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Ivan Kiguradze

OSCILLATORY SOLUTIONS OF HIGHER ORDER NONLINEAR NONAUTONOMOUS DIFFERENTIAL SYSTEMS

Abstract. Oscillatory properties of solutions of higher order nonlinear nonautonomous differential systems are considered. In particular, unimprovable in a certain sense conditions are found under which all proper solutions of those systems are oscillatory.

რეზიუმე. გამოკვლეულია მაღალი რიგის არაავტონომიური, არაწრფივი დიფერენციალური სისტემების ამონახსნების ოსცილაციური თვისებები. კერძოდ, ნაპოვნია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც უზრუნველყოფენ ამ სისტემების წესიერი ამონახსნების რხევადობას.

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On an infinite interval $[a,+\infty[$, we consider the differential system

$$u_i^{(n_i)} = g_i\big(t, u_1, \dots, u_1^{(n_1-1)}, u_2, \dots, u_2^{(n_2-1)}\big) \ \ (i=1,2), \tag{1}$$

where $n_1 \geq 1$, $n_2 \geq 2$, $a > 0$, $g_i : [a,+\infty[\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ $(i=1,2)$ are continuous functions, satisfying on $[a,+\infty[\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ one of the following two conditions

$$\begin{aligned} &g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1) \geq f_1(t,y_1)\operatorname{sgn}(y_1),\\ &g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1) \leq -f_2(t,x_1)\operatorname{sgn}(x_1), \end{aligned} \tag{2}$$

or

$$\begin{aligned} &g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1) \geq f_1(t,y_1)\operatorname{sgn}(y_1),\\ &g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1) \geq f_2(t,x_1)\operatorname{sgn}(x_1). \end{aligned} \tag{3}$$

Here $f_i[a,+\infty[\times\mathbb{R} \to \mathbb{R}$ $(i=1,2)$ are nondecreasing in the second argument continuous functions such that

$$f_i(t,x)\operatorname{sgn}(x) \geq 0 \ \ (i=1,2).$$

The present paper is devoted to the investigation of oscillatory properties of solutions of system (1). Previously, such properties have been investigated only in the cases when system (1) can be reduced to one differential equation of order $n = n_1 + n_2$ (see, [1–13, 15] and the references therein), or when $n_1 = n_2 = 1$ (see, [14]).

A solution of system (1) defined on some interval $[a_0,+\infty[\subset[a,+\infty[$ is said to be **proper** if it does not identically equal to zero in any neighbourhood of $+\infty$.

A proper solution (u_1,u_2) of system (1) is said to be **oscillatory** if at least one of its components changes sign in any neighbourhood of $+\infty$, and is said to be **Kneser** solution if in the interval $[a_0,+\infty[$ it satisfies the inequalities

$$\begin{aligned} &(-1)^i u_1^{(i)}(t)u_1(t) \geq 0 \quad (i=1,\dots,n_1),\\ &(-1)^k u_2^{(k)}(t)u_2(t) \geq 0 \ \ (k=1,\dots,n_2). \end{aligned}$$

Assume

$$n = n_1 + n_2,$$

and introduce the definitions.

Definition 1. System (1) has the **property** A_0 if every its proper solution for even n is oscillatory, and for odd n either is oscillatory or is a Kneser solution.

Definition 2. System (1) has the **property** B_0 if every its proper solution for even n is either oscillatory, or is a Kneser solution, or satisfies the condition

$$\lim_{t\to+\infty}|u^{(n_i-1)}(t)|>0 \ \ (i=1,2), \tag{4}$$

and for n odd either is oscillatory or satisfies condition (4).

If m is a natural number, then by $\mathcal{N}^0_m$ we denote the set of those $k\in\{1,\dots,m\}$ for which $m+k$ is even.

For an arbitrary natural k, we put

$$I_k(t,x)=x\bigg[t^{n_1-1}+\int\limits_a^t(t-s)^{n_1-1}\big|f_1(s,xs^{k-1})\big|\,ds\bigg].$$

Theorem 1. *Let condition* (2) *be satisfied and for any $x\neq 0$ and $k\in\mathcal{N}^0_{n_2-1}$ the equalities*

$$\int\limits_a^{+\infty}|f_1(t,x)|\,dt=+\infty,\quad \int\limits_a^{+\infty}t^{n_2-1}|f_2(t,x)|\,dt=+\infty, \tag{5}$$

$$\int\limits_a^{+\infty}t^{n_2-k-1}\big|f_2(t,I_k(t,x))\big|\,dt=+\infty \tag{6}$$

be fulfilled. Then system (1) *has the property A_0.*

Theorem 2. *Let condition* (3) *be satisfied. If, moreover, $n_2>2$ $(n_2=2)$ and for any $x\neq 0$ and $k\in\mathcal{N}^0_{n_2-2}$ equalities* (5) *and* (6) *hold* (*for any $x\neq 0$ equalities* (5) *is fulfilled*), *then system* (1) *has the property B_0.*

If $n_1=1$, $n_2=n-1$,

$$g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})=y_1,\quad g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})=f(t,x_1),$$

then system (1) is equivalent to the differential equation

$$u^{(n)}=f(t,u). \tag{7}$$

We consider the last equation in the case where $f:[a,+\infty[\,\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying either the condition

$$f(t,0)=0,\quad f(t,x)\leq f(t,y)\ \text{ for }\ t>a,\ \ x<y, \tag{8}$$

or the condition

$$f(t,0)=0,\quad f(t,x)\geq f(t,y)\ \text{ for }\ t>a,\ \ x<y. \tag{9}$$

A solution u of the equation (1), defined on some interval $[a_0,+\infty[\,\subset[a,+\infty[$, is said to be **proper** if is not identically zero in any neighborhood of $+\infty$.

A proper solution $u:[a_0+\infty[\,\to\mathbb{R}$ is said to be oscillatory if it changes the sign in any neighborhood of $+\infty$ and side to be **Kneser solution**

$$(-1)^iu^{(i)}(t)u(t)\geq 0\ \text{ for }\ t\geq a_0\ \ (i=1,\dots,n).$$

For equation (6), Definitions 1, 2 and Theorems 1 and 2 have the following forms.

Definition 3. Equation (7) has the property A_0 if any proper solution of this equation in case n even is oscillatory and in case n odd either is oscillatory or is a Kneser solution.

Definition 4. Equation (7) has the property B_0 if any proper solution of this equation in case n even either is oscillatory, or is a Kneser solution, or satisfies the condition

$$\lim_{t\to+\infty}|u^{(n-2)}(t)|=+\infty, \tag{10}$$

and in case n odd either is oscillatory or satisfies condition (10).

Theorem 3. *If along with* (8) *the condition*

$$\int\limits_a^{+\infty} t^{n-k-1}\big|f(t,xt^{k-1})\big|\,dt = +\infty \ \ \textit{for} \ \ x \neq 0, \ \ k \in \mathcal{N}_{n-1}^0 \tag{11}$$

holds, then equation (7) *has the property* A_0.

Theorem 4. *If* $n \geq 3$ *and along with* (9) *the condition*

$$\int\limits_a^{+\infty} t^{n-k-1}\big|f(t,xt^{k-1})\big|\,dt = +\infty \ \ \textit{for} \ \ x \neq 0, \ \ k \in \mathcal{N}_{n-2}^0 \tag{12}$$

holds, then equation (7) *has the property* B_0.

The conditions of Theorems 1–4 are in a certain sense unimprovable. Moreover, the following statements are valid.

Theorem 5. *Let condition* (8) *be satisfied and for any* $x \neq 0$ *there exist numbers* $t_x \geq a$ *and* $\delta(x) > 0$ *such that*

$$t^{n-k-1}\big|f(t,xt^{k-1})\big| \geq \delta(x)\big|f(t,xt^{n-1})\big| \ \ \textit{for} \ \ t \geq t_x, \ \ k \in \mathcal{N}_{n-1}^0.$$

Then for the differential equation (6) *to have the property* A_0 *it is necessary and sufficient equalities* (11) *to be fulfilled.*

Theorem 6. *Let conditions* (9) *be fulfilled,* $n \geq 3$ *and for any* $x \neq 0$ *there exist numbers* $t_x \geq a$ *and* $\delta(x) > 0$ *such that*

$$t^{n-k-2}\big|f(t,xt^{k-1})\big| \geq \delta(x)\big|f(t,xt^{n-2})\big| \ \ \textit{for} \ \ t \geq t_x, \ \ k \in \mathcal{N}_{n-2}^0.$$

Then for the differential equation (2) *to have the property* B_0 *it is necessary and sufficient equalities* (12) *to be fulfilled.*

An essential difference between the above formulated theorems and the results obtained earlier (see, e.g., [1–15]) is that they cover the case, where the right-hand sides of system (1) and of equation (7) are slowly increasing with respect to the phase variable functions.

As an example, let us consider the differential equation

$$u^{(n)} = g_0(t)f_0(u) + g_1(t)\ln\big(1+|u|\big)\operatorname{sign}(u), \tag{13}$$

$g_i : [a,+\infty[\, \to \mathbb{R}$ $(i = 0,1)$ are continuous functions, $f_0 : \mathbb{R} \to \mathbb{R}$ is a continuous, nondecreasing function such that

$$f_0(x)x > 0 \ \text{ for } \ x \neq 0, \quad \sup\big\{|f_0(x)| : \ x \in \mathbb{R}\big\} < +\infty.$$

Theorems 5 and 6 result in the following corollaries.

Corollary 1. *If* $n \geq 3$ *and* $g_0(t) \leq 0$, $g_1(t) \leq 0$ *for* $t \geq a$, *then for equation* (13) *to have property* A_0 *it is necessary and sufficient the equality*

$$\int\limits_a^{+\infty} \big[g_0(t) + g_1(t)\ln t\big]\,dt = -\infty$$

to be fulfilled.

Corollary 2. *If* $n \geq 4$ *and* $g_0(t) \geq 0$, $g_1(t) \geq 0$ *for* $t \geq a$, *then for differential equation* (13) *to have property* B_0 *it is necessary and sufficient the equality*

$$\int\limits_a^{+\infty} t\big[g_0(t) + g_1(t)\ln t\big]\,dt = +\infty$$

to be satisfied.

Consider now the case where the right-hand sides of system (1) on the set $[a,+\infty[\,\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ satisfy either the inequalities

$$\begin{aligned} g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1) &\geq p_1(t)|y_1|^{\lambda_1},\\ g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1) &\leq -p_2(t)|x_1|^{\lambda_2}, \end{aligned} \tag{14}$$

or the inequalities

$$\begin{aligned} g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1) &\geq p_1(t)|y_1|^{\lambda_1},\\ g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1) &\geq p_2(t)|x_1|^{\lambda_2}, \end{aligned} \tag{15}$$

where

$$\lambda_1>0,\ \ \lambda_2>0,\ \ \lambda_1\lambda_2>1,$$

and $p_i:[a,+\infty[\,\to[0,+\infty[$ are continuous functions.

Along with system (1), let us consider its particular cases

$$u_1^{(n_1)}=p_1(t)|u_2|^{\lambda_1}\operatorname{sgn}(u_2),\quad u_2^{(n_2)}=-p_2(t)|u_1|^{\lambda_2}\operatorname{sgn}(u_1), \tag{16}$$

and

$$u_1^{(n_1)}=p_1(t)|u_2|^{\lambda_1}\operatorname{sgn}(u_2),\quad u_2^{(n_2)}=p_2(t)|u_1|^{\lambda_2}\operatorname{sgn}(u_1). \tag{17}$$

Theorem 7. *If along with* (14) (*along with* (15)) *the conditions*

$$\int_a^{+\infty} p_1(t)\,dt=+\infty, \tag{18}$$

$$\int_a^{+\infty} t^{n_2-1}\bigg[\int_a^t (t-s)^{n_1-1}\Big(\frac{s}{t}\Big)^{(n_2-1)\lambda_1}p_1(s)\,ds\bigg]^{\lambda_2}p_2(t)\,dt=+\infty, \tag{19}$$

$$\lim_{x\to+\infty}\int_a^x t^{n_1-1}\bigg[\int_t^x (s-t)^{n_2-1}p_2(s)\,ds\bigg]^{\lambda_1}p_1(t)\,dt=+\infty \tag{20}$$

are fulfilled, then system (1) *has the property* A_0 (*the property* B_0).

Note that if

$$\liminf_{t\to+\infty}\frac{\int_a^t (t-s)^{n_1-1}s^{(n_2-1)\lambda_1}p_1(s)\,ds}{t^{(n_2-1)\lambda_1}\int_a^t (t-s)^{n_1-1}p_1(s)\,ds}>0, \tag{21}$$

then condition (19) takes the form

$$\int_a^{+\infty} t^{n_2-1}\bigg[\int_a^t (t-s)^{n_2-1}p_1(s)\,ds\bigg]^{\lambda_2}p_2(t)\,dt=+\infty. \tag{22}$$

For system (16), from Theorem 5 it follows

Corollary 3. *If conditions* (18) *and* (21) *are fulfilled, then for system* (16) (*system* (17)) *to have the property* A_0 (*the property* B_0), *it is necessary and sufficient the equalities* (20) *and* (22) *to be satisfied.*

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Author's address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: kig@rmi.ge