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THE GLOBAL INDICATOR OF CLASSICALITY OF AN ARBITRARY N-LEVEL QUANTUM SYSTEM

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It is commonly accepted that the deviation of the Wigner quasiprobability distribution of a quantum state from a proper statistical distribution signifies its nonclassicality. Following this ideology, we introduce a global indicator $\mathcal{Q}_N$ for quantifying the "classicality-quantumness" correspondence in the form of a functional on the orbit space $\mathcal{O}[\mathfrak{P}_N]$ of the adjoint action of the group SU(N) *on the state space $\mathfrak{P}_N$ of an N-dimensional quantum system. The indicator $\mathcal{Q}_N$ is defined as the relative volume of the subspace $\mathcal{O}[\mathfrak{P}_N^{(+)}] \subset \mathcal{O}[\mathfrak{P}_N]$ where the Wigner quasiprobability distribution is positive. The algebraic structure of $\mathcal{O}[\mathfrak{P}_N^{(+)}]$ is revealed and exemplified by the case of a single qubit ($N = 2$) and a single qutrit ($N = 3$). For the Hilbert–Schmidt ensemble of qutrits, the dependence of the global indicator on the moduli parameter of the Wigner quasiprobability distribution is found. Bibliography: 18 titles.*

1. Introduction

Over the past decades, a number of witnesses and measures of the nonclassicality of quantum systems have been formulated (see, e.g., [1–3]). Most of them are based on the primary impossibility of a classical statistical description of quantum systems. In particular, the nonexistence of positive definite probability distributions serves as a certain indication of the nonclassicality of a physical system.[1]

In the present note, we will focus on the problem of quantifying the nonclassicality of quantum systems associated with a finite-dimensional Hilbert space by studying the nonpositivity of the Wigner quasiprobability distributions (the Wigner function, or, in short, WF) [6–9]. Our treatment is based on the recent publications [10, 11], where the Wigner quasiprobability distribution $W^{(\boldsymbol{\nu})}_{\varrho}(\Omega_N)$ of an N-level quantum system is constructed via the dual pairing,

$$W^{(\boldsymbol{\nu})}_{\varrho}(\Omega_N) = \operatorname{tr}\left[\varrho\, \Delta(\Omega_N \,|\, \boldsymbol{\nu})\right], \tag{1}$$

of a density matrix ϱ, which is an element of the quantum state space

$$\mathfrak{P}_N = \{X \in M_N(\mathbb{C}) \mid X = X^{\dagger}, \quad X \geq 0, \quad \operatorname{tr}(X) = 1\}, \tag{2}$$

and an element of the dual space $\Delta(\Omega_N \,|\, \boldsymbol{\nu}) \in \mathfrak{P}^*_N$, the so-called Stratonovich–Weyl (SW) kernel. The dual space $\mathfrak{P}^*_N$ is defined as[2]

$$\mathfrak{P}^*_N = \{X \in M_N(\mathbb{C}) \mid X = X^{\dagger}, \quad \operatorname{tr}(X) = 1, \quad \operatorname{tr}\left(X^2\right) = N\}, \tag{3}$$

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[1] Furthermore, the negativity of quasiprobability distributions was shown to be a resource for quantum computation [4, 5].

[2] The algebraic equations in (3) define a family of s-parameter SW kernels. Further in the text, the s-dimensional moduli parameter $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_s)$, $s \leq N - 2$ (for details, see [11]), will be used to distinguish the corresponding Wigner distributions (1).

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and the SW kernel is a mapping between the phase space Ω_N and the dual space $\mathfrak{P}_N^*$. Assuming that the SW kernel $\Delta(\Omega_N)$ has isotropy group $H \subset U(N)$ of the form

$$H = U(k_1) \times U(k_2) \times \ldots \times U(k_{s+1}),$$

we identify the phase space Ω_N with a complex flag manifold:

$$\Omega_N \to \mathbb{F}^N_{d_1,d_2,\ldots,d_s} = U(N)/H,$$

where $(d_1, d_2, \ldots, d_s)$ is a sequence of positive integers with sum N such that $k_1 = d_1$ and $k_{i+1} = d_{i+1} - d_i$ with $d_{s+1} = N$.

The Wigner function defined in Eqs. (1)–(3) possesses all the properties of a proper statistical distribution except for nonnegativity. From a physical point of view, the positiveness of probability distributions is a fundamental element of the classical statistical paradigm. Therefore, if the WF attains negative values, it is undeniable that the physical system shows some "nonclassical" behavior. Following this observation, we introduce the *global indicator of classicality* $\mathcal{Q}_N$ characterizing the degree of closeness of a quasiprobability distribution to a proper one. Commonly used measures of deviation from classicality are defined as functionals either on the quantum state space (measures based on the distance from the base "classical state" [12–14]), or on the phase space (measures that depend on the volume of the phase space region where the WF is negative [2]). In contrast to this approach, we follow an alternative one, the so-called "minimal description," when characteristics of quantum systems are given solely in terms of SU(N)-invariants. In other words, we intend to define the global indicator $\mathcal{Q}_N$ as a functional on the unitary orbit space $\mathcal{O}[\mathfrak{P}_N]$. With this aim, we introduce the following definitions.

Definition 1. The unitary orbit space $\mathcal{O}[\mathfrak{P}_N]$ is the quotient space under the equivalence relation induced by the adjoint action of SU(N) on the state space $\mathfrak{P}_N$ with the quotient (canonical) mapping

$$\pi: \ \mathfrak{P}_N \longrightarrow \mathcal{O}[\mathfrak{P}_N] = \mathfrak{P}_N/\mathrm{SU}(N). \tag{4}$$

Definition 2. The set $\Omega_N^{(+)}[\varrho]$ is the subset of the phase space Ω_N where the Wigner function of a given state ϱ is nonnegative:

$$\Omega_N^{(+)}[\varrho] = \{x \in \Omega_N \,|\, W_\varrho(\Omega_N) \geq 0\}. \tag{5}$$

Definition 3. The subspace $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ consists of the states ϱ such that

$$\mathfrak{P}_N^{(+)} = \{\varrho \in \mathfrak{P}_N \,|\, \Omega_N^{(+)}[\varrho] = \Omega_N\}. \tag{6}$$

Definition 4. The subset $\mathcal{O}[\mathfrak{P}_N^{(+)}]$ is the image of $\mathfrak{P}_N^{(+)}$ under the quotient mapping (4):

$$\mathcal{O}[\mathfrak{P}_N^{(+)}] = \pi[\mathfrak{P}_N^{(+)}] = \{\pi(x) \,|\, x \in \mathfrak{P}_N^{(+)}\}. \tag{7}$$

Using the definitions above, we introduce the *global indicator of nonclassicality* $\mathcal{Q}_N$ of an N-dimensional quantum system as the following ratio:

$$\mathcal{Q}_N = \frac{\text{volume of the orbit subspace } \mathcal{O}[\mathfrak{P}_N^{(+)}]}{\text{volume of the orbit space } \mathcal{O}[\mathfrak{P}_N]}. \tag{8}$$

In order to make this definition self-consistent, we assume that

- $\mathcal{O}[\mathfrak{P}_N]$, $\Omega_N^{(+)}[\varrho]$, $\mathfrak{P}_N^{(+)}$, and $\mathcal{O}[\mathfrak{P}_N^{(+)}]$ are open connected sets in $\mathbb{R}^n$;[3]

[3] In support of this assumption, note that the WF is certainly nonnegative for any state whose Bloch vector lies inside the ball of radius $r_*(N) = \sqrt{N+1}/(N^2-1)$.

- the volume of the orbit space in (8) is associated with the measure induced by the quotient mapping π from a certain Riemannian metric on $\mathfrak{P}_N$.[4]

In order to perform efficient computations of $\mathcal{Q}_N$, it is necessary to have, instead of the implicit definitions (6) and (7), a more constructive representation of the space $\mathcal{O}[\mathfrak{P}^{(+)}_N]$. With this aim, we remind the reader of some facts about the stratified structure of the state space $\mathfrak{P}_N$. First of all, note that a $U(N)$-automorphism of the Hilbert space of an N-level quantum system induces the adjoint action of SU(N) on the state space:

$$g \cdot \varrho = g\varrho g^\dagger, \qquad g \in \mathrm{SU}(N). \tag{9}$$

The group action (9) establishes an equivalence relation on $\mathfrak{P}_N$ and gives rise to an SU(N) orbit classification. Formally, a subgroup $H_x \subset \mathrm{SU}(N)$ is defined as the *isotropy group (stabilizer)* of a point $x \in \mathfrak{P}_N$,

$$H_x = \{g \in \mathrm{SU}(N) \mid g \cdot x = x\},$$

and points $x, y \in \mathfrak{P}_N$ are said to be of the same type if their stabilizers H_x and H_y are conjugate subgroups of SU(N). The *orbit type of a point* $x \in \mathfrak{P}_N$ is given by the conjugacy class $[H_x]$ of the corresponding isotropy group. Up to conjugation in SU(N), the isotropy groups H_x are in a one-to-one correspondence with the Young diagrams corresponding to the possible decompositions of N into nonnegative integers. Hence, for given N, with any $[H_\alpha]$, $\alpha = 1, 2, \dots, P(N)$, one can associate a *stratum* $\mathfrak{P}_{[H_\alpha]}$, defined as the set of all points of $\mathfrak{P}_N$ whose stabilizer is conjugate to H_α:[5]

$$\mathfrak{P}_{[H_\alpha]} := \big\{x \in \mathfrak{P}_N \mid H_x \text{ is conjugate to } H_\alpha\big\}. \tag{10}$$

The union of the sets $\mathfrak{P}_{[H_\alpha]}$ gives the decomposition of the state space $\mathfrak{P}_N$ into orbit types:

$$\mathfrak{P}_N = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_\alpha]}. \tag{11}$$

Having in mind the above notions and argumentation, we can formulate the following assertion.

Proposition I. *Let $\boldsymbol{r}^\downarrow = \{r_1, r_2, \dots, r_N\}$ and $\boldsymbol{\pi}^\uparrow = \{\pi_N, \pi_{N-1}, \dots, \pi_1\}$ be the eigenvalues of a density matrix ϱ and the SW kernel $\Delta(\Omega_N \,|\, \boldsymbol{\nu})$, arranged in decreasing and increasing order, respectively. Then*:

(i) *The Wigner function $W_\varrho(\boldsymbol{\theta})$ of any state $\varrho \in \mathfrak{P}_N$ is bounded, and there exist $\boldsymbol{\theta}_-, \boldsymbol{\theta}_+ \in \Omega_N$ such that*

$$W_\varrho(\boldsymbol{\theta}_-) = \inf_{\boldsymbol{\theta}\in\Omega_N} W_\varrho(\boldsymbol{\theta}), \qquad W_\varrho(\boldsymbol{\theta}_+) = \sup_{\boldsymbol{\theta}\in\Omega_N} W_\varrho(\boldsymbol{\theta}).$$

(ii) *If $\varrho_1, \varrho_2 \in \mathfrak{P}_{[H_\alpha]}$, then the extreme values of the corresponding Wigner functions are related as follows*:

$$\inf_{\boldsymbol{\theta}} W_{\varrho_1}(\boldsymbol{\theta}) = \inf_{\boldsymbol{\theta}} W_{\varrho_2}(\boldsymbol{\theta}), \qquad \sup_{\boldsymbol{\theta}} W_{\varrho_1}(\boldsymbol{\theta}) = \sup_{\boldsymbol{\theta}} W_{\varrho_2}(\boldsymbol{\theta}). \tag{12}$$

(iii) *$\mathcal{O}[\mathfrak{P}^{(+)}_N]$ can be identified with the dual cone of the subset $\mathcal{O}[\mathfrak{P}_N] \subset \mathbb{R}^{N-1}$*:

$$\mathcal{O}[\mathfrak{P}^{(+)}_N] = \Big\{ \boldsymbol{\pi} \in \mathcal{O}[\mathfrak{P}^*_N] \;\mid\; (\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) \geq 0 \quad \text{for every } \boldsymbol{r} \in \mathcal{O}[\mathfrak{P}_N] \Big\}, \tag{13}$$

where the dual pairing (,) *in* (13) *is*

$$(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) = r_1\pi_N + r_2\pi_{N-1} + \cdots + r_N\pi_1. \tag{14}$$

[4] In the next section, the global indicator will be computed with respect to the metric corresponding to the Hilbert–Schmidt distance between density matrices [15].

[5] The strata $\mathfrak{P}_{(H_\alpha)}$ are determined by this set of equations and inequalities.

The correctness of the above proposition stems from the following observations. First, according to our construction, an N-level system is associated with a symplectic manifold Ω_N, which is compact. Second, the Wigner distributions of trace-class operators are continuous functions (cf. the discussion in [16]). Hence, according to the multivariable Weierstrass extreme value theorem, the Wigner function attains its extreme values on Ω_N. Moreover, the absolute maximum and minimum must occur at a critical point of the WF in Ω_N or at a boundary point of Ω_N. Some technical details of the proof of Proposition I are given in the appendix.

The article is organized as follows. The next section is devoted to a brief exposition of necessary facts about the WF of finite-dimensional systems, mainly borrowed from our recent articles [10,11]. In Sec. 3, we present a reinterpretation of the Wigner distributions as functions defined on the space of unistochastic matrices and describe their continuation to the whole Birkhoff polytope. With the aid of this extension, the global extrema of the WF are derived. In Sec. 4, using the lower and upper bounds on the WF for the orbit subspace $\mathcal{O}[\mathfrak{P}_N^{(+)}]$, the global $\mathcal{Q}$-indicators for $N = 2$ (qubit) and $N = 3$ are obtained. Final remarks are collected in Sec. 5.

2. Basic settings

The Wigner function of an N-level system. A density matrix ϱ and the Stratonovich–Weyl kernel $\Delta(\Omega_N \,|\, \boldsymbol{\nu})$ obey the following decompositions into the Lie algebra $\mathfrak{su}(N)$ and its dual $\mathfrak{su}(N)^*$:

$$\varrho = \frac{1}{N}\mathbb{I}_N + \frac{1}{N}\,\imath\,\mathfrak{su}(N), \tag{15}$$

$$\Delta(\Omega_N \,|\, \boldsymbol{\nu}) = \frac{1}{N}\mathbb{I}_N + \kappa\,\frac{1}{N}\,\imath\,\mathfrak{su}(N)^*, \tag{16}$$

where $\kappa = \sqrt{N(N^2-1)/2}$ is a normalization constant. It is convenient to use the orthonormal Hermitian basis $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{N^2-1})$ of $\mathfrak{su}(N)$ and rewrite the density matrix (15) in the Bloch form

$$\varrho_{\boldsymbol{\xi}} = \frac{1}{N}\left(I + \sqrt{\frac{N\,(N-1)}{2}}\,(\boldsymbol{\xi}, \boldsymbol{\lambda})\right), \tag{17}$$

where $\boldsymbol{\xi}$ stands for the (N^2-1)-dimensional Bloch vector. In parallel to (17), we will extensively use the singular value decomposition (SVD) of the SW kernel:

$$\Delta(\Omega_N \,|\, \boldsymbol{\nu}) = \frac{1}{N}U(\Omega_N)\left(I + \kappa \sum_{\lambda_s \in \mathfrak{h}} \mu_s(\boldsymbol{\nu})\lambda_s\right)U^\dagger(\Omega_N), \tag{18}$$

where $\mathfrak{h}$ is the Cartan subalgebra in $\mathfrak{su}(N)$. Under these conventions, the algebraic equations in (3) define the following family of Wigner functions:

$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\theta_1, \theta_2, \ldots, \theta_d) = \frac{1}{N}\left[1 + \frac{N^2-1}{\sqrt{N+1}}\,(\boldsymbol{n}, \boldsymbol{\xi})\right]. \tag{19}$$

In (19), the dependence of the Wigner function on a point of the phase space Ω_N with coordinates[6] $(\theta_1, \theta_2, \ldots, \theta_d)$ is encoded in the (N^2-1)-dimensional vector $\boldsymbol{n}$ given by the linear superposition

$$\boldsymbol{n} = \mu_3(\boldsymbol{\nu})\boldsymbol{n}^{(3)} + \mu_8(\boldsymbol{\nu})\boldsymbol{n}^{(8)} + \cdots + \mu_{N^2-1}(\boldsymbol{\nu})\boldsymbol{n}^{(N^2-1)}. \tag{20}$$

[6]The number d of independent variables θ in the Wigner function varies depending on the dimension of the isotropy group of the SW kernel: $d = \dim_{\mathbb{C}} \mathbb{F}^N_{d_1, d_2, \ldots, d_s}$.

The real coefficients $\mu_3(\boldsymbol{\nu}), \mu_8(\boldsymbol{\nu}), \ldots, \mu_{N^2-1}(\boldsymbol{\nu})$ characterize a family of Wigner functions through their dependence on the coordinates $\boldsymbol{\nu}$ of the moduli space $\mathcal{P}_N(\boldsymbol{\nu})$. The moduli space $\mathcal{P}_N(\boldsymbol{\nu})$ is a spherical polyhedron on the unit sphere,

$$\mu_3^2(\boldsymbol{\nu}) + \mu_8^2(\boldsymbol{\nu}) + \cdots + \mu_{N^2-1}^2(\boldsymbol{\nu}) = 1, \tag{21}$$

which corresponds to a chosen order of the eigenvalues of the SW kernel.[7] The orthonormal vectors $\boldsymbol{n}^{(3)}, \boldsymbol{n}^{(8)}, \ldots, \boldsymbol{n}^{(N^2-1)}$ in (20) are specified by $N-1$ basis elements $\lambda_3, \lambda_8, \ldots, \lambda_{N^2-1}$ of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{su}(N)$:

$$\boldsymbol{n}_\mu^{(s^2-1)} = \frac{1}{2}\operatorname{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right).$$

Finally, it is worth mentioning that the Wigner function (19) is a normalized distribution,

$$\int\limits_{\Omega_N} \mathrm{d}\Omega_N \, W_\varrho(\Omega_N) = 1, \tag{22}$$

with the measure $\mathrm{d}\Omega_N$ determined from the normalized Haar measure $\mathrm{d}\mu_{\mathrm{SU}(N)}$ on the $\mathrm{SU}(N)$ group manifold:

$$\mathrm{d}\mu_{\mathrm{SU}(N)} = \frac{1}{N\mathrm{Vol}(H)}\,\mathrm{d}\Omega_N \times \mathrm{d}\mu(H).$$

Here, $\mathrm{Vol}(H)$ is the volume of the isotropy group of the SW kernel computed with respect to the measure $\mathrm{d}\mu(H)$ induced by the corresponding embedding of H into $\mathrm{SU}(N)$.

The orbit space of an N-level system. Similarly to (18), writing down the SVD of a density matrix ϱ with fixed, say decreasing, order of the eigenvalues $\boldsymbol{r} = (r_1, r_2, \ldots, r_N)$,

$$\varrho = U \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_N \end{pmatrix} U^\dagger, \tag{23}$$

we realize the orbit space $\mathcal{O}[\mathfrak{P}_N]$ as an ordered $(N-1)$-simplex:

$$C_{N-1} = \{\boldsymbol{r} \in \mathbb{R}^N \,\Big|\, \sum_{i=1}^N r_i = 1,\;\; 1 \geq r_1 \geq r_2 \geq \ldots \geq r_{N-1} \geq r_N \geq 0\}. \tag{24}$$

In the present note, we mainly focus on the Wigner functions (19) of a qubit ($N = 2$) and qutrit ($N = 3$) and thus deal with a 1-simplex (line segment) and a 2-simplex (triangle), respectively.

3. The Wigner distribution as a function on the Birkhoff polytope

In this section, we rewrite the Wigner distribution in the form of a function on the so-called Birkhoff polytope $\mathcal{B}_N$, see [17]. The Birkhoff polytope $\mathcal{B}_N$ is the polytope of *bistochastic, or doubly stochastic, $N \times N$ complex matrices*, obeying the following conditions:

$$B_{ij} \geq 0, \quad \sum_{i=1}^N B_{ij} = 1, \quad \sum_{j=1}^N B_{ij} = 1.$$

Precisely speaking, the Wigner function of an N-level system is defined on the subset of bistochastic matrices called *unistochastic*. If a matrix B is expressible via a unitary matrix U as

$$B_{ij} = |\, U_{ij}\,|^2 \quad \text{for any } \; i, j = 1, 2, \ldots N,$$

[7] A detailed description of the moduli space $\mathcal{P}_N(\boldsymbol{\nu})$ is presented in [11].

then it is said to be unistochastic. The following proposition establishes this relation.

Proposition II. *Assign to a matrix $B \in \mathcal{B}_N$ the following bilinear form on $\mathbb{R}_+^N$:*

$$(\boldsymbol{x}, \boldsymbol{y})_B = (\boldsymbol{x}, B\boldsymbol{y}) = \sum_{ij} B_{ij} x_i y_j. \tag{25}$$

Then the Wigner quasiprobability distribution of an N-level system can be identified with the bilinear form (25) with a matrix B from the subset $\mathcal{U}_N \subset \mathcal{B}_N$ of unistochastic matrices[8]:

$$W_\varrho(\Omega_N) = \left(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\downarrow\right)_B \bigg|_{B=|U|^2}, \tag{26}$$

evaluated at the ordered vectors $\boldsymbol{r}^\downarrow$ and $\boldsymbol{\pi}^\downarrow$ whose components are the eigenvalues of a density matrix ϱ and the SW kernel Δ, respectively.

Using Proposition II, we can study the problem of finding the global extrema of the WF as follows. Observing that an analogous problem for the bilinear form $(\,\cdot\,,\,\cdot\,)_B$ is well studied, we define the *continuation of the Wigner distribution* as a function $W(B)$ whose domain of definition is the whole Birkhoff polytope:

$$W(B) := \left(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\downarrow\right)_B. \tag{27}$$

Applying the Birkhoff–von Neumann theorem to the function $W(B)$, one can find its global maximum and minimum. The next step is to analyze the fate of the extrema after the restriction of (27) to the subspace of unistochastic matrices. The following conjecture aims to answer this question.

Proposition III. *The Wigner quasiprobability distribution function defined on the set of unistochastic matrices attains the global maximum $W^{(+)}$ and the global minimum $W^{(-)}$ at the permutation matrices*

$$P_{\min} = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & 1 & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, \qquad P_{\max} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \tag{28}$$

with the following values:

$$W^{(-)} = \lim_{B \to P_{\min}} W = \left(\boldsymbol{r}^\uparrow, \boldsymbol{\pi}^\downarrow\right), \tag{29}$$

$$W^{(+)} = \lim_{B \to P_{\max}} W = \left(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\downarrow\right). \tag{30}$$

For a formal discussion of this conjecture, we refer to the appendix, while here we only give two arguments in its support. The first one is the Birkhoff–von Neumann theorem [18, p. 36], according to which $\mathcal{B}$ is the convex hull of all $N \times N$ permutation matrices. There is at least one decomposition of $\mathcal{B}$ of the form

$$\mathcal{B}_N = \sum_i^k \kappa_i P_i, \qquad \sum_i \kappa_i = 1, \quad \kappa_i \geq 0, \tag{31}$$

with $k \leq (n-1)^2 + 1$ permutation matrices P_i corresponding to the vertices of the Birkhoff polytope. Due to this theorem, the bilinear form $(\,.\,,\,.\,)_B$ assumes its extremum on the set of

[8]Note that for $N \geq 3$ the set of unistochastic matrices is not convex.

extreme points consisting of the permutations (28) mentioned in the conjecture:

$$\min_B (x,y)_B = (x,y)_{P_{\min}} = \sum_i x_i^{\uparrow} y_i^{\downarrow}, \tag{32}$$

$$\max_B (x,y)_B = (x,y)_{P_{\max}} = \sum_i x_i^{\downarrow} y_i^{\downarrow}. \tag{33}$$

The second argument in support of the conjecture is that the space of unistochastic matrices contains all permutation matrices, and $P_{\min}$ and $P_{\max}$ are among them.

Therefore, for a given SW kernel with eigenvalues $\boldsymbol{\pi}^{\downarrow} = \{\pi_1, \pi_2, \ldots, \pi_N\}$ and a density matrix with spectrum $\boldsymbol{r}^{\downarrow} = \{r_1, r_2, \ldots, r_N\}$, the knowledge of the global minimum of the WF provides information on the subset

$$\mathcal{O}[\mathfrak{P}_N^{(+)} = \{\boldsymbol{r} \in C_{N-1} \mid \left(\boldsymbol{r}^{\uparrow}, \boldsymbol{\pi}^{\downarrow}\right) \geq 0\}. \tag{34}$$

Using these results, in the next section we explicitly evaluate the rate of quantumness-classicality for low-dimensional systems, such as a qubit and a qutrit.

4. The global indicator of classicality of a qubit and a qutrit

Summarizing the discussions of the previous section, the Wigner function satisfies the following inequality:

$$W_N^{(-)} \leq W(\Omega_N) \leq W_N^{(+)}, \tag{35}$$

where

$$W_N^{(-)} = \sum_{i=1}^{N} \pi_i r_{N-i+1}, \qquad W_N^{(+)} = \sum_{i=1}^{N} \pi_i r_i. \tag{36}$$

Below, considering inequalities (35) for two low cases, $N = 2$ and $N = 3$, we will obtain an explicit parametrization of the subspaces $\mathcal{O}[\mathfrak{P}_2^{(+)}]$ and $\mathcal{O}[\mathfrak{P}_3^{(+)}]$ of the orbit space corresponding to a positive WF of a single qubit and a single qutrit.

The positivity of the lower bound $W_2^{(-)}$. For a simplest $N = 2$ level system, a single qubit, the density matrix expanded in terms of the Pauli $\boldsymbol{\sigma}$-matrices is characterized by a three-dimensional Bloch vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$:

$$\varrho = \frac{1}{2}\left(I + (\boldsymbol{\xi}, \boldsymbol{\sigma})\right). \tag{37}$$

The spectrum of the SW kernel for a qubit is defined from (18) up to permutations, and, assuming that the eigenvalues are arranged in descending order, it is

$$\operatorname{spec}(\Delta_2) = \left\{\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right\}. \tag{38}$$

Taking into account the above expressions, the lower and upper bounds (36) for a qubit are

$$W_2^{(\mp)} = \frac{1}{2} \mp \frac{\sqrt{3}}{2}\,|\boldsymbol{\xi}|. \tag{39}$$

Therefore, the Wigner function of a qubit is positive definite inside the Bloch ball of radius $r_*(2) < 1/\sqrt{3}$.

The $\mathcal{Q}$-indicator of a single qubit. Using the constraint on the states of a qubit with nonnegative WF derived above, the global indicator of quantumness $\mathcal{Q}$ can be evaluated after specifying a measure on the orbit space $\mathcal{O}[\mathfrak{P}_2]$. The measure $\mathrm{d}\mu_{\text{H-S}}$ on $\mathfrak{P}_2$ associated with the Hilbert–Schmidt ensemble of qubits has a product form

$$\mathrm{d}\mu_{\text{H-S}} = (r_1 - r_2)^2\, \mathrm{d}r_1 \wedge \mathrm{d}r_2 \times \mathrm{d}\mu_{\frac{\mathrm{SU}(2)}{U(1)}}\,, \tag{40}$$

where $\mathrm{d}\mu_{\frac{\mathrm{SU}(2)}{U(1)}}$ is the measure on the coset $\mathrm{SU}(2)/U(1)$ induced by the normalized Haar measure on SU(2). The factor in (40), which depends on the 1-simplex coordinates r_1 and r_2, defines a measure on the orbit space $\mathcal{O}[\mathfrak{P}_2]$. Thus, the computation of the indicator $\mathcal{Q}$ of a qubit reduces to the evaluation of the ratio of two simple integrals,

$$\mathcal{Q}_2 = \frac{\mathrm{Vol}\left(\mathcal{O}[\mathfrak{P}_2^{(+)}]\right)}{\mathrm{Vol}\left(\mathcal{O}[\mathfrak{P}_2]\right)} = \frac{\int\limits_0^{\frac{1}{\sqrt{3}}} r^2 dr}{\int\limits_0^{1} r^2 dr} = \frac{1}{3\sqrt{3}} = 0.19245. \tag{41}$$

The positivity of the lower bound $W_3^{(-)}$. For further study, we introduce two types of coordinates on the orbit space of a qutrit. The first parametrization takes into account the algebraic structure of the density matrix of a qutrit state:

$$r_1 = \frac{1}{3} + \frac{1}{\sqrt{3}}\xi_3 + \frac{1}{3}\xi_8, \quad r_2 = \frac{1}{3} - \frac{1}{\sqrt{3}}\xi_3 + \frac{1}{3}\xi_8, \quad r_3 = \frac{1}{3} - \frac{2}{3}\,\xi_8. \tag{42}$$

In terms of ξ_3 and ξ_8, the ordered 2-simplex is mapped to the domain $\mathcal{O}[\mathfrak{P}_3]$ defined by the following set of inequalities:

$$\mathcal{O}[\mathfrak{P}_3]: \ \left\{\xi_3, \xi_8 \in \mathbb{R} \;\middle|\; 0 \le \xi_3 \le \frac{\sqrt{3}}{2}, \quad \frac{\xi_3}{\sqrt{3}} \le \xi_8 \le \frac{1}{2}\right\}. \tag{43}$$

The second useful set of coordinates, (r, φ), on the orbit space of a qutrit is given by the following map:

$$\xi_3 = \sqrt{3} r \sin\left(\frac{\varphi}{3}\right), \qquad \xi_8 = \sqrt{3} r \cos\left(\frac{\varphi}{3}\right), \qquad 0 \le \varphi \le \pi. \tag{44}$$

Under the transformation (44), the ordered 2-simplex of a qutrit is mapped to the domain on the upper half-plane with coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ outlined by the trisectrix of Maclaurin (see the grey region in Fig. 1):

$$\mathcal{O}[\mathfrak{P}_3]: \ \left\{ r \ge 0, \varphi \in [0, \pi] \;\middle|\; \cos\left(\frac{\varphi}{3}\right) \le \frac{1}{2\sqrt{3}r}\right\}. \tag{45}$$

According to the analysis in [11], the algebraic equations (3) for the eigenvalues of the SW kernel of a qutrit have a one-parameter solution which can be written as

$$\pi_1 = \frac{1}{3} + \frac{2}{\sqrt{3}}\mu_3 + \frac{2}{3}\,\mu_8, \quad \pi_2 = \frac{1}{3} - \frac{2}{\sqrt{3}}\mu_3 + \frac{2}{3}\,\mu_8, \quad \pi_3 = \frac{1}{3} - \frac{4}{3}\,\mu_8. \tag{46}$$

Here, the parameters μ_3 and μ_8 are the Cartesian coordinates of a segment of the unit circle with apex angle ζ:

$$\mu_3 = \sin\zeta, \qquad \mu_8 = \cos\zeta, \qquad 0 \le \zeta \le \frac{\pi}{3}. \tag{47}$$

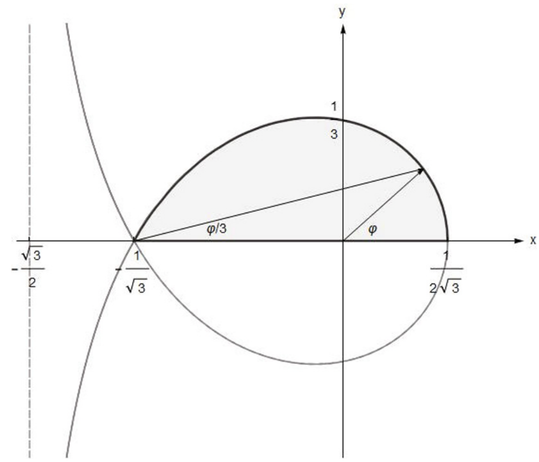


Fig. 1. The trisectrix of Maclaurin intersecting the x-axis at two points, $(\frac{1}{2\sqrt{3}}, 0)$ and $(-\frac{1}{\sqrt{3}}, 0)$. On the (x, y) plane, the equation of this curve in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ reads as $r(\varphi, \frac{1}{\sqrt{3}}) = \frac{1}{2\sqrt{3}\cos(\varphi/3)}$. The orbit space $\mathcal{O}[\mathfrak{P}_3]$ of a qutrit is given by the grey domain.

It is worth noting that the apex angle ζ determines the value of a polynomial SU(3)-invariant of degree 3 of the SW kernel $\Delta(\Omega_3)|\nu)$:

$$\cos(3\zeta) = -\frac{27}{16}\det\left(\Delta(\Omega_3|\nu)\right) - \frac{11}{16},$$

with the moduli parameter

$$\nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta), \quad \zeta \in [0, \ \pi/3]. \tag{48}$$

Having these ingredients for a density matrix (42) and the SW kernel (46), a straightforward evaluation of (36) for $N = 3$ gives

$$W_3^{(-)} = \frac{1}{3} - \frac{4r}{\sqrt{3}}\cos\left(\zeta + \frac{\varphi}{3} - \frac{\pi}{3}\right), \tag{49}$$

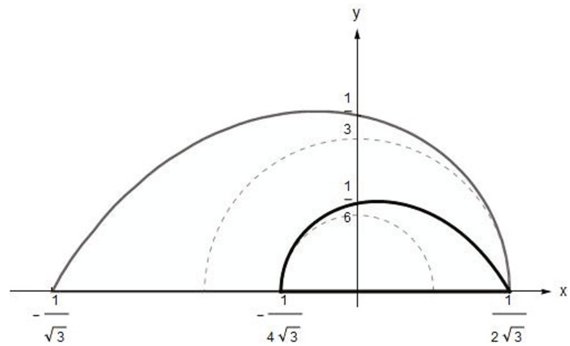
$$W_3^{(+)} = \frac{1}{3} + \frac{4r}{\sqrt{3}}\cos\left(\zeta - \frac{\varphi}{3}\right). \tag{50}$$

From (49) it follows that the subspace of the orbit space $\mathcal{O}[\mathfrak{P}_3^{(+)}]$ where the WF is positive is given by

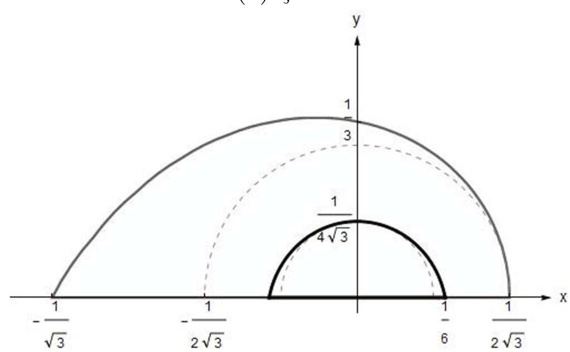
$$\mathcal{O}[\mathfrak{P}_3^{(+)}] \, : \left\{ r \geq 0, \varphi \in [0, \pi] \;\middle|\; \cos\left(\frac{\varphi}{3} + \zeta - \frac{\pi}{3}\right) \leq \frac{1}{4\sqrt{3}r} \right\}. \tag{51}$$

Comparing (51) with the qutrit orbit space (45), we conclude that $\mathcal{O}[\mathfrak{P}_3^{(+)}]$ lies inside the qutrit orbit space $\mathcal{O}[\mathfrak{P}_3]$ as shown in Fig. 2. Here, a few comments on the shape of $\mathcal{O}[\mathfrak{P}_3^{(+)}]$ are in order:

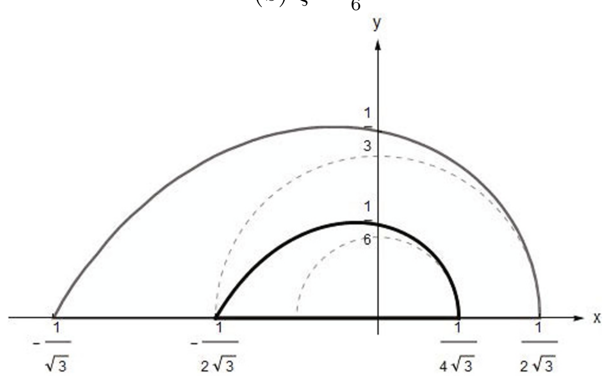
- for $0 \leq r \leq \frac{1}{4\sqrt{3}}$, the lower bound $W_3^{(-)}$ is positive for all ζ and φ;
- for $\frac{1}{2\sqrt{3}} \leq r \leq \frac{1}{\sqrt{3}}$, the lower bound $W_3^{(-)}$ is always negative;
- for intermediate values $\frac{1}{4\sqrt{3}} \leq r \leq \frac{1}{2\sqrt{3}}$, the lower bound $W_3^{(-)}$ becomes negative only for certain values of ζ and φ.



(a) $\zeta = 0$



(b) $\zeta = \frac{\pi}{6}$



(c) $\zeta = \frac{\pi}{3}$

Fig. 2. The state space of a qutrit divided into bands. The Wigner function is always positive (necessarily has some negative values) inside (outside) the region enclosed by the dashed inner (outer) semicircle independent of the choice of the kernel. Inside the region enclosed by the kernel-dependent inner solid curve, the Wigner function is always positive for a specific choice of the kernel.

The $\mathcal{Q}$-indicator of a single qutrit. The global indicator of classicality of a qutrit is given by the ratio of volumes

$$\mathcal{Q}_3 = \frac{\mathrm{Vol}\left(\mathcal{O}[\mathfrak{P}_3^{(+)}]\right)}{\mathrm{Vol}\left(\mathcal{O}[\mathfrak{P}_3]\right)}. \tag{52}$$

To evaluate these volume integrals, we need to specify a measure on the orbit space $\mathcal{O}[\mathfrak{P}_3]$. Similarly to the qubit case, we assume that the qutrit state space $\mathfrak{P}_3$ is endowed with the

Hilbert–Schmidt metric:

$$\mathrm{g} = 4\,\mathrm{tr}\left(\mathrm{d}\varrho \otimes \mathrm{d}\varrho\right). \tag{53}$$

In terms of the Bloch coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_8)$ of a qutrit,

$$\varrho = \frac{1}{3}\left(\mathbb{I} + \sqrt{3}\,(\boldsymbol{\lambda}, \boldsymbol{\xi})\right), \tag{54}$$

the metric (53) gives the standard Euclidean volume form on $\mathfrak{P}_3$:

$$\omega = \left(\frac{8}{3}\right)^4 \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 \wedge \dots \wedge \mathrm{d}\xi_8. \tag{55}$$

Now, in order to compute the corresponding induced form on the orbit space $\mathcal{O}[\mathfrak{P}_3]$, we rewrite (55) in terms of the SVD of the density matrix

$$\varrho = UDU^\dagger. \tag{56}$$

Since the measure of singular and degenerate matrices is zero, we consider a generic spectrum $D = \mathrm{diag}||r_1, r_2, r_3||$ with eigenvalues $1 > r_1 > r_2 > r_3 > 0$ arranged in decreasing order. This means that U is determined up to an element of the torus T of SU(3). Therefore, the volume form in adaptive SVD coordinates is

$$\omega = (r_1 - r_2)^2 (r_1 - r_3)^2 (r_2 - r_3)^2\, \mathrm{d}r_1 \wedge \mathrm{d}r_2 \wedge \mathrm{d}r_3 \wedge \omega_{\mathrm{SU(3)/T}}. \tag{57}$$

For illustrative reasons, it is convenient to pass from the 2-simplex Cartesian coordinates r_1, r_2, r_3 to the polar variables r and φ introduced in (44). As a result, the volume form (57) on the orbit space $\mathcal{O}[\mathfrak{P}_3]$ reduces to the following expression:

$$\omega_{\mathcal{O}[\mathfrak{P}_3]} = r^7 \sin^2\varphi\, \mathrm{d}r \wedge \mathrm{d}\varphi. \tag{58}$$

Computing the volume integrals in (52) with respect to the measure (58) on the orbit space of a qutrit (45) and its subspace where the WF is positive, we find an explicit dependence of the global indicator of classicality on the moduli parameter ζ of the SW kernel:

$$\mathcal{Q}_3(\zeta) = \frac{\int_0^\pi d\varphi \int_0^{\frac{1}{4\sqrt{3}\cos\left(\frac{\varphi}{3}+\zeta-\frac{\pi}{3}\right)}} r^7 \sin^2(\varphi) dr}{\int_0^\pi d\varphi \int_0^{\frac{1}{2\sqrt{3}\cos\frac{\varphi}{3}}} r^7 \sin^2(\varphi) dr} = \frac{1}{128}\,\frac{1+20\cos^2\left(\zeta-\pi/6\right)}{\left(-1+4\cos^2\left(\zeta-\pi/6\right)\right)^5}. \tag{59}$$

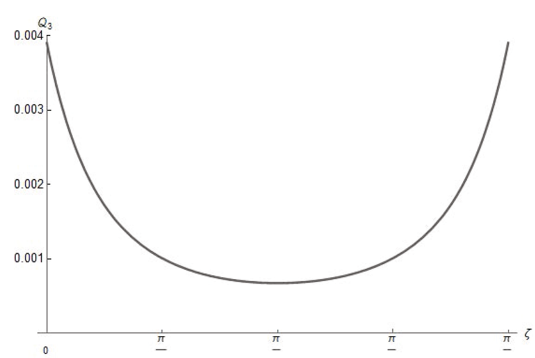


Fig. 3. The $\mathcal{Q}$-indicator as a function of the moduli parameter ζ of the SW kernel for the Hilbert–Schmidt ensemble of qutrits.

Straightforward calculations show that the indicator $\mathcal{Q}_3(\zeta)$ attains the absolute minimum

$$\min_{\zeta\in[0,\frac{\pi}{3}]} \mathcal{Q}_3(\zeta) = \mathcal{Q}_3\left(\frac{\pi}{6}\right) = \frac{7}{2^7\,3^4} \approx 0.000675$$

at the qutrit moduli parameter $\zeta = \pi/6$, corresponding to the SW kernel with spectrum

$$\operatorname{spec}(\Delta_3) = \left\| \frac{1+2\sqrt{3}}{3}, \frac{1}{3}, \frac{1-2\sqrt{3}}{3} \right\|. \tag{60}$$

In Fig. 3, the dependence of $\mathcal{Q}_3$ on the moduli parameter ζ is shown.

5. Summary

In the present article, we introduce a global indicator of classicality of a quantum N-dimensional system. This indicator directly measures the portion of its unitary orbit space that is associated with states admitting a conventional statistical interpretation in terms of true probability distributions. The study revealed an interesting relation between the properties of Wigner quasiprobability distributions and the structure of Birkhoff polytopes. It seems that this relation deserves attention, and in our future publication we will return to the problem of classical-quantum correspondence from this point of view.

Appendix

In this appendix, we discuss the problem of finding the global extrema for a function on the unitary orbits of a Hermitian matrix.

Problem. *Let A be a positive definite Hermitian matrix and B be a Hermitian matrix. Consider the adjoint unitary orbit $\mathcal{O}_B = gBg^\dagger$ with $g \in \mathrm{SU}(N)$. Find the global extrema of the function*

$$\Phi(g) = \operatorname{tr}(AgBg^\dagger). \tag{61}$$

To find the extrema of (61), one can apply a standard method from calculus used to find the critical points of functions. To be precise, consider matrices A and B whose spectrum is of the following form:

$$\boldsymbol{\mu}^{\downarrow}(A) = \{\mu_1(A)\overbrace{(1,\dots,1)}^{k_1(A)};\ \mu_2(A)\overbrace{(1,\dots,1)}^{k_2(A)};\ \dots;\ \mu_s(A)\overbrace{(1,\dots,1)}^{k_s(A)}\}, \tag{62}$$

$$\boldsymbol{\mu}^{\downarrow}(B) = \{\mu_1(B)\overbrace{(1,\dots,1)}^{k_1(B)};\ \mu_2(B)\overbrace{(1,\dots,1)}^{k_2(B)};\ \dots;\ \mu_s(B)\overbrace{(1,\dots,1)}^{k_s(B)}\}. \tag{63}$$

The elements of the spectra of both matrices are arranged in decreasing order:

$$\mu_1(A) > \mu_2(A) > \cdots > \mu_s(A) \quad \text{and} \quad \mu_1(B) > \mu_2(B) > \cdots > \mu_s(B). \tag{64}$$

The degrees of degeneracy $(k(A), k(B))$ of the matrices A and B are constrained by the relations $\sum\limits_{i=1}^{s} k_i(A) = r_A$ and $\sum\limits_{i=1}^{s} k_i(B) = r_B$. The SVD decompositions

$$A = VD_AV^\dagger, \qquad B = WD_BW^\dagger \tag{65}$$

are not unique, and a family of unitary matrices V and W in (65) can be built as follows. Denote by $V^{\downarrow}$ the unitary matrix constructed from the right eigenvectors of the matrix A

arranged in accordance with the decreasing order of its eigenvalues. Then the most general family of unitary matrices diagonalizing A reads as

$$V = V^{\downarrow} \begin{pmatrix} V_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V_s \end{pmatrix} P, \tag{66}$$

where $V_1, \dots, V_s$ are arbitrary unitary matrices of order $k_1, \dots, k_s$, respectively, and P is a transposition matrix

$$P = \|\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \dots, \boldsymbol{e}_{i_N}\|,$$

with $\boldsymbol{e}_j$ the N-dimensional vector having zeros everywhere except 1 in the jth position. The right multiplication by P transposes the columns as $j \to i_j$, $j = 1, \dots, N$. Below, the same construction will be used for a unitary matrix W as well.

Straightforward computations show that the necessary condition for an extremum of $\Phi(g)$ can be written as

$$\mathrm{d}\Phi(g) = \mathrm{tr}\left([O_B, A] w_g\right) = 0, \tag{67}$$

where

$$w_g = \mathrm{d}g g^{\dagger} = \frac{\imath}{2} \sum_{a,i=1}^{N^2-1} (w_g)_i^a \lambda_a \mathrm{d}\vartheta^i \tag{68}$$

is the Maurer–Cartan 1-form on $\mathrm{SU}(N)$. Equation (67) tells us that extrema of $\Phi(g)$ are realized for all points of the orbits $\mathcal{O}_B = g_c B g_c^{\dagger}$ commuting with A:[9]

$$[A, \mathcal{O}_B] = 0. \tag{69}$$

This equation has a solution $g_c = VW^{\dagger}$ with unitary matrices V and W diagonalizing A and B, respectively. According to (66), the matrices V and W constitute a family of diagonalizing unitary matrices. One can see that the set of corresponding critical points $g = g_c$ of $\Phi(g)$ is discrete. As a result of (66), for given $\mathrm{spec}(A)$ and $\mathrm{spec}(B)$ the extrema are determined by permutations P:

$$\Phi(g)\Big|_{g=g_c} = \mathrm{tr}(D_A D_B) = \mathrm{tr}\left(\boldsymbol{\mu}^{\downarrow}(A) P^T \boldsymbol{\mu}^{\downarrow}(B) P\right).$$

Among these extrema, the minimum and maximum can be identified using a well-known result on the majorization of two vectors $x, y \in \mathbb{R}^N$ (cf. [18, p. 49]):

$$\langle x^{\downarrow}, y^{\uparrow}\rangle \le \langle x, y\rangle \le \langle x^{\downarrow}, y^{\downarrow}\rangle. \tag{70}$$

Hence, finally, the global extrema of $\Phi(g)$ read

$$\min_{g \in g_c} \Phi(g) = \mathrm{tr}\left(\boldsymbol{\mu}^{\downarrow}(A)\, \boldsymbol{\mu}^{\uparrow}(B)\right), \tag{71}$$

$$\max_{g \in g_c} \Phi(g) = \mathrm{tr}\left(\boldsymbol{\mu}^{\downarrow}(A)\, \boldsymbol{\mu}^{\downarrow}(B)\right). \tag{72}$$

[9] Condition (67) is a system of linear homogeneous equations $(w_g)_i^a x_a = 0$ with unknowns x_a and, apart from the trivial solution $x_a = 0$, can have other solutions corresponding to singular points occurring at $\det ||(w_g)_i^a|| = 0$. Recalling that $\det \|w_g\| = \sqrt{\det \|\mathrm{g}_{\mathrm{U(N)}}\|}$ and recalling the explicit expression for the Haar measure $\sqrt{\det \|\mathrm{g}_{\mathrm{U(N)}}\|}\, \mathrm{d}\vartheta_1 \cdots \mathrm{d}\vartheta_N$ in terms of the eigenvalues of a $U(N)$ element

$$\sqrt{\det ||\mathrm{g}_{\mathrm{U(N)}}||} = \frac{1}{(2\pi)^N N!} \prod_{1 \le i < j \le n} \left| e^{i\vartheta_i} - e^{i\vartheta_j} \right|^2,$$

we associate the set of singular solutions to (67) with the variety of possible types of degeneracies of the eigenvalues of unitary matrices, $\vartheta_{i_1} = \vartheta_{i_2} = \cdots = \vartheta_{i_k}$.

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