საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 5, №3, 2011
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 5, no. 3, 2011

Mathematics

Positive Solutions of Two-Point Boundary Value Problems for Higher Order Nonlinear Singular Differential Equations

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ABSTRACT. For higher order nonlinear differential equations with singularities with respect to the time and phase variables, the sufficient conditions for the existence of positive solutions of the Dirichlet and Focal boundary value problems are established. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *nonlinear differential equation, strong singularity with respect to the time variable, strong singularity with respect to the phase variable, Dirichlet problem, focal boundary value problem, positive solution.*

In an open interval $]a,b[$, we consider the nonlinear differential equation

$$u^{(2m)} = f(t,u) \tag{1}$$

with the boundary conditions of one of the following two types:

$$u^{(i-1)}(a+) = 0, \quad u^{(i-1)}(b-) = 0 \quad (i = 1,\dots,m) \tag{2_1}$$

and

$$u^{(i-1)}(a+) = 0, \quad u^{(m+i-1)}(b-) = 0 \quad (i = 1,\dots,m). \tag{2_2}$$

(2_1) is called the conjugate boundary conditions, or the Dirichlet conditions, and (2_2) is called the focal boundary conditions, or the Nicoletti conditions. The problems of the type (1), (2_1) and (1), (2_2) take central place in the theory of boundary value problems for ordinary differential equations and are the subject of numerous investigations (see [1-11] and the references therein). In the present paper we give new sufficient conditions for the existence of positive solutions of the above-mentioned problems.

Throughout the paper, it is assumed that m is an arbitrary natural number, and $f :]a,b[\times]0,+\infty[\to R$ is a continuous function.

A function $u :]a,b[\to]0,+\infty[$ **is said to be a positive solution of equation** (1), if it is $2m$-times continuously differentiable and at every point of the interval $]a,b[$ satisfies this equation.

A positive solution of equation (1) **is said to be a positive solution of problem** (1), (2_1) (**of problem** (1), (2_2)), if it satisfies the boundary conditions (2_1) (the boundary conditions (2_2)), where by $u^{(j)}(a+)$ and $u^{(j)}(b-)$ there are understood, respectively, the right and the left limits of the function $u^{(j)}$ at the points a and b.

We are especially interested in the case where equation (1) has singularities with respect to the time and phase variables. Following [1], [5] we will say that equation (1) with respect to the time variable has **strong singularities (strong singularity) at the points *a* and *b* (at the point *a*)**, if

$$\int_a^{t_0} (t-a)^{2m-1} |f(t,x)|\, dt = \int_{t_0}^{b} (t-a)^{2m-1} |f(t,x)|\, dt = +\infty \quad \left(\int_a^{t_0} (t-a)^{2m-1} |f(t,x)|\, dx = +\infty \right)$$

$$\text{for} \quad a < t_0 < b, \quad x > 0.$$

If, however, there is a set $I_0 \subset]a,b[$ such that

$$\limsup_{x \to 0} \left(x^k |f(t,x)| \right) = +\infty \quad \text{for arbitrary} \quad t \in I_0 \text{ and } k > 0,$$

then we will say that equation (1) has strong **singularity with respect to the phase variable**.

Unlike the previous well-known results (see [2, 3, 8, 10, 11]), the theorems below on the existence and uniqueness of a positive solution of problem (1), (2_1) (of problem (1), (2_2)) cover the case where equation (1), along with strong singularities with respect to the time variable at the points a and b (at the point a), has strong singularity with respect to the phase variable as well.

By $C^{2m,m}(]a,b[)$ we denote the space of $2m$-times continuously differentiable functions, satisfying the condition

$$\int_a^b \left| u^{(m)}(t) \right|^2 dt < +\infty .$$

First we consider problem (1), (2_1). The following theorem is valid.

Theorem 1 (A principle of a priori boundedness). *Let in the domain* $]a,b[\times]0,+\infty[$ *the inequality*

$$(-1)^m f(t,x) \ge h(t) x^{\mu} \tag{3}$$

be fulfilled, where $\mu \in [0,1[$ *is a constant, and* $h:[a,b] \to [0,+\infty[$ *is a continuous function, not identically equal to zero. Let, moreover, there exist a positive number r such that for any continuous function* $\lambda:[a,b] \to [0,1]$ *every positive solution of the differential equation*

$$u^{(n)} = \lambda(t) f(t,u) + (-1)^m (1-\lambda(t)) h(t) u^{\mu}, \tag{4}$$

belonging to the space $C^{2m,m}(]a,b[)$ *and satisfying the boundary conditions* (2_1)*, admits the estimate*

$$\int_a^b \left| u^{(m)}(t) \right|^2 dt \le r . \tag{5}$$

Then problem (1), (2_1) *in the space* $C^{2m,m}(]a,b[)$ *has at least one positive solution.*

On the basis of this theorem, the following theorem on the solvability and unique solvability of problem (1), (2_1) is proved.

Theorem 2. *Let in the domain* $]a,b[\times]0,+\infty[$ *the inequality*

$$h(t) x^{\mu} \le (-1)^m f(t,x) \le \ell \left(\frac{1}{(t-a)^{2m}} + \frac{1}{(b-t)^{2m}} \right) x + h_0(t) x^{\mu_0} + q(t,x) \tag{6}$$

be satisfied, where $\mu \in [0,1[$, $\mu_0 \in [0,1[$ *and* $\ell \ge 0$ *are constants, h and* $h_0 :]a,b[\to [0,+\infty[$ *are continuous functions, and* $q :]a,b[\times]0,+\infty[\to [0,+\infty[$ *is a continuous and nonincreasing in the second argument function. If, moreover,*

$$\ell < 4^{-m}\left[(2m-1)!!\right]^2, \tag{7}$$

$$h(t) \not\equiv 0, \quad \int_a^b \left[(t-a)(b-t)\right]^{(1+\mu_0)\left(m-\frac{1}{2}\right)} h_0(t)\,dt < +\infty,$$

and

$$\int_a^b \left[(t-a)(b-t)\right]^{m-\frac{1}{2}} q\left(t,(t-a)^m (b-t)^m x\right)dt < +\infty \quad for \quad x > 0, \tag{8}$$

then problem (1), (2_1) *in the space* $C^{2m,m}(]a,b[)$ *has at least one positive solution.*

Theorem 3. *If*

$$(-1)^m\left[f(t,x) - f(t,y)\right] \le \ell\left(\frac{1}{(t-a)^{2m}} + \frac{1}{(b-t)^{2m}}\right)(x-y) \quad for \quad a < t < b,\ x > y > 0,$$

where ℓ *is a nonnegative constant, satisfying inequality* (7), *then problem* (1), (2_2) *in the space* $C^{2m,m}(]a,b[)$ *has at most one positive solution.*

Let us now consider problem (1), (2_2). Assume

$$f^*(t;x_0,x) = \max\left\{|f(t,y)| : x_0 \le y \le x\right\} \quad \text{for} \quad a < t < b,\ 0 < x_0 < x.$$

The following theorems are valid.

Theorem 4. (A principle of a priori boundedness). *Let*

$$\int_{t_0}^b f^*(t;x_0,x)\,dt < +\infty \quad for\ any\ t_0 \in]a,b[,\ x_0 > 0\ and\ x > x_0,$$

and let in the domain $]a,b[\times]0,+\infty[$ *inequality* (3) *be fulfilled, where* $\mu \in [0,1[$ *is a constant and* $h : [a,b] \to [0,+\infty[$ *is a continuous function, not identically equal to zero. Let, moreover, there exist a positive number r such that every positive solution of problem* (4), (2_2), *belonging to the space* $C^{2m,m}(]a,b[)$, *admits estimate* (5). *Then problem* (1), (2_2) *in the space* $C^{2m,m}(]a,b[)$ *has at least one positive solution.*

Theorem 5. *Let in the domain* $]a,b[\times]0,+\infty[$ *the inequality*

$$h(t)x^{\mu} \le (-1)^m f(t,x) \le \frac{\ell}{(t-a)^{2m}}x + h_0(t)x^{\mu_0} + q(t,x)$$

be fulfilled, where $\mu \in [0,1[$, $\mu_0 \in [0,1[$, $\ell \ge 0$ *are constants, h and* $h_0 :]a,b[\to [0,+\infty[$ *are continuous functions, and* $q :]a,b[\times]0,+\infty[\to [0,+\infty[$ *is a continuous and nonincreasing in the second argument function. If, moreover,*

$$h(t) \not\equiv 0, \quad \int_a^b (t-a)^{(1+\mu_0)\left(m-\frac{1}{2}\right)} h_0(t)\,dt < +\infty,$$

$$\int_a^b (t-a)^{m-\frac{1}{2}} q\left(t,(t-a)^m x\right)dt < +\infty \quad for \quad x > 0, \tag{9}$$

and inequality (7) *is fulfilled, then problem* (1), (2_2) *in the space* $C^{2m,m}\left(\left]a,b\right[\right)$ *has at least one positive solution.*

Theorem 6. *If*

$$(-1)^m\left[f(t,x)-f(t,y)\right]\le\frac{\ell}{(t-a)^{2m}}(x-y)\quad for\ a<t<b,\ x>y>0,$$

where ℓ *is a nonnegative constant, satisfying inequality* (7)*, then problem* (1), (2_2) *in the space* $C^{2m,m}\left(\left]a,b\right[\right)$ *has at most one positive solution.*

As examples, let us consider the differential equations

$$u^{(2m)}=(-1)^m\left[p(t)u+p_1(t)u^{\mu}+p_2(t)u^{-\nu}\right],\tag{10}$$

$$u^{(2m)}=(-1)^m\left[p(t)u+q(t,u)\right],\tag{11}$$

where $\mu\in\left[0,1\right[$ and $\nu\ge 0$ are constants, $p,p_1,p_2:\left]a,b\right[\to\left[0,+\infty\right[$ are continuous functions, and $q:\left]a,b\right[\times\left]0,+\infty\right[\to\left[0,+\infty\right[$ is the continuous and nonincreasing in the second argument function.

From Theorems 2 and 3 follow the following corollaries.

Corollary 1. *Let*

$$p(t)\le\ell\left(\frac{1}{(t-a)^{2m}}+\frac{1}{(b-t)^{2m}}\right)\quad for\ \ a<t<b,\tag{12}$$

where ℓ *is the nonnegative constant satisfying inequality* (7)*. If, moreover,*

$$p_1(t)\not\equiv 0,\quad \int_a^b\left[(t-a)(b-t)\right]^{(1+\mu)\left(m-\frac{1}{2}\right)}p_1(t)\,dt<+\infty,$$
$$\int_a^b\left[(t-a)(b-t)\right]^{(1-\nu)m-\frac{1}{2}}p_2(t)\,dt<+\infty,$$

then the problem (10), (2_1) *in the space* $C^{2m,m}\left(\left]a,b\right[\right)$ *has at least one positive solution.*

Corollary 2. *Let in the domain* $\left]a,b\right[\times\left]0,+\infty\right[$ *the inequality*

$$q(t,x)\ge q_0(t)\tag{13}$$

be fulfilled, where $q_0:\left]a,b\right[\to\left[0,+\infty\right[$ *is the continuous function, not identically equal to zero. If, moreover, the conditions* (7), (8) *and* (12) *are fulfilled, then the problem* (11), (2_1) *in the space* $C^{2m,m}\left(\left]a,b\right[\right)$ *has a unique positive solution.*

From Theorems 5 and 6 follow the following corollaries.

Corollary 3. *Let*

$$p(t)\le\frac{\ell}{(t-a)^{2m}}\quad for\ \ a<t<b,\tag{14}$$

where ℓ *is a nonnegative constant satisfying inequality* (7)*. If, moreover,*

$$p_1(t) \not\equiv 0, \quad \int_a^b (t-a)^{(1+\mu)\left(m-\frac{1}{2}\right)} p_1(t)\,dt < +\infty,$$

$$\int_a^b (t-a)^{(1-\nu)m-\frac{1}{2}} p_2(t)\,dt < +\infty,$$

then the problem (10), (2_2) *in the space* $C^{2m,m}(]a,b[)$ *has at least one positive solution.*

Corollary 4. *Let in the domain* $]a,b[\times]0,+\infty[$ *inequality* (13) *be fulfilled, where* $q_0 :]a,b[\to[0,+\infty[$ *is a continuous function, not identically equal to zero. If, moreover, the conditions* (7), (9) *and* (14) *are fulfilled, then the problem* (11), (2_2) *in the space* $C^{2m,m}(]a,b[)$ *has a unique positive solution.*

Remark. Let

$$p(t) \equiv \ell\left(\frac{1}{(t-a)^{2m}} + \frac{1}{(b-t)^{2m}}\right), \quad q(t,x) \equiv \exp\left(\frac{(t-a)^m (b-t)^m}{x}\right)$$

$$\left(p(t) \equiv \frac{\ell}{(t-a)^{2m}}, \quad q(t,x) \equiv \exp\left(\frac{(t-a)^m}{x}\right)\right)$$

where ℓ is a positive number satisfying inequality (7). Then all conditions of Corollary 2 (of Corollary 4) are fulfilled. On the other hand, in the case under consideration, equation (11) along with strong singularities with respect to the time variable at the points *a* and *b* (at the point *a*) has strong singularity with respect to the phase variable as well.

Acknowledgement. This work is supported by the Shota Rustaveli National Science Foundation (Project No. GNSF/ST09_175_3-101).

მათემატიკა

მაღალი რიგის არაწრფივ სინგულარულ დიფერენციალურ განტოლებათა ორწერტილოვანი სასაზღვრო ამოცანების დადებითი ამონახსნები

ი. კიღურაძე

აკადემიკოსი, ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის ა. რაზმაძის მათემატიკის ინსტიტუტი

მაღალი რიგის არაწრფივი დიფერენციალური განტოლებებისათვის სინგულარობებით დროითი და ფაზური ცლადების მიმართ დადგენილია დირიხლესა და ფოკალური სასაზღვრო ამოცანების დადებითი ამონახსნების არსებობის საკმარისი პირობები.

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Received July, 2011