Mathematics

Solvability Conditions of Nonlocal Problems for Singular in Phase Variables Higher Order Differential Equations

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ABSTRACT. The unimprovable in a certain sense conditions guaranteeing the solvability of nonlocal problems for singular in phase variables higher order differential equations are established.

Key words and phrases. *Higher order differential equation, nonlinear, singular in phase variables, principle of a priori boundedness, unimprovable conditions of solvability.*

Let $R_+ = [0,+\infty[$, $R_{0+} =]0,+\infty[$, $R_{0+}^k = \{(x_1,\dots,x_k) : x_1 > 0,\dots,x_n > 0\}$. In a finite interval $[a,b]$ we consider the differential equation

$$u^{(n)} = f\left(t,u,\dots,u^{(n-1)}\right) \tag{1}$$

with a continuous right-hand side $f : [a,b]\times R_{0+}^n \to R_+$. We are mainly interested in the case where the equation (1) is singular in phase variables, i.e. the case where it is impossible to extend the function f continuously to $[a,b]\times R_+^n$. For example, if for any $i \in \{1,\dots,n\}$ and $t \in [a,b]$ the equality

$$\lim_{x_i \to 0} f\left(t,x_1,\dots,x_n\right) = +\infty \quad \text{for} \quad x_k > 0 \quad \left(k \neq i; \quad k = 1,\dots,n\right)$$

is satisfied, then the equation (1) is singular in phase variables.

The boundary value problems for singular in phase variables second order differential equations find a wide application in different areas of natural science and are the subject of numerous investigations (see, e.g., [1-8] and the references therein). As for the singular differential equation (1), for it only the initial problem is studied when $n > 2$ [9], whereas the boundary value problems, including the problem with the nonlocal boundary conditions

$$u^{(i-1)}(a) = \varphi_i\left(u^{(i-1)}(b_1),\dots,u^{(i-1)}(b_m)\right) \quad \left(i = 1,\dots,m\right), \tag{2}$$

remain still unstudied. The present paper is devoted, namely, to singular problems of the type (1), (2).

Throughout the paper, it is assumed that

$$a \le b_i \le b \quad (i = 1, \dots, m-1), \quad b_m = b,$$

and along with $f : [a,b] \times R_{0+}^m \to R_+$, the functions

$$\varphi_i : R_{0+}^m \to R_+ \quad (i = 1, \dots, m)$$

are also continuous.

By $C^-([a,b] \times R_{0+}^k; R_+)$ we denote the set of continuous nonnegative functions defined on $[a,b] \times R_{0+}^k$, which are nonincreasing in phase variables. Consequently, a continuous function $g : [a,b] \times R_{0+}^k \to R_+$ belongs to the set $C^-([a,b] \times R_{0+}^k; R_+)$ if

$$g(t, y_1, \dots, y_k) \le g(t, x_1, \dots, x_k) \quad \text{for} \quad a \le t \le b, \quad y_i \ge x_i > 0 \quad (i = 1, \dots, k).$$

A function $u : [a,b] \to R_{0+}$ is said to be a solution of the equation (1) if it is n-times continuously differentiable and at every point of the interval $[a,b]$ together with the inequalities

$$u^{(i-1)}(t) > 0 \quad (i = 1, \dots, n)$$

satisfies the above differential equation.

A solution of the differential equation (1) satisfying the boundary conditions (2) is called a solution of the problem (1), (2).

We investigate the problem (1), (2) in the cases, where the function f on the set $[a,b] \times R_{0+}^n$ admits either the one-sided estimate of one of the following two types

$$f(t, x_1, \dots, x_n) \ge f_0(t, x_1, \dots, x_n), \tag{3}$$

$$f(t, x_1, \dots, x_n) \ge \sum_{i=1}^n h_i(t) x_i + f_0(t, x_1, \dots, x_n), \tag{4}$$

or the two-sided estimate

$$f_0(t, x_1, \dots, x_n) \le f(t, x_1, \dots, x_n) \le \sum_{i=1}^n \big(h_i(t) + f_i(t, x_1, \dots, x_n)\big) x_i \,. \tag{5}$$

As for the functions $\varphi_i \ (i = 1, \dots, n)$, on the set

$$\{(x_1, \dots, x_m) \in R_{0+}^m : \ x_k \le x_m \quad (k = 1, \dots, m)\}$$

they admit either the one-sided estimates

$$\varphi_i(t, x_1, \dots, x_m) \ge \alpha_i x_m \quad (i = 1, \dots, n), \tag{6}$$

or the two-sided estimates

$$\beta_i x_m \le \varphi_i(x_1, \dots, x_m) \le \alpha_i x_m + \alpha \quad (i = 1, \dots, n). \tag{7}$$

Moreover, it is assumed that

$$\alpha \ge 0, \quad 0 < \alpha_i < 1, \quad 0 < \beta_i < \alpha_i \quad (i = 1, \dots, n),$$

the functions $h_i : [a,b] \to R_+ \ (i = 1, \dots, n)$ are continuous, and the functions $f_i \ (i = 0, 1, \dots, n)$ satisfy the conditions

$$f_0 \in C^-([a,b] \times R_{0+}^n; R_+), \quad \max\{f_0(t, x, \dots, x) : \ a \le t \le b\} > 0 \quad \text{for} \quad x > 0, \tag{8}$$

$$f_i \in C^-([a,b] \times R_{0+}^n; R_+), \quad \lim_{x \to +\infty} \int_a^b f_i(t, x, \dots, x)\, dt = 0 \quad (i = 1, \dots, n). \tag{9}$$

Along with the problem (1), (2), we consider the auxiliary problem

$$u^{(n)}=(1-\lambda)f_0\left(t,u,\ldots,u^{(n-1)}\right)+\lambda f\left(t,u,\ldots,u^{(n-1)}\right), \tag{10}$$

$$u^{(i-1)}(a)=(1-\lambda)\alpha_i u^{(i-1)}(b)+\lambda\varphi_i\left(u^{(i-1)}(b_1),\ldots,u^{(i-1)}(b_m)\right)+\varepsilon \quad (i=1,\ldots,n), \tag{11}$$

depending on the parameters $\lambda\in\left]0,1\right]$ and $\varepsilon>0$.

The following proposition holds.

Proposition 1 (The principle of a priori boundedness). *Let the conditions* (3)*,* (6)*,* (8) *be fulfilled and there exist positive constants* ε_0 *and* r *such that for arbitrary* $\varepsilon\in\left]0,\varepsilon_0\right]$ *and* $\lambda\in\left]0,1\right]$ *every solution* u *of the problem* (10)*,* (11) *admits the estimates*

$$u^{(i-1)}(t)\le r \quad \textit{for} \quad a\le t\le b \quad (i=1,\ldots,n).$$

Then the problem (1)*,* (2) *has at least one solution.*

The above proposition allows one to state in a certain sense unimprovable sufficient conditions of solvability of the problem (1), (2). To formulate the theorem containing these conditions, we have to introduce the following notation.

$$v_{n+1}(t)=1,\quad v_k(t)=\frac{\alpha_k}{1-\alpha_k}\int_a^b v_{k+1}(s)\,ds+\int_a^t v_{k+1}(s)\,ds \quad (k=1,\ldots,n),$$

$$w_n(t)=\frac{1}{1-\alpha_n},\quad w_k(t)=\frac{\alpha_k}{1-\alpha_k}\int_a^b w_{k+1}(s)\,ds+\int_a^t w_{k+1}(s)\,ds \quad (k=1,\ldots,n-1).$$

Theorem 1. *Let the conditions* (5) *and* (7)-(9) *be fulfilled. If, moreover, the functions* h_i $(i=1,\ldots,n)$ *satisfy either the inequalities*

$$\max\left\{\sum_{i=1}^n v_i(t)h_i(t):\ \ a\le t\le b\right\}\le 1,\qquad \min\left\{\sum_{i=1}^n v_i(t)h_i(t):\ \ a\le t\le b\right\}<1, \tag{12}$$

or the inequality

$$\sum_{i=1}^n\int_a^b w_i(t)h_i(t)\,dt\le 1, \tag{13}$$

then the problem (1)*,* (2) *has at least one solution.*

Theorem 2. *Let the conditions* (4)*,* (6)*,* (8) *be fulfilled and the functions* h_i $(i=1,\ldots,n)$ *satisfy one of the following two inequalities:*

$$\min\left\{\sum_{i=1}^n v_i(t)h_i(t):\ \ a\le t\le b\right\}\ge 1, \tag{14}$$

$$\sum_{i=1}^n\int_a^b w_i(t)h_i(t)\,dt\ge\frac{1}{\alpha_n}. \tag{15}$$

Then the problem (1)*,* (2) *has no solution.*

A particular case of the problem (1), (2) is the problem

$$u^{(n)}=\sum_{i=1}^n\left(h_i(t)+f_i\left(t,u,\ldots,u^{(n-1)}\right)\right)u^{(i-1)}+f_0\left(t,u,\ldots,u^{(n-1)}\right), \tag{16}$$

$$u^{(i-1)}(a)=\alpha_i u^{(i-1)}(b) \quad (i=1,\dots,n), \tag{17}$$

where $h_i:[a,b]\to R_+$ $(i=1,\dots,n)$ and $f_i:[a,b]\times R_{0+}^n\to R_+$ $(i=0,1,\dots,n)$ are continuous functions, and $\alpha_i\in\,]0,1[$ $(i=1,\dots,n)$.

Theorems 1 and 2 imply the following corollary.

Corollary 1. *If the conditions* (8)*,* (9) *hold and*

$$\max\left\{\sum_{i=1}^{n} v_i(t)h_i(t):\quad a\le t\le b\right\}\le 1, \tag{18}$$

then for the solvability of (16)*,* (17)*, it is necessary and sufficient that the inequality*

$$\min\left\{\sum_{i=1}^{n} v_i(t)h_i(t):\quad a\le t\le b\right\}<1$$

be fulfilled.

Corollary 2. *Let the functions* f_i $(i=0,1,\dots,n)$ *satisfy the conditions* (8)*,* (9)*. If, moreover, the inequality* (13) *is fulfilled, then the problem* (16)*,* (17) *has at least one solution, but if instead of* (13) *the inequality* (15) *is fulfilled, then this problem has no solution.*

Remark 1. According to Corollary 1, the condition (12) in Theorem 1 is unimprovable and it cannot be replaced by the condition (18). On the other hand, the condition (13) in this theorem is likewise unimprovable in the sense that it cannot be replaced by the condition

$$\sum_{i=1}^{n}\int_a^b w_i(t)h_i(t)\,dt\le 1+\varepsilon,$$

no matter how small is $\varepsilon>0$. Indeed, if

$$\alpha_n=\frac{1}{1+\varepsilon},\qquad \sum_{i=1}^{n}\int_a^b w_i(t)h_i(t)\,dt=1+\varepsilon,$$

then the condition (15) is fulfilled, and by Corollary 2, the problem (16), (17) has no solution.

Remark 2. The conditions (8), (9) are satisfied with the functions

$$f_i(t,x_1,\dots,x_n)=\sum_{k=1}^{n} f_{ik}(t)x_k^{-\ell_{ik}}\exp\left(\frac{m_{ik}}{x_k}\right) \quad (i=0,1,\dots,n),$$

where $f_{ik}:[a,b]\to R_{0+}$ $(i=0,1,\dots,n;\ k=1,\dots,n)$ are continuous functions, and ℓ_{ik} and m_{ik} are positive constants. Consequently, Proposition 1, Theorems 1, 2 and their corollaries cover the case in which the differential equations (1) and (16) in phase variables have singularities of arbitrary orders.

We investigate the unique solvability of the problem (1), (2) in the case, where the function f on the set $[a,b]\times R_{0+}^n$ admits the estimate

$$f(t,x_1,\dots,x_n)\ge f_0(t,x_n), \tag{19}$$

and the boundary conditions (2) are of the form

$$u^{(i-1)}(a)=\psi_i\left(u^{(i-1)}(b)\right) \quad (i=1,\dots,n). \tag{20}$$

Moreover, it is assumed that

$$f_0\in C^-\left([a,b]\times R_{0+};R_+\right),\quad \max\left\{f_0(t,x):\ a\le t\le b\right\}>0 \quad \text{for} \quad x>0, \tag{21}$$

and

$$\psi_i(x) \geq \beta_i x, \quad |\psi_i(x) - \psi_i(y)| \leq \alpha_i |x - y| \quad \text{for} \quad x > 0, \ y > 0 \ (i = 1, \dots, n), \tag{22}$$

where

$$0 < \beta_i \leq \alpha_i < 1 \ (i = 1, \dots, n).$$

By Theorem 2.1 from [10], the boundary value problem

$$\frac{dz}{dt} = f_0(t, z), \quad z(a) = \beta_n z(b)$$

has a unique solution. We denote this solution by z_n and assume that

$$z_k(t) = \frac{\beta_k}{1 - \beta_k} \int_a^b z_{k+1}(s)\, ds + \int_a^t z_{k+1}(s)\, ds \ (k = 1, \dots, n-1).$$

Theorem 3. *Let the conditions* (19), (21) *and* (22) *be fulfilled and*

$$\left(f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)\right) \operatorname{sgn}(x_n - y_n) \leq$$
$$\leq \sum_{i=1}^n h_i(t) |x_i - y_i| \quad \text{for} \quad a \leq t \leq b, \quad x_k \geq z_k(t), \quad y_k \geq z_k(t) \quad (k = 1, \dots, n),$$

where $h_i : [a, b] \to R_+$ $(i = 1, \dots, n)$ *are continuous functions satisfying either the inequalities* (12)*, or the inequality* (13)*. Then the problem* (1)*,* (20) *has one and only one solution.*

As an example, we consider the differential equation

$$u^{(n)} = \sum_{i=1}^n \ell_i(t) \left(u^{(i-1)}\right)^{\mu_i} + \ell_0(t) \left(u^{(n-1)}\right)^{-\mu} \tag{23}$$

with the boundary conditions (17), where

$$-\infty < \mu_i \leq 1 \quad (i = 1, \dots, n-1), \quad 0 < \mu_n \leq 1, \quad \mu > 0, \quad 0 < \alpha_i < 1 \quad (i = 1, \dots, n),$$

$\ell_i : [a, b] \to R_+$ $(i = 0, 1, \dots, n)$ are continuous functions, and $\max\{\ell_0(t) : \ a \leq t \leq b\} > 0$.

Suppose

$$z_n(t) = \left[\left(\alpha_n^{-\mu-1} - 1\right)^{-1} \int_a^b \ell_0(s)\, ds + \int_a^t \ell_0(s)\, ds \right]^{\frac{1}{\mu+1}},$$

$$z_i(t) = \frac{\alpha_i}{1 - \alpha_i} \int_a^b z_{i+1}(s)\, ds + \int_a^t z_{i+1}(s)\, ds \quad (i = 1, \dots, n-1),$$

$$h_i(t) = |\mu_i| \left(z_i(t)\right)^{\mu_i - 1} \ell_i(t) \quad (i = 1, \dots, n).$$

From Theorem 3 we have the following corollary.

Corollary 3. *If the functions* h_i $(i = 1, \dots, n)$ *satisfy either the inequalities* (12)*, or the inequality* (13)*, then the problem* (1), (2) *has one and only one solution.*

Acknowledgement. This work is supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

მათემატიკა

არალოკალურ ამოცანათა ამოხსნადობის პირობები ფაზური ცვლადების მიმართ სინგულარული მაღალი რიგის დიფერენციალური განტოლებებისათვის

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აკადემიის წევრი, ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის ანდრია რაზმაძის მათემატიკის ინსტიტუტი, თბილისი

დადგენილია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც უზრუნველყოფენ ფაზური ცვლადების მიმართ სინგულარული მაღალი რიგის დიფერენციალური განტოლებებისათვის არალოკალურ ამოცანათა ამოხსნადობას.

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Received May, 2015