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OPTIMAL CONDITIONS OF SOLVABILITY AND UNSOLVABILITY OF NONLOCAL PROBLEMS FOR ESSENTIALLY NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract

Unimprovable, in a certain sense, sufficient conditions of solvability and unsolvability of nonlocal problems are found for the differential system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n),$$

where each of the functions $f_i : [a,b] \times R^n \to R$ $(i = 1, \dots, n)$ may be superlinear or sublinear with respect to phase variables.

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1 Introduction

In the present paper, for the nonlinear differential system

$$\frac{du_i}{dt} = f_i(t, u_1, \dots, u_n) \;\; (i = 1, \dots, n) \tag{1.1}$$

on a finite interval $[a,b]$ we study the nonlocal boundary value problem

$$u_i(t_i) = \varphi_i(u_i) \;\; (i = 1, \dots, n) \tag{1.2}$$

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and its particular case

$$u_i(t_i) = \int_a^b u(s)\,d\alpha_i(s) + c_i \;\; (i = 1,\dots,n). \tag{1.3}$$

Here, $f_i : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1,\dots,n)$ are the functions satisfying the local Carathéodory conditions, $t_i \in [a,b]$, $c_i \in \mathbb{R}$ $(i = 1,\dots,n)$, $\varphi_i : C([a,b]) \to \mathbb{R}$ $(i = 1,\dots,n)$ are continuous functionals bounded on every compact set of the space $C([a,b])$, and $\alpha_i : [a,b] \to \mathbb{R}$ $(i = 1,\dots,n)$ are the functions of bounded variations.

The boundary value problems of the type (1.1), (1.2) have been mainly investigated in the case where the functions f_i $(i = 1,\dots,n)$ admit the one-sided estimates

$$f_i(t,x_1,\dots,x_n)\operatorname{sgn}\big((t-t_i)x_i\big) \le \sum_{k=1}^n p_{ik}(t)|x_k| + q_i(t) \;\; (i = 1,\dots,n),$$

that is, when orders of growth of the functions

$$(t,x_1,\dots,x_n) \to \big[f_i(t,x_1,\dots,x_n)\operatorname{sgn}((t-t_i)x_i)\big]_+ \;\; (i = 1,\dots,n)$$

with respect to the phase variables do not exceed 1 (see, e.g., [1]–[5], [9]–[11] and the references therein).

In case for which this condition is violated, the above-mentioned problems are, as a matter of fact, being unstudied. The theorems proven below fill to some extent this gap.

Throughout the paper, the use will be made of the following notation: $\mathbb{R}^n$ is the n-dimensional real Euclidean space; $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ is the column-vector with components $x_i \in \mathbb{R}$ $(i = 1,\dots,n)$; δ_{ik} is the Kronecker symbol; $X = (x_{ik})_{i,k=1}^n$ is the $n \times n$-matrix with components $x_{ik} \in \mathbb{R}$ $(i,k = 1,\dots,n)$ and with the norm $\|X\| = \sum_{i,k=1}^n |x_{ik}|$; X^{-1} is the inverse to X matrix; $r(X)$ is a spectral radius of the matrix X; E is the unit matrix; $C([a,b])$ and $L([a,b];\mathbb{R})$ are the spaces of continuous and Lebesgue integrable functions $u : [a,b] \to \mathbb{R}$ and $v : [a,b] \to \mathbb{R}$, respectively, with the norms

$$\|u\|_C = \max\big\{|u(t)| : \; a \le t \le b\big\}, \quad \|v\|_L = \int_a^b |v(s)|\,ds;$$

$\int_a^t |d\alpha(s)|$ is a full variation of the function $\alpha : [a,t] \to \mathbb{R}$ on $[a,t]$.[1]

Definition 1.1. The real matrix $H = (h_{ik})_{i,k=1}^n$ belongs to the set A_s if it is quasi-nonnegative and asymptotically stable, i.e., if $h_{ik} \ge 0$ for $i \ne k$, and if real parts of eigen-values of the matrix H are negative.

[1] $\int_a^t |d\alpha(s)| = 0$ for $t = a$.

Definition 1.2. The function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ belongs to the set $\mathcal{K}_{loc}([a,b] \times \mathbb{R}^n)$ if it satisfies the local Carathéodory conditions[2], i.e., if $f(t,\cdot,\dots,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost all $t \in [a,b]$, $f(\cdot,x_1,\dots,x_n) : [a,b] \to \mathbb{R}$, measurable for all $(x_i)_{i=1}^n \in \mathbb{R}^n$ and

$$\max\left\{ \big|f(\cdot,x_1,\dots,x_n)\big| : \ \sum_{i=1}^n |x_i| \le \rho \right\} \in L([a,b];\mathbb{R})$$

for any $\rho \in]0,+\infty[$.

Everywhere in the sequel, it will be assumed that

$$f_i \in \mathcal{K}_{loc}([a,b] \times \mathbb{R}^n) \ \ (i = 1,\dots,n).$$

Along with the problem (1.1) (1.2) we consider the auxiliary boundary value problem

$$\frac{du_i}{dt} = (1-\lambda)p_i(t,u_1,\dots,u_n)u_i + \lambda f_i(t,u_1,\dots,u_n) \ \ (i = 1,\dots,n), \tag{1.4}$$

$$u_i(t_i) = \lambda\varphi_i(u)(i = 1,\dots,n), \tag{1.5}$$

depending on the parameter $\lambda \in]0,1[$. From Theorem 1 of [8] it follows

Proposition 1.3 (The principle of a priori boundedness)**.** *Let there exist functions $p_i \in \mathcal{K}_{loc}([a,b] \times \mathbb{R}^n)$ $(i = 1,\dots,n)$, a set of zero measure $I \subset [a,b]$ and a positive constant ρ such that on the set $([a,b] \times I) \times \mathbb{R}^n$ the inequalities*

$$p_i(t,x_1,\dots,x_n)(t-t_i) \ge 0 \ \ (i = 1,\dots,n) \tag{1.6}$$

are fulfilled, and for an arbitrary $\lambda \in]0,1[$, every solution of the problem (1.4), (1.5) *admits the estimate*

$$\sum_{i=1}^n \|u_i\|_C \le \rho. \tag{1.7}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Besides the above proposition, in the sequel we will need the following three lemmas.

Lemma 1.4. *The quasi-nonnegative matrix $H = (h_{ik})_{i,k=1}^n$ belongs to the set A_s iff*

$$h_{ii} < 0 \quad (i = 1,\dots,n), \tag{1.8}$$

$$r(H_0) < 1, \tag{1.9}$$

where

$$H_0 = \left((1-\delta_{ik}) \frac{h_{ik}}{|h_{ii}|} \right)_{i,k=1}^n. \tag{1.10}$$

[2]in Greek $K\alpha\rho\alpha\theta\varepsilon o\partial\omega\rho\eta$.

Lemma 1.5. *Let $H_0 = (h_{0ik})_{i,k=1}^n$ be a nonnegative matrix satisfying the inequality* (1.9) *and ρ_i and h_{0i} $(i = 1,\dots,n)$ be nonnegative numbers such that*

$$\rho_i \le \sum_{k=1}^n h_{0ik}\rho_k + h_{0i} \quad (i = 1,\dots,n). \tag{1.11}$$

Then

$$\sum_{i=1}^n \rho_i \le \|(E - H_0)^{-1}\| \sum_{i=1}^n h_{0i}\,. \tag{1.12}$$

Lemma 1.6. *Let $H_0 = (h_{0ik})_{i,k=1}^n$ be a nonnegative matrix satisfying the inequality*

$$r(H_0) \ge 1. \tag{1.13}$$

Then for any $\varepsilon > 0$ there exist the numbers $h_{0i} \in [0,\varepsilon]$ $(i = 1,\dots,n)$ such that

$$\sum_{i=1}^n h_{0i} > 0, \tag{1.14}$$

$$\sum_{k=1}^n h_{0ik}h_{0k} \ge h_{0i} \quad (i = 1,\dots,n). \tag{1.15}$$

Lemma 1.4 follows from Theorem 1.18 of monograph [6], and the proof of Lemma 1.5 can be found in [7]. As for Lemma 1.6, it is obvious and we omit its proof.

2 Main Results

First of all, we consider the case where the functions f_i $(i = 1,\dots,n)$ and the functionals φ_i $(i = 1,\dots,n)$ satisfy the inequalities

$$\begin{aligned} &f_i(t,x_1,\dots,x_n)\,\mathrm{sgn}\,((t-t_i)x_i) \\ &\le g_i(t,x_1,\dots,x_n)\left(\sum_{k=1}^n h_{ik}|x_k| + h_i\right) \quad \text{for} \quad t\in[a,b]\setminus I, \quad (x_k)_{k=1}^n \in R^n \quad (i=1,\dots,n), \end{aligned} \tag{2.1}$$

$$|\varphi_i(u)| \le \int_a^b |u(t)|\,d\beta_i(t) + \gamma_i \quad \text{for} \quad u \in C([a,b]) \quad (i = 1,\dots,n), \tag{2.2}$$

where $I \subset [a,b]$ is a set of zero measure, h_i and γ_i $(i = 1,\dots,n)$ are nonnegative numbers,

$$H = (h_{ik})_{i,k=1}^n \in A_s, \tag{2.3}$$

$g_i \in \mathcal{K}_{loc}([a,b]\times R^n)$ $(i = 1,\dots,n)$ are nonnegative and $\beta_i : [a,b] \to R$ $(i = 1,\dots,n)$ are nondecreasing functions such that

$$\beta_i(b) - \beta_i(a) \le 1 \quad (i = 1,\dots,n). \tag{2.4}$$

Assume

$$g_{0i}(t) = \inf\{g_i(t,x_1,\dots,x_n) : (x_k)_{k=1}^n \in R^n\} \quad (i = 1,\dots,n), \tag{2.5}$$

$$\eta_i = \int_a^b \exp\left(h_{ii}\left|\int_{t_i}^t g_{0i}(s)\,ds\right|\right) d\beta_i(t) \quad (i = 1,\dots,n). \tag{2.6}$$

Theorem 2.1. *Let the conditions* (2.1)–(2.4) *be fulfilled and*

$$\text{either } \eta_i < 1, \text{ or } \gamma_i = 1-\eta_i = 0 \text{ for every } i \in \{1,\dots,n\}, \tag{2.7}$$

where η_i $(i=1,\dots,n)$ *are the numbers given by the equalities* (2.5) *and* (2.6)*. Then the problem* (1.1)*,* (1.2) *has at least one solution.*

Proof. First, we note that the condition (2.3) by Lemma 1.4 ensures the fulfilment of the inequalities (1.8) and (1.9), where H_0 is the matrix given the inequality (1.10). On the other hand, by the condition (2.7), we have

$$\gamma_i = (1-\eta_i)\gamma_{0i} \quad (i=1,\dots,n), \tag{2.8}$$

where

$$\gamma_{0i} = \begin{cases} \gamma_i/(1-\eta_i) & \text{for } \eta_i < 1, \\ 0 & \text{for } \eta_i = 1. \end{cases}$$

Suppose

$$h_{0i} = \frac{h_i}{|h_{ii}|} + \gamma_{0i} \quad (i=1,\dots,n), \tag{2.9}$$

$$\rho = \big\|(E-H_0)^{-1}\big\| \sum_{i=1}^{n} h_{0i}, \tag{2.10}$$

and

$$p_i(t,x_1,\dots,x_n) \equiv h_{ii} g_i(t,x_1,\dots,x_n)\,\mathrm{sgn}(t-t_i) \quad (i=1,\dots,n). \tag{2.11}$$

By (1.8) the functions p_i $(i=1,\dots,n)$ satisfy the inequalities (1.6), since g_i $(i=1,\dots,n)$ are nonnegative.

Let $(u_i)_{i=1}^n$ be a solution of the problem (1.4), (1.5) for an arbitrary $\lambda \in]0,1[$. According to Proposition 1.3, to prove the theorem, it suffices to state that $(u_i)_{i=1}^n$ admits the estimate (1.7).

In view of (1.8), (2.1) and (2.11), almost everywhere on $[a,b]$ the inequalities

$$|u_i(t)|'\,\mathrm{sgn}(t-t_i) \le -p_i(t)|u_i(t)| + p_i(t)\bigg(\sum_{k=1}^{n} h_{0ik}\rho_k + h_i/|h_{ii}|\bigg) \quad (i=1,\dots,n) \tag{2.12}$$

are fulfilled, where $h_{0ik} = (1-\delta_{ik})h_{ik}/|h_{ii}|$ $(i=1,\dots,n)$,

$$p_i(t) = |h_{ii}| g_i(t,u_1(t),\dots,u_n(t)) \ge |h_{ii}| g_{0i}(t) \ge 0 \quad (i=1,\dots,n), \tag{2.13}$$

and

$$\rho_i = \|u_i\|_C \quad (i=1,\dots,n). \tag{2.14}$$

On the other hand, it follows from (1.5), (2.2) and (2.8) that

$$|u_i(t_i)| \le \lambda \int_a^b |u(t)|\,d\beta_i(t) + \lambda(1-\eta_i)\gamma_{0i} \quad (i=1,\dots,n). \tag{2.15}$$

From (2.12) we have

$$|u_i(t)| \le \exp\left(-\left|\int_{t_i}^t p_i(s)\,ds\right|\right)|u_i(t_i)| + \left(\sum_{k=1}^n h_{0ik}\rho_k + h_i/|h_{ii}|\right)$$
$$\times\left(1-\exp\left(-\left|\int_{t_i}^t p_i(s)\,ds\right|\right)\right) \quad \text{for } a \le t \le b \;\; (i=1,\dots,n). \quad (2.16)$$

If along with (2.16) we take into account that $\beta_i : [a,b] \to R$ $(i=1,\dots,n)$ are the nondecreasing functions satisfying the inequalities (2.4), then from (2.15) we find

$$|u_i(t)| \le \zeta|u_i(t_i)| + \left(\sum_{k=1}^n h_{0ik}\rho_k + h_i/|h_{ii}|\right)(1-\zeta_i) + \lambda(1-\eta_i)\gamma_{0i} \quad (i=1,\dots,n), \quad (2.17)$$

where

$$\zeta_i = \lambda\int_a^b \exp\left(-\left|\int_{t_i}^t p_i(s)\,ds\right|\right)d\beta_i(t) \quad (i=1,\dots,n).$$

On the other hand, by virtue of (2.6) and (2.13), it is clear that

$$\zeta_i \le \lambda\eta_i < 1, \quad \lambda(1-\eta_i) < 1-\zeta_i \quad (i=1,\dots,n).$$

Taking into account the above inequalities and the notation (2.9), from (2.17) we get

$$|u_i(t_i)| \le \sum_{k=1}^n h_{0ik}\rho_k + h_{0i} \quad (i=1,\dots,n).$$

Thus it follows from (2.16) that

$$|u_i(t)| \le \sum_{k=1}^n h_{0ik}\rho_k + h_{0i} \;\; \text{for} \;\; a \le t \le b \;\; (i=1,\dots,n).$$

Consequently, the inequalities (1.11) are fulfilled. However, by Lemma 1.5, the inequalities (1.9) and (1.11) result in (1.12). Taking now into account the notations (2.10) and (2.14), the validity of the estimate (1.7) becomes obvious. □

If

$$\varphi_i(u) = \int_a^b u(t)\,d\alpha_i(t) + c_i \quad (i=1,\dots,n),$$

then the boundary conditions (1.2) take the form (1.3). On the other hand, in this case the inequalities (2.2) are fulfilled, where $\gamma_i = |c_i|$ $(i=1,\dots,n)$ and

$$\beta_i(t) = \int_a^t |d\alpha_i(s)| \quad \text{for} \quad a \le t \le b \quad (i=1,\dots,n). \quad (2.18)$$

Therefore from Theorem 2.1 we have

Corollary 2.2. *Let the conditions* (2.1), (2.3) *and*

$$\int_a^b |d\alpha_i(t)| \le 1 \quad (i=1,\dots,n) \quad (2.19)$$

be fulfilled. If, moreover,

$$\textit{either } \eta_i < 1, \textit{ or } c_i = 1-\eta_i = 0 \textit{ for every } i \in \{1,\dots,n\}, \quad (2.20)$$

where η_i $(i=1,\dots,n)$ *are the numbers given by the equalities* (2.5), (2.6), *and* (2.18), *then the problem* (1.1), (1.3) *has at least one solution.*

If $\gamma_1 = \cdots = \gamma_n = 0$ ($c_1 = \cdots = c_n = 0$), then the condition (2.7) (the condition (2.20)) in Theorem 2.1 (in Corollary 2.2) is fulfilled automatically. Consequently, the following corollary is valid.

Corollary 2.3. *Let the conditions* (2.1)–(2.4) *(the conditions* (2.1), (2.3), *and* (2.19)*) be fulfilled, and* $\gamma_1 = \cdots = \gamma_n = 0$ $(c_1 = \cdots = c_n = 0)$. *Then the problem* (1.1), (1.2) *(the problem* (1.1), (1.3)*) has at least one solution.*

Theorem 2.4. *Let there exist constants* $h_i > 0$ *and* $h_{ik} \in R$ $(i,k = 1,\dots,n)$, *functions* $g_i \in \mathcal{K}_{loc}([a,b] \times R^n)$ $(i = 1,\dots,n)$ *and a set of zero measure* $I \subset [a,b]$ *such that*

$$(1-\delta_{ik})h_{ik} \geq 0, \;\; h_{ii} < 0 \;\; (i = 1,\dots,n), \quad H = (h_{ik})_{i,k=1}^{n} \notin A_s \tag{2.21}$$

and on the set $([a,b] \setminus I) \times R^n$ *the inequalities*

$$g_i(t,x_1,\dots,x_n) > 0 \quad (i = 1,\dots,n), \tag{2.22}$$

$$f_i(t,x_1,\dots,x_n)\operatorname{sgn}(t-t_i) \geq g_i(t,x_1,\dots,x_n)\left(\sum_{k=1}^{n} h_{ik}|x_k| + h_i\right) \quad (i = 1,\dots,n) \tag{2.23}$$

are fulfilled. If, moreover,

$$\varphi_i(u) \geq \int_a^b |u(t)|\, d\beta_i(t) \;\; \textit{for} \;\; u \in C([a,b]), \tag{2.24}$$

where $\beta_i : [a,b] \to R$ $(i = 1,\dots,n)$ *are the nondecreasing functions satisfying the conditions*

$$\beta_i(b) - \beta_i(a) = 1, \quad \lim_{t \to t_i} \beta_i(t) = \beta_i(t_i) \;\; (i = 1,\dots,n), \tag{2.25}$$

then the problem (1.1), (1.2) *has no solution.*

Proof. Assume

$$h_{0ik} = (1-\delta_{ik})h_{ik}/|h_{ii}| \;\; (i = 1,\dots,n), \quad H_0 = (h_{0ik})_{i,k=1}^{n}.$$

Then according to Lemmas 1.4, 1.6 and the condition (2.21), the matrix H_0 satisfies the inequality (1.13), and there exist the numbers

$$h_{0i} \in [0, h_i/|h_{ii}|] \quad (i = 1,\dots,n) \tag{2.26}$$

satisfying the inequalities (1.14) and (1.15).

Assume now that the theorem is invalid, i.e., the problem (1.1), (1.2) has a solution $(u_i)_{i=1}^{n}$. Then by the conditions (2.21)–(2.23) and (2.26), almost everywhere on $[a,b]$ the inequalities

$$p_i(t) \overset{def}{=} |h_{ii}| g_i(t,u_1(t),\dots,u_n(t)) > 0 \quad (i = 1,\dots,n), \tag{2.27}$$

$$u_i'(t)\operatorname{sgn}(t-t_i) \geq p_i(t)|u_i(t)| + p_i(t)\left(\sum_{k=1}^{n} h_{0ik}\mu_k + h_{0i}\right) \quad (i = 1,\dots,n), \tag{2.28}$$

$$\big(u_i'(t) - \widetilde{p}_i(t)u_i(t)\big)\operatorname{sgn}(t-t_i) > 0 \quad (i = 1,\dots,n) \tag{2.29}$$

are fulfilled, where

$$\widetilde{p}_i(t) = p_i(t)\,\mathrm{sgn}\,((t-t_i)u_i(t)), \quad \mu_i = \min\{u_i(t) : a \le t \le b\} \quad (i=1,\dots,n).$$

On the other hand, in view of (2.24) we have

$$u_i(t_i) \ge \int_a^b |u_i(t)|\,d\beta_i(t) \ge 0 \quad (i=1,\dots,n). \tag{2.30}$$

Since $u_i(t_i)$ $(i=1,\dots,n)$ are nonnegative, it follows from (2.29) that

$$u_i(t) \ge u_i(t_i)\exp\left(\int_{t_i}^t \widetilde{p}_i(s)\,ds\right) \ge 0 \quad \text{for} \quad a \le t \le b \quad (i=1,\dots,n). \tag{2.31}$$

If along with (2.27) and (2.31) we take into account that β_i $(i=1,\dots,n)$ are the nondecreasing functions satisfying the equalities $\beta_i(b) = \beta_i(a)$ $(i=1,\dots,n)$, then form (2.28) and (2.30) we find

$$\begin{aligned} u_i(t) &\ge u_i(t_i)\exp\left(-\left|\int_{t_i}^t p_i(s)\,ds\right|\right) \\ &+ \left(\sum_{k=1}^n h_{0ik}\mu_k + h_{0i}\right)\left(1-\exp\left(-\left|\int_{t_i}^t p_i(s)\,ds\right|\right)\right) \quad \text{for} \quad a \le t \le b \quad (i=1,\dots,n) \end{aligned} \tag{2.32}$$

and

$$u_i(t_i) \ge \zeta_i u_i(t_i) + \left(\sum_{k=1}^n h_{0ik}\mu_k + h_{0i}\right)(1-\zeta_i) \quad (i=1,\dots,n), \tag{2.33}$$

where

$$\zeta_i = \int_a^b \exp\left(-\left|\int_{t_i}^t p_i(s)\,ds\right|\right) d\beta_i(t) \quad (i=1,\dots,n).$$

However, by virtue of (2.25) and (2.27) it is clear that

$$\zeta_i < 1 \quad (i=1,\dots,n).$$

Therefore it follows from (2.33) and (2.32) that

$$u_i(t_i) \ge \sum_{k=1}^n h_{0ik}\mu_k + h_{0i} \quad (i=1,\dots,n)$$

and

$$u_i(t) \ge \sum_{k=1}^n h_{0ik}\mu_k + h_{0i} \quad \text{for} \quad a \le t \le b \quad (i=1,\dots,n).$$

Consequently,

$$\mu_i \ge \sum_{k=1}^n h_{0ik}\mu_k + h_{0i} \quad (i=1,\dots,n),$$

whence by the inequalities (1.15) we find

$$\mu_i \ge m h_{0i} \quad (i=1,\dots,n;\ m=1,2,\dots).$$

Therefore,

$$h_{0i} \le \lim_{m\to\infty} \frac{\mu_i}{m} = 0 \quad (i=1,\dots,n),$$

which contradicts the inequality (1.14). The obtained contradiction proves the theorem. $\square$

As an example, we consider the problems

$$\frac{du_i}{dt} = p_i(t,u_1,\dots,u_n)\Big(h_{ii}u_i + \sum_{k=1}^{n} h_{ik}|u_k| + q_i(t,u_1,\dots,u_n)\Big) \quad (i=1,\dots,n), \tag{2.34}$$

$$u_i(t_i) = \int_a^b |u_i(t)|\,d\beta_i(t) \quad (i=1,\dots,n) \tag{2.35}$$

and

$$\frac{du_i}{dt} = p_i(t,u_1,\dots,u_n)\Big(\sum_{k=1}^{n} h_{ik}u_k + q_i(t,u_1,\dots,u_n)\Big) \quad (i=1,\dots,n), \tag{2.36}$$

$$u_i(t_i) = \int_a^b u_i(t)\,d\beta_i(t) \quad (i=1,\dots,n). \tag{2.37}$$

Here

$$(1-\delta_{ik})h_{ik} \geq 0, \quad h_{ii} < 0 \quad (i=1,\dots,n),$$

p_i and $q_i \in \mathcal{K}_{loc}([a,b]\times R^n)$ $(i=1,\dots,n)$, and $\beta_i : [a,b] \to R$ $(i=1,\dots,n)$ are the nondecreasing functions satisfying the conditions (2.25). Moreover, there exist a set of zero measure $I \subset [a,b]$ and the constants $\ell_0 > 0$ and $\ell > \ell_0$ such that on the set $([a,b]\setminus I)\times R^n$ the inequalities

$$p_i(t,x_1,\dots,x_n)\,\mathrm{sgn}(t-t_i) \geq 0, \quad \ell_0 \leq q_i(t,x_1,\dots,x_n) \leq \ell \quad (i=1,\dots,n)$$

are fulfilled.

The above problems are tightly connected with each other, since as it is not difficult to see, the problem (2.34), (2.35) is solvable if and only if the problem (2.36), (2.37) has at least one positive solution.[3]

From Theorems 2.1 and 2.4 we have the following corollaries.

Corollary 2.5. *The problem* (2.34), (2.35) *is solvable iff* $H = (h_{ik})_{i,k=1}^n \in A_s$.

Corollary 2.6. *The problem* (2.36), (2.37) *has at least one positive solution iff* $H = (h_{ik})_{i,k=1}^n \in A_s$.

According to Corollary 2.5, the condition $H = (h_{ik})_{i,k=1}^n \in A_s$ (the condition $H = (h_{ik})_{i,k=1}^n \notin A_s$) in Theorem 2.1 (in Theorem 2.4) is unimprovable.

Note also that in the conditions of Corollary 2.5 we might meet with the possibility when for arbitrary $i \in \{1,\dots,n\}$ and $\nu \geq 0$ one of the conditions

$$\Big(\sum_{k=1}^{n}|x_k|\Big)^{-\nu}|p_i(t,x_1,\dots,x_n)| \to +\infty \quad \text{as} \quad \sum_{k=1}^{n}|x_k| \to +\infty$$

or

$$\Big(\sum_{k=1}^{n}|x_k|\Big)^{\nu}|p_i(t,x_1,\dots,x_n)| \to 0 \quad \text{as} \quad \sum_{k=1}^{n}|x_k| \to +\infty$$

is fulfilled.

Consequently, unlike the well-known earlier results, the theorems proved by us cover the cases where each of the functions f_i $(i=1,\dots,n)$ is either superlinear or sublinear with respect to the phase variables.

[3]That is a solution with positive components.

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