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ON ONE NEUMANN TYPE PROBLEM FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to the Blessed Memory of Professor A. Kharadze

Abstract. Optimal in a certain sense conditions guaranteeing the existence of a unique solution of the differential equation

$$u'' = p(t)u + q(t),$$

satisfying the Neumann type boundary conditions

$$u'(a) = \ell_1 u(a) + c_1, \quad u'(b) = \ell_2 u(b) + c_2,$$

are established.

On a finite interval $[a, b]$, we consider the differential equation

$$u'' = p(t)u + q(t) \tag{1}$$

with the boundary conditions

$$u'(a) = \ell_1 u(a) + c_1, \quad u'(b) = \ell_2 u(b) + c_2, \tag{2}$$

where $p, q \in [a, b] \to \mathbb{R}$ are Lebesgue integrable functions, $c_i \in \mathbb{R}$ $(i = 1, 2)$,

$$\ell_1 \geq 0, \quad \ell_2 \leq 0.$$

For $\ell_i = 0$ $(i = 1, 2)$, the boundary conditions (2) are the Neumann ones. In this case, problem (1), (2) is studied in detail (see, e.g., [1, 3–5] and the references therein). However, this problem in a general case remains still insufficiently studied. The present paper is devoted to fill up this gap.

Assume

$$p_+(t) \equiv \big(|p(t)| + p(t)\big)/2, \quad p_-(t) \equiv \big(|p(t)| - p(t)\big)/2,$$

$$\mathcal{P}_+ = \int_a^b p_+(t)\,dt, \quad \mathcal{P}_- = \int_a^b p_-(t)\,dt.$$

Theorem 1. *Let $\ell_1 \geq 0$, $\ell_2 \leq 0$,*

$$\ell_1 - \ell_2 + \operatorname{mes}\big\{t \in [a, b] :\ p(t) \neq 0\big\} > 0, \tag{3}$$

$$\ell_1 - \ell_2 + \mathcal{P}_+ \leq \mathcal{P}_-, \tag{4}$$

and there exist a number $\lambda \geq 1$ such that

$$\int_a^b [p(t)]_-^\lambda\,dt \leq \frac{4}{b-a}\Big(\frac{\pi}{b-a}\Big)^{2\lambda-2}. \tag{5}$$

Then problem (1), (2) *has one and only one solution.*

To prove this theorem, we need the following

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Lemma 1. *Let $\ell_1 \geq 0$, $\ell_2 \leq 0$, conditions* (3), (4) *be fulfilled and the homogeneous problem*

$$u'' = p(t)u, \tag{1_0}$$

$$u'(a) = \ell_1 u(a), \quad u'(b) = \ell_2 u(b) \tag{2_0}$$

have a nontrivial solution u. Then there exist points $t_0 \in]a, b[$, $t_1 \in [a, t_0[$, and $t_2 \in]t_0, b]$ such that

$$u(t_0) = 0, \quad u'(t_1) = 0, \quad u'(t_2) = 0. \tag{6}$$

Proof. In view of (2_0), it is obvious that

$$u(a) \neq 0, \quad u(b) \neq 0. \tag{7}$$

First, let us show that the solution u in the interval $]a, b[$ has at least one zero. Assume the contrary, i.e., $u(t) \neq 0$ for $a < t < b$. Then by virtue of inequality (7), we have

$$u(t) \neq 0 \;\text{ for }\; a \leq t \leq b.$$

Consequently,

$$\frac{u''(t)}{u(t)} + p(t) = 0 \;\text{ for almost all }\; t \in [a, b]. \tag{8}$$

Integrating this identity from a to b, and taking into account equality (2_0), we find

$$\int_a^b \frac{u'^2(t)}{u^2(t)}\, dt = \ell_1 - \ell_2 + \mathcal{P}_+ - \mathcal{P}_-,$$

whence, owing to conditions (2)–(4) and (8), it follows that $u'(t) \equiv 0$,

$$p(t) = 0 \;\text{ for almost all }\; t \in]a, b], \;\; \ell_1 - \ell_2 > 0,$$

and either $u(a) = 0$, or $u(b) = 0$. But this contradicts condition (7). The obtained contradiction proves that for some $t_0 \in]a, b[$ the equality

$$u(t_0) = 0 \tag{9}$$

is fulfilled.

Without loss of generality, we can assume that

$$u'(t_0) > 0. \tag{10}$$

If $\ell_1 = 0$, then

$$u'(t_1) = 0, \tag{11}$$

where $t_1 = a$. Let us show that if $\ell_1 > 0$, then this equality is fulfilled for some $t_1 \in]a, t_0[$. Assume the contrary, i.e.,

$$u'(t) > 0 \;\text{ for }\; a < t \leq t_0.$$

Then, in view of (9), we have

$$u(t) < 0 \;\text{ for }\; a \leq t < t_0.$$

But this is impossible since

$$u(a) = u'(a)/\ell_1 \geq 0.$$

Thus we have proved that for some $t_1 \in [a, t_0[$ equality (11) is fulfilled.

Analogously we can show that for some $t_2 \in]t_0, b]$, the equality

$$u'(t_2) = 0$$

is fulfilled. □

Lemma 2 (T. Kiguradze [2]). *Let for some $a_0 \in [a, b[$ and $b_0 \in]a_0, b]$ the differential equation (1_0) have a nontrivial solution u satisfying either the boundary conditions*

$$u'(a_0) = 0, \quad u(b_0) = 0,$$

or the boundary conditions

$$u(a_0) = 0, \quad u'(b_0) = 0.$$

Then

$$(b_0 - a_0)^{2\lambda-1} \int\limits_{a_0}^{b_0} [p(t)]_-^\lambda \, dt > \Big(\frac{\pi}{2}\Big)^{2\lambda-2} \quad \text{for } \lambda \geq 1.$$

This lemma is a corollary of Theorem 1.3 from [2].

Proof of Theorem 1. Assume that the theorem is not true. Then, owing to the Fredholmicity of problem $(1), (2)$, the homogeneous problem $(1_0), (2_0)$ has a nontrivial solution u.

According to Lemma 1, there exist points $t_0 \in]a, b[$, $t_1 \in [a, t_0[$ and $t_2 \in]t_0, b]$ such that the solution u satisfies equalities (6). Thus by Lemma 2, we have the inequalities

$$(t_0 - t_1)^{2\lambda-1} \int\limits_{t_1}^{t_0} [p(t)]_-^\lambda \, dt > \Big(\frac{\pi}{2}\Big)^{2\lambda-2}, \quad (t_2 - t_0)^{2\lambda-1} \int\limits_{t_0}^{t_2} [p(t)]_-^\lambda \, dt > \Big(\frac{\pi}{2}\Big)^{2\lambda-2}.$$

Consequently,

$$\big[(t_0 - t_1)(t_2 - t_0)\big]^{2\lambda-1} \Big(\int\limits_{t_1}^{t_0} [p(t)]_-^\lambda \, dt\Big)\Big(\int\limits_{t_0}^{t_2} [p(t)]_-^\lambda \, dt\Big) > \Big(\frac{\pi}{2}\Big)^{4\lambda-4}.$$

On the other hand,

$$(t_0 - t_1)(t_2 - t_0) \leq \frac{(t_2 - t_1)^2}{4} \leq \frac{(b-a)^2}{4},$$

$$\Big(\int\limits_{t_1}^{t_0} [p(t)]_-^\lambda \, dt\Big)\Big(\int\limits_{t_0}^{t_2} [p(t)]_-^\lambda \, dt\Big) \leq \Big(\int\limits_{t_1}^{t_2} [p(t)]_-^\lambda \, dt\Big)^2 \Big/ 4 \leq \Big(\int\limits_{a}^{b} [p(t)]_-^\lambda \, dt\Big)^2 \Big/ 4.$$

Therefore,

$$\frac{1}{4}\Big(\frac{b-a}{2}\Big)^{4\lambda-2}\Big(\int\limits_a^b [p(s)]_-^\lambda \, ds\Big)^2 > \Big(\frac{\pi}{2}\Big)^{4\lambda-4}.$$

However, this inequality contradicts inequality (5). The obtained contrdiction proves the theorem. □

Remark 1. If $\ell_1 \geq 0$ and $\ell_2 \leq 0$, then for problem $(1), (2)$ to be uniquely solvable, it is necessary that inequality (3) is fulfilled. Indeed, if the above-mentioned inequality is violated, then $p(t) = 0$ for almost all $t \in]a, b[$, $\ell_1 = \ell_2 = 0$ and, consequently, the homogeneous problem $(1_0), (2_0)$ has an infinite set of solutions.

Remark 2. Examples 1 and 2 below show that conditions (4) and (5) in Theorem 1 are unimprovable and they cannot be replaced by the conditions

$$\ell_1 - \ell_2 + \mathcal{P}_+ < \mathcal{P}_- + \varepsilon, \tag{12}$$

$$\int\limits_a^b [p(t)]_-^\lambda \, dt < \frac{4}{b-a}\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{2\lambda-2} \tag{13}$$

no matter how small $\varepsilon > 0$ is.

Example 1. Let ε be an arbitrary positive constant,

$$p(t) \equiv -\Big(\frac{2x}{b-a}\Big)^2, \quad \ell_1 = \frac{2x}{b-a}\ \mathrm{tg}(x), \quad \ell_2 = -\frac{2x}{b-a}\ \mathrm{tg}(x), \tag{14}$$

and $x \in]0,1[$ be so small that

$$\frac{4x}{b-a}\ \mathrm{tg}(x) < \frac{4x^2}{b-a} + \varepsilon.$$

Then $\ell_1 > 0$, $\ell_2 < 0$,

$$\ell_1 - \ell_2 + \mathcal{P}_+ = \frac{4x}{b-a}\ \mathrm{tg}(x) < \mathcal{P}_- + \varepsilon,$$

$$\int_a^b [p(t)]_-^\lambda\, dt = (b-a)\Big(\frac{2x}{b-a}\Big)^{2\lambda} < (b-a)\Big(\frac{2}{b-a}\Big)^{2\lambda} < \frac{4}{b-a}\,\Big(\frac{\pi}{b-a}\Big)^{2\lambda-2}.$$

Consequently, all the conditions of Theorem 1 are fulfilled, except inequality (4), instead of which inequality (12) holds. Nevertheless, in this case the homogeneous problem $(1_0), (2_0)$ has the nontrivial solution

$$u(t) \equiv \cos\Big(\frac{2x(t-a)}{b-a} - x\Big).$$

Example 2. Let $\varepsilon \in]0, \frac{\pi}{4}[$, $x \in]0,\varepsilon[$,

$$p(t) \equiv -\Big(\frac{\pi+x}{b-a}\Big)^2, \quad \ell_1 = \frac{\pi+x}{b-a}\ \mathrm{tg}(x), \quad \ell_2 = 0,$$

and the number $\lambda \in [1,+\infty[$ be such that

$$\frac{\pi+x}{b-a} < \Big(\frac{2}{b-a}\Big)^{\frac{1}{\lambda}}\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{1-\frac{1}{\lambda}}.$$

Then

$$\ell_1 - \ell_2 + \mathcal{P}_+ = \frac{\pi+x}{b-a}\ \mathrm{tg}(x) < \frac{\pi+x}{b-a} < \frac{(\pi+x)^2}{b-a} = \mathcal{P}_-,$$

$$\int_a^b [p(t)]_-^\lambda\, dt = (b-a)\Big(\frac{\pi+x}{b-a}\Big)^{2\lambda} < \frac{4}{b-a}\,\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{2\lambda-2}.$$

Consequently, all the conditions of Theorem 1 are fulfilled, except inequality (5), instead of which inequality (13) holds. On the other hand, the homogeneous problem $(1_0), (2_0)$ has the nontrivial solution

$$u(t) \equiv \cos\Big(\frac{(\pi+x)(t-a)}{b-a} - x\Big).$$

Theorem 2. *If $\ell_1 \geq 0$, $\ell_2 \leq 0$, and*

$$\mathcal{P}_- < \frac{\ell_1 - \ell_2 + \mathcal{P}_+}{1 + (b-a)(\ell_1 - \ell_2 + \mathcal{P}_+)}\,, \tag{15}$$

then problem $(1), (2)$ *has one and only one solution.*

Proof. First note that inequality (15) yields the following inequalities

$$\delta = \ell_1 - \ell_2 + \mathcal{P}_+ - \mathcal{P}_- > 0, \tag{16}$$

$$r = 1 - (b-a)(\mathcal{P}_- + \delta^{-1}\mathcal{P}_-^2) > 0. \tag{17}$$

Assume that the theorem is not true. Then the homogeneous problem $(1_0), (2_0)$ has a nontrivial solution u. Put

$$x = \min\big\{|u(t)|:\ a \leq t \leq b\big\}, \quad y = \Big(\int_a^b u'^2(t)\, dt\Big)^{\frac{1}{2}}.$$

Then
$$x^2 \le u^2(t) \le x^2 + 2(b-a)^{\frac{1}{2}} y + (b-a)y^2 \ \text{ for } \ a \le t \le b. \tag{18}$$
On the other hand, in view of inequality (16), it is obvious that
$$y > 0. \tag{19}$$

Integrating both sides of the identity
$$u''(t)u(t) = p(t)u^2(t) \ \text{ for almost all } \ t \in [a,b]$$
from a to b and taking into account the boundary conditions (2_0), we obtain
$$\ell_1 u^2(a) - \ell_2 u^2(b) + \int\limits_a^b [p(t)]_+ u^2(t)\,dt + \int\limits_a^b u'^2(t)\,dt = \int\limits_a^b [p(t)]_- u^2(t)\,dt.$$
Thus, by inequality (18), it follows that
$$(\ell_1 - \ell_2 + \mathcal{P}_+)x^2 + y^2 \le \mathcal{P}_-\big(x^2 + 2(b-a)^{\frac{1}{2}} xy + (b-a)y^2\big),$$
that is,
$$\big(\delta^{\frac{1}{2}} x - (b-a)^{\frac{1}{2}} \delta^{-\frac{1}{2}} \mathcal{P}_- y\big)^2 + ry^2 \le 0.$$
However, this inequality contradicts inequalities (17) and (19). The obtained contradiction proves the theorem. □

Remark 3. Condition (15) is unimprovable and it cannot be replaced by the condition
$$\mathcal{P}_- < \frac{\ell_1 - \ell_2 + \mathcal{P}_+}{1 + (b-a)(\ell_1 - \ell_2 + \mathcal{P}_+)} + \varepsilon \tag{20}$$
no matter how small is $\varepsilon > 0$.

Indeed, if
$$0 < \varepsilon < \frac{4}{b-a}\,, \ \ 0 < x < \varepsilon^{\frac{1}{2}} (b-a)^{\frac{1}{2}}/2,$$
and the function p and numbers ℓ_i $(i = 1,2)$ are defined by equalities (14), then instead of (15) inequality (20) holds, but nevertheless, the homogeneous problem $(1_0), (2_0)$ has the nontrivial solution
$$u(t) \equiv \cos\Big(\frac{2x(t-a)}{b-a} - x\Big).$$

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