{2\pi t}{\omega} + c_{21} \cos \frac{2\pi t}{\omega}.$$

However, the latter identity is equivalent to (1.6) for a suitable choice of c and t_0.

Now we will prove inequality (1.4). Choose $t_0 \in [0, \omega]$ such that

$$\|u - c_0\|_{C_\omega} = |u(t_0) - c_0|.$$

By virtue of Hölder's inequality, it follows from (1.9) that

$$\|u - c_0\|_{C_\omega}^2 \le \zeta(2\ell) \sum_{k=1}^{+\infty} k^{2\ell} h_k^2(t_0). \tag{1.12}$$

Moreover, the equality

$$\|u - c_0\|^2_{C_\omega} = \zeta(2\ell)\sum_{k=1}^{+\infty} k^{2\ell} h_k^2(t_0)$$

holds if and only if there exists $c \in \mathbb{R}$ such that

$$h_k(t_0) = \frac{c}{k^{2\ell}} \quad \text{for} \quad k = 1, 2, \dots . \tag{1.13}$$

On the other hand,

$$h_k^2(t_0) = c_{1k}^2 + c_{2k}^2 - \left(c_{1k}\cos\frac{2k\pi t_0}{\omega} - c_{2k}\sin\frac{2k\pi t_0}{\omega}\right)^2.$$

Hence, from (1.11) and (1.12) we get (1.4). Moreover, equality (1.7) holds if and only if (1.13) and

$$c_{1k}\cos\frac{2k\pi}{\omega}t_0 - c_{2k}\sin\frac{2k\pi}{\omega}t_0 = 0 \quad \text{for} \quad k = 1, 2, \dots \tag{1.14}$$

are fulfilled. However, (1.13) and (1.14) imply that

$$c_{1k} = \frac{c}{k^{2\ell}}\sin\frac{2k\pi}{\omega}t_0, \quad c_{2k} = \frac{c}{k^{2\ell}}\cos\frac{2k\pi}{\omega}t_0, \quad \text{for} \quad k = 1, 2, \dots,$$

which, together with (1.9), yields (1.8). □

Remark 1.1 For $\ell = 1$, inequality (1.3) is well-known Wirtinger's inequality (see, e.g., [8, Theorem 258]).

Lemma 1.3 *Let ℓ be a natural number,*

$$u \in AC^{2\ell-1}_\omega, \qquad u(t) \not\equiv c_0, \tag{1.15}$$

where c_0 is the mean value of the function u. Then

$$\|u - c_0\|^2_{C_\omega} < \frac{\zeta(2\ell)}{\pi}\left(\frac{\omega}{2\pi}\right)^{2\ell-1}\int_0^\omega \left|u^{(\ell)}(t)\right|^2 dt. \tag{1.16}$$

Proof Assume the contrary that (1.16) does not hold. Then, by virtue of (1.15) and Lemma 1.2, identity (1.8) is fulfilled with $c \neq 0$. Hence,

$$u^{(2\ell-1)}(t) = c(-1)^\ell\left(\frac{2\pi}{\omega}\right)^{2\ell-1}\sum_{k=1}^{+\infty}\frac{1}{k}\sin\frac{2k\pi}{\omega}(t - t_0) \qquad \text{for} \quad 0 < t - t_0 < \omega.$$

Therefore,

$$u^{(2\ell-1)}(t) = c(-1)^{\ell}\frac{\pi}{\omega}\left(\frac{2\pi}{\omega}\right)^{2\ell-1}(\omega - t + t_0) \quad \text{for} \quad 0 < t - t_0 < \omega,$$

which contradicts the condition $u^{(2\ell-1)} \in C_\omega$. □

Lemma 1.4 *Let u be a nontrivial ω-periodic solution of the homogeneous equation* (1.1_0). *Then*

$$\int_0^\omega p(t)u(t)dt = 0, \tag{1.17}$$

$$\int_0^\omega \left|u^{(m)}(t)\right|^2 dt = (-1)^m \int_0^\omega p(t)u^2(t)dt \quad \textit{for} \quad n = 2m, \tag{1.18}$$

and

$$\int_0^\omega p(t)u^2(t)dt = 0 \quad \textit{for} \quad n = 2m+1. \tag{1.19}$$

If, moreover, $p(t) \not\equiv 0$, then for any $k \in \{1, 2, \ldots, n\}$,

$$u^{(k)}(t) \not\equiv 0, \tag{1.20}$$

while if

$$\int_0^\omega p(t)dt \neq 0, \tag{1.21}$$

then for any $c_0 \in \mathbb{R}$, the inequality

$$\|u\|^2_{C_\omega} \le \gamma_0(p)\|u - c_0\|^2_{C_\omega} \tag{1.22}$$

holds.

Proof Let u be a nontrivial ω-periodic solution of (1.1_0). Integrating (1.1_0) from 0 to ω we get (1.17).

Let now $n = 2m$ ($n = 2m + 1$). Multiplying both sides of (1.1_0) by $(-1)^m u$ (by u) and integrating it from 0 to ω we get relation (1.18) [relation (1.19)].

Suppose now that $u^{(k)}(t) \equiv 0$ for some $k \in \{1, 2, \ldots, n\}$. Then evidently $u(t) \equiv c_0$, where $c_0 \neq 0$. Hence, it follows from (1.1_0) that $p(t) \equiv 0$. Therefore, if $p(t) \not\equiv 0$, then for each $k \in \{1, 2, \ldots, n\}$ relation (1.20) is fulfilled.

Now assume that (1.21) holds. By virtue of (1.17), for any $c_0 \in \mathbb{R}$ we have

$$c_0 \int_0^\omega p(t)dt = -\int_0^\omega p(t)\,(u(t) - c_0)\,dt.$$

Hence,

$$|c_0| \le \frac{\int_0^\omega |p(t)|dt}{\left|\int_0^\omega p(t)dt\right|}\,\|u - c_0\|_{C_\omega}.$$

Taking now into account the inequality

$$\|u\|_{C_\omega} \le \|u - c_0\|_{C_\omega} + |c_0|,$$

we easily get (1.22). □

Theorem 1.1 *Let $n = 2m$ and*

$$p(t) \not\equiv 0, \qquad (-1)^m \int_0^\omega p(t)dt \ge 0. \tag{1.23}$$

Let, moreover, one of the following two conditions

$$(-1)^m p(t) \le \left(\frac{2\pi}{\omega}\right)^n \quad \text{for} \quad t \in \mathbb{R}, \qquad (-1)^m p(t) \not\equiv \left(\frac{2\pi}{\omega}\right)^n \tag{1.24}$$

and

$$\int_0^\omega \left[(-1)^m p(s)\right]_+ ds \le \frac{\pi}{\zeta(n)}\left(\frac{2\pi}{\omega}\right)^{n-1} \tag{1.25}$$

be fulfilled. Then, (1.1) has one and only one ω-periodic solution.

Proof By virtue of Lemma 1.1, it is sufficient to show that the homogeneous equation (1.1_0) has no nontrivial ω-periodic solution.

Assume the contrary that u is a nontrivial ω-periodic solution of (1.1_0). Then, by virtue of Lemma 1.4, $u^{(m)}(t) \not\equiv 0$ and relations (1.17) and (1.18) hold. Denote by c_0 the mean value of the function u. Then, in view of (1.17), it easily follows from (1.18) that

$$0 < \int_0^\omega \left|u^{(m)}(t)\right|^2 dt = (-1)^m \int_0^\omega p(t)\,(u(t) - c_0)^2\,dt - (-1)^m c_0^2 \int_0^\omega p(t)dt,$$

whence, on account of (1.23), we get

$$0 < \int_0^\omega \left|u^{(m)}(t)\right|^2 dt \le \int_0^\omega \left[(-1)^m p(t)\right]_+ (u(t) - c_0)^2\, dt. \tag{1.26}$$

Suppose now that (1.24) holds. According to Lemma 1.2, either

$$\int_0^\omega |u(t) - c_0|^2\, dt < \left(\frac{\omega}{2\pi}\right)^n \int_0^\omega \left|u^{(m)}(t)\right|^2 dt,$$

or there exist $c, t_0 \in \mathbb{R}$ such that (1.6) is fulfilled and, moreover, $c \ne 0$. In both cases, by virtue of (1.24) and (1.26), we obtain a contradiction

$$\int_0^\omega \left|u^{(m)}(t)\right|^2 dt < \int_0^\omega \left|(u^{(m)}(t)\right|^2 dt.$$

Therefore, if (1.24) holds then (1.1_0) has no nontrivial ω-periodic solution.

Suppose now that (1.25) is fulfilled. Then, it follows from (1.26) that

$$\int_0^\omega \left|u^{(m)}(t)\right|^2 dt \le \frac{\pi}{\zeta(n)} \left(\frac{2\pi}{\omega}\right)^{n-1} \|u - c_0\|^2_{C_\omega}.$$

But this is impossible since by Lemma 1.3,

$$\|u - c_0\|^2_{C_\omega} < \frac{\zeta(n)}{\pi} \left(\frac{\omega}{2\pi}\right)^{n-1} \int_0^\omega |u^{(m)}(t)|^2\, dt.$$

Therefore, (1.1_0) has no nontrivial ω-periodic solution. □

Remark 1.2 Condition (1.23) in Theorem 1.1 cannot be replaced by the condition

$$p(t) \not\equiv 0, \qquad (-1)^m \int_0^\omega p(t)dt > -\varepsilon, \tag{1.27}$$

no matter how small $\varepsilon > 0$ would be. Indeed, let

$$u(t) = 1 + \varepsilon + \varepsilon \sin \frac{2\pi t}{\omega}, \quad p(t) = (-1)^m \left(\frac{2\pi}{\omega}\right)^n \frac{u(t) - 1 - \varepsilon}{u(t)},$$

where

$$0 < \varepsilon < \min\left\{\left(\frac{\omega}{2\pi}\right)^n \frac{1}{\omega}, \frac{1}{2\zeta(n)}\right\}.$$

Then conditions (1.24) and (1.25) hold, but instead of (1.23) condition (1.27) is fulfilled. However, the function u is a nontrivial ω-periodic solution of (1.1_0). Therefore, by virtue of Lemma 1.1, equation (1.1) either has no ω-periodic solution or has infinitely many ω-periodic solutions.

Remark 1.3 Condition (1.24) is optimal and cannot be weakened. Indeed, if $p(t) \equiv (-1)^m \left(\frac{2\pi}{\omega}\right)^n$, then for any $c_1, c_2 \in \mathbb{R}\backslash\{0\}$ the function

$$u(t) = c_1 \sin\frac{2\pi t}{\omega} + c_2 \cos\frac{2\pi t}{\omega}$$

is a nontrivial ω-periodic solution of the homogeneous equation (1.1_0).

Remark 1.4 It follows from Theorem 1.1 that the second order equation

$$u'' = p(t)u + q(t) \tag{1.28}$$

possesses a unique ω-periodic solution provided

$$p(t) \not\equiv 0, \quad \int_0^\omega p(t)dt \le 0, \tag{1.29}$$

and

$$p(t) \ge -\left(\frac{2\pi}{\omega}\right)^2 \quad \text{for} \quad t \in \mathbb{R}, \quad p(t) \not\equiv -\left(\frac{2\pi}{\omega}\right)^2.$$

This result belongs to Mawhin [21] and Mawhin and Ward [22]. For $n = 2$, condition (1.25) is not a new as well, since as it is shown by Lasota and Opial [17] condition (1.29), together with

$$\int_0^\omega [p(t)]_- dt \le \frac{16}{\omega},$$

guarantee the existence and uniqueness of an ω-periodic solution of (1.28).

Remark 1.5 In the case when

$$(-1)^m p(t) \ge 0 \quad \text{for} \quad t \in \mathbb{R},$$

Theorem 1.1 implies a result stated in [18] and Theorem 1.1 established in [13]. Note that in [18] the condition

$$\int_0^\omega |p(t)|dt \le \frac{2}{\pi}\left(\frac{2\pi}{\omega}\right)^{n-1}$$

is supposed. However, (1.25) is more general, because

$$\zeta(n) \le \zeta(2) = \frac{\pi^2}{6} \quad \text{for} \quad n \ge 2.$$

Theorem 1.2 *Let $n = 2m$,*

$$(-1)^m \int_0^\omega p(t)dt < 0, \tag{1.30}$$

and

$$\gamma_0(p)\int_0^\omega \left[(-1)^m p(t)\right]_+ dt \le \frac{\pi}{\zeta(n)}\left(\frac{2\pi}{\omega}\right)^{n-1}. \tag{1.31}$$

Then, (1.1) *has one and only one ω-periodic solution.*

Proof Let u be a nontrivial ω-periodic solution of (1.1_0). Denote by c_0 the mean value of the function u. By virtue of Lemma 1.4 and condition (1.30), $u^{(m)}(t) \not\equiv 0$ and (1.18) and (1.22) are fulfilled. Hence,

$$0 < \int_0^\omega \left|u^{(m)}(t)\right|^2 dt < \int_0^\omega \left[(-1)^m p(t)\right]_+ u^2(t)dt$$

$$\le \gamma_0(p)\|u - c_0\|^2_{C_\omega}\int_0^\omega \left[(-1)^m p(t)\right]_+ dt.$$

On the other hand, by virtue of Lemma 1.3,

$$\|u - c_0\|^2_{C_\omega} \le \frac{\zeta(n)}{\pi}\left(\frac{\omega}{2\pi}\right)^{n-1}\int_0^\omega \left|u^{(m)}(t)\right|^2 dt.$$

The latter two inequalities imply

$$\frac{\zeta(n)}{\pi}\left(\frac{\omega}{2\pi}\right)^{n-1}\gamma_0(p)\int_0^\omega \left[(-1)^m p(t)\right]_+ dt > 1,$$

which contradicts (1.31). Thus, (1.1_0) has no nontrivial ω-periodic solution. Therefore, according to Lemma 1.1, (1.1) has one and only one ω-periodic solution. □

Lemma 1.5 *If $v \in AC_\omega$, then*

$$\|v - c_0\|_{C_\omega} \le \frac{1}{2}\int_0^\omega |v'(t)|dt, \tag{1.32}$$

where c_0 is the mean value of the function v.

Proof By virtue of the condition $v \in AC_\omega$, there exist $t_0 \in [0, \omega]$ and $t_1 \in (t_0, t_0 + \omega)$ such that

$$v(t_0) = v(t_0 + \omega) = c_0, \quad |v(t_1) - c_0| = \|v - c_0\|_{C_\omega}.$$

Thus

$$\|v - c_0\|_{C_\omega} = \left|\int_{t_0}^{t_1} v'(s)ds\right| \le \int_{t_0}^{t_1} |v'(s)|ds,$$

$$\|v - c_0\|_{C_\omega} = \left|\int_{t_1}^{t_0+\omega} v'(s)ds\right| \le \int_{t_1}^{t_0+\omega} |v'(s)|ds.$$

If we add these two inequalities, we obtain

$$2\|v - c_0\|_{C_\omega} \le \int_{t_0}^{t_0+\omega} |v'(s)|ds = \int_0^\omega |v'(s)|ds.$$

Consequently, inequality (1.32) is valid. □

Theorem 1.3 *Let $n = 2m + 1$, $\sigma \in \{-1, 1\}$,*

$$\sigma \int_0^\omega p(t)dt < 0, \tag{1.33}$$

and

$$\gamma(p)\int_0^\omega [\sigma p(t)]_+\, dt \le \frac{1}{\zeta(2n-2)}\left(\frac{2\pi}{\omega}\right)^{2n-2}. \tag{1.34}$$

Then, (1.1) has one and only one ω-periodic solution.

Proof Let u be a nontrivial ω-periodic solution of (1.1_0). It follows from Lemmas 1.2 and 1.4, and condition (1.33) that (1.19) and (1.22) hold and

$$0 < \|u - c_0\|^2_{C_\omega} \leq \frac{2\zeta(2n-2)}{\omega} \left(\frac{\omega}{2\pi}\right)^{2n-2} \int\limits_0^\omega \left|u^{(n-1)}(t)\right|^2 dt$$
$$< 2\zeta(2n-2) \left(\frac{\omega}{2\pi}\right)^{2n-2} \|u^{(n-1)}\|^2_{C_\omega}, \tag{1.35}$$

where c_0 is the mean value of the function u.

By Lemma 1.5,

$$\|u^{(n-1)}\|_{C_\omega} \leq \frac{1}{2} \int\limits_0^\omega \left|u^{(n)}(t)\right| dt = \frac{1}{2} \int\limits_0^\omega |p(t)u(t)| dt.$$

Hence, by virtue of Schwartz's inequality, we get

$$\|u^{(n-1)}\|^2_{C_\omega} \leq \frac{\varrho}{4} \int\limits_0^\omega |p(t)| dt,$$

where

$$\varrho = \int\limits_0^\omega |p(t)| u^2(t) dt.$$

Now it follows from (1.35) that

$$0 < \|u - c_0\|^2_{C_\omega} < \frac{\zeta(2n-2)}{2} \left(\frac{\omega}{2\pi}\right)^{2n-2} \varrho \int\limits_0^\omega |p(t)| dt. \tag{1.36}$$

On the other hand, in view of (1.19) and (1.22),

$$\varrho = \int\limits_0^\omega (2[\sigma p(t)]_+ - \sigma p(t))\, u^2(t) dt = 2 \int\limits_0^\omega [\sigma p(t)]_+ u^2(t) dt$$
$$\leq 2\|u\|^2_{C_\omega} \int\limits_0^\omega [\sigma p(t)]_+ dt \leq 2\gamma_0(p) \|u - c_0\|^2_{C_\omega} \int\limits_0^\omega [\sigma p(t)]_+ dt.$$

The latter inequality, together with (1.36), implies

$$\gamma(p)\zeta(2n-2) \left(\frac{\omega}{2\pi}\right)^{2n-2} \int\limits_0^\omega [\sigma p(t)]_+ \, dt > 1,$$

which contradicts (1.34). Thus, (1.1_0) has no nontrivial ω-periodic solution. Therefore, by virtue of Lemma 1.1, (1.1) has one and only one ω-periodic solution. □

Remark 1.6 In the case when $n = 2m$, $(-1)^m p(t) \le 0$ ($n = 2m + 1$, $\sigma p(t) \ge 0$) and $p(t) \not\equiv 0$, conditions (1.30) and (1.31) [conditions (1.33) and (1.34)] hold automatically. In this case, Theorem 1.2 (Theorem 1.3) coincides with Proposition 1.1 in [13]. Note also that if p is not of a constant sign, then Theorems 1.2 and 1.3 as well as Theorem 1.1 are new.

2 Nonlinear problem

In this section, we consider the nonlinear differential equation

$$u^{(n)} = f\left(t, u, u', \dots, u^{(n-1)}\right), \tag{2.1}$$

where the function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the local Carathéodory conditions and is periodic in the first argument with the period $\omega > 0$, i.e.,

$$f(t + \omega, x_1, \dots, x_n) \equiv f(t, x_1, \dots, x_n).$$

Lemma 2.1 *Let $\sigma \in \{-1, 1\}$ and, on the set $\mathbb{R} \times \mathbb{R}^n$, the inequalities*

$$\begin{aligned} p_1(t)|x_1| - \delta\left(t, \sum_{k=1}^n |x_k|\right) &\le \sigma f(t, x_1, x_2, \dots, x_n) \operatorname{sgn} x_1 \\ &\le p_2(t)|x_1| + \delta\left(t, \sum_{k=1}^n |x_k|\right), \end{aligned} \tag{2.2}$$

be fulfilled, where $p_1, p_2 \in L_\omega$ and $\delta \in Z_\omega$. Let, moreover, for any $p \in L_\omega$, satisfying the condition

$$p_1(t) \le \sigma p(t) \le p_2(t) \quad \textit{for} \quad t \in \mathbb{R}, \tag{2.3}$$

(1.1_0) have no nontrivial ω-periodic solution. Then, (2.1) has at least one ω-periodic solution.

For $\sigma = 1$, this lemma is proved in [13]. For $\sigma = -1$, the lemma can be proved analogously.

Theorem 2.1 *Let $n = 2m$ and, on the set $\mathbb{R} \times \mathbb{R}^n$, the inequalities*

$$\begin{aligned} p_1(t)|x_1| - \delta\left(t, \sum_{k=1}^n |x_k|\right) &\le (-1)^m f(t, x_1, x_2, \dots, x_n) \operatorname{sgn} x_1 \\ &\le p_2(t)|x_1| + \delta\left(t, \sum_{k=1}^n |x_k|\right) \end{aligned} \tag{2.4}$$

be fulfilled, where $p_1, p_2 \in L_\omega$*,*

$$p_1(t) \not\equiv 0, \qquad \int\limits_0^\omega p_1(t)dt \geq 0, \tag{2.5}$$

and $\delta \in Z_\omega$*. Let, moreover, one of the following two conditions*

$$p_2(t) \leq \left(\frac{2\pi}{\omega}\right)^n \quad \textit{for} \quad t \in \mathbb{R}, \qquad p_2(t) \not\equiv \left(\frac{2\pi}{\omega}\right)^n \tag{2.6}$$

and

$$\int\limits_0^\omega [p_2(t)]_+ \, dt \leq \frac{\pi}{\zeta(n)} \left(\frac{2\pi}{\omega}\right)^{n-1} \tag{2.7}$$

hold. Then, (2.1) *has at least one* ω*-periodic solution.*

Proof By virtue of Lemma 2.1 with $\sigma = (-1)^m$, it is sufficient to show that for any $p \in L_\omega$, satisfying the condition

$$p_1(t) \leq (-1)^m p(t) \leq p_2(t) \quad \text{for} \quad t \in \mathbb{R}, \tag{2.8}$$

(1.1_0) has no nontrivial ω-periodic solution.

It is clear that (2.5) and (2.8) imply (1.23), while conditions (2.6) and (2.8) [conditions (2.7) and (2.8)] yield (1.24) [condition (1.25)]. Therefore, by virtue of Theorem 1.1, (1.1_0) has no nontrivial ω-periodic solution for any p satisfying (2.8). □

Remark 2.1 For the second order equation

$$u'' = f_1(t, u) + f_2(u)u' + q(t)$$

the result close to Theorem 2.1 is contained in the paper by Mawhin [21] and Mawhin and Ward [22].

Theorem 2.2 *Let, on the set* $\mathbb{R} \times \mathbb{R}^n$*, inequalities* (2.2) *be fulfilled, where* $p_1, p_2 \in L_\omega$*,*

$$\int\limits_0^\omega p_2(t)dt < 0, \tag{2.9}$$

and $\delta \in Z_\omega$*. Let, moreover, either* $n = 2m$*,* $\sigma = (-1)^m$*, and*

$$\eta_0(p_1, p_2) \int\limits_0^\omega [p_2(t)]_+ dt \leq \frac{\pi}{\zeta(n)} \left(\frac{2\pi}{\omega}\right)^{n-1}, \tag{2.10}$$

or $n = 2m + 1$, $\sigma \in \{-1, 1\}$, *and*

$$\eta(p_1, p_2) \int_0^\omega [p_2(t)]_+ dt \le \frac{1}{\zeta(2n-2)} \left(\frac{2\pi}{\omega}\right)^{n-2}. \tag{2.11}$$

Then, (2.1) *has at least one* ω*-periodic solution.*

Proof Let the function $p \in L_\omega$ satisfy (2.3). Then clearly

$$|p(t)| \le \max\{|p_1(t)|, |p_2(t)|\} = \frac{1}{2}\left(|p_1(t)| + |p_2(t)| + ||p_1(t)| - |p_2(t)||\right).$$

On the other hand, in view of (2.9), we have

$$\left|\int_0^\omega p(t)dt\right| \ge \left|\int_0^\omega p_2(t)dt\right|.$$

Hence,

$$\gamma_0(p) \le \eta_0(p_1, p_2) \tag{2.12}$$

and

$$\gamma(p) \le \eta(p_1, p_2). \tag{2.13}$$

Suppose now that $n = 2m$, $\sigma = (-1)^m$ ($n = 2m + 1$, $\sigma \in \{-1, 1\}$) and condition (2.10) [condition (2.11)] holds. Then, in view of (2.3) and (2.12) [(2.3) and (2.13)], inequalities (1.30) and (1.31) [(1.33) and (1.34)] hold as well. Hence, by virtue of Theorem 1.2 (Theorem 1.3), (1.1_0) has no nontrivial ω-periodic solution. Therefore, by virtue of Lemma 2.1, (2.1) has at least one ω-periodic solution. □

Let us now pass to the case where $f(t, x_1, \dots, x_n) \equiv f(t, x_1)$, and thus (2.1) has the form

$$u^{(n)} = f(t, u). \tag{2.14}$$

As above we assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the local Carathéodory conditions and

$$f(t + \omega, x) \equiv f(t, x).$$

Theorem 2.3 *Let, on the set* $\mathbb{R} \times \mathbb{R}$*, the inequalities*

$$p_1(t)|x - y| \le \sigma [f(t, x) - f(t, y)] \operatorname{sgn}(x - y) \le p_2(t)|x - y| \tag{2.15}$$

be fulfilled, where $p_1, p_2 \in L_\omega$. *Let, moreover, either* $n = 2m$, $\sigma = (-1)^m$, *and along with* (2.5) *one of conditions* (2.6) *and* (2.7) *hold, or* $n = 2m$, $\sigma = (-1)^m$, *and inequalities* (2.9) *and* (2.10) *hold, or* $n = 2m + 1$, $\sigma \in \{-1, 1\}$, *and inequalities* (2.9) *and* (2.11) *be satisfied. Then,* (2.14) *has a unique* ω*-periodic solution.*

Proof From (2.15) it follows (2.2), where $f(t, x_1, \dots, x_n) \equiv f(t, x_1)$ and $\delta(t, \varrho) \equiv |f(t, 0)|$. Therefore, by virtue of Theorems 2.1 and 2.2, (2.14) has at least one ω-periodic solution. To complete the proof of the theorem it remains to show that this equation has no more then one ω-periodic solution.

Let u_1 and u_2 be any ω-periodic solutions of (2.14). Then the function $u(t) = u_2(t) - u_1(t)$ is an ω-periodic solution of (1.1_0), where

$$p(t) = \begin{cases} \frac{f(t,u_2(t))-f(t,u_1(t))}{u_2(t)-u_1(t)} & \text{if} \quad u_2(t) \neq u_1(t), \\ \sigma p_1(t) & \text{if} \quad u_2(t) = u_1(t). \end{cases}$$

By (2.15), the function p satisfies inequalities (2.3). However, as it is shown above, the restrictions imposed on p_1, p_2, n, and σ imply that (1.1_0) with p, satisfying (2.3), has no nontrivial ω-periodic solution. Consequently, $u(t) \equiv 0$, i.e., $u_1(t) \equiv u_2(t)$. □

Remark 2.2 In the case when the functions p_1 and p_2 are not of constant signs, Theorems 2.1–2.3 are new even for the second order equation (see, e.g., [3]). Note also that if p_1 and p_2 are of constant signs, then Theorems 2.1–2.3 imply Theorems 2.1–2.4 established in [13].

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