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ON THE MODULI SPACE OF WIGNER QUASIPROBABILITY DISTRIBUTIONS FOR N-DIMENSIONAL QUANTUM SYSTEMS

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A mapping between operators on the Hilbert space of an N-dimensional quantum system and Wigner quasiprobability distributions defined on the symplectic flag manifold is discussed. The Wigner quasiprobability distribution is constructed as a dual pairing between the density matrix and the Stratonovich–Weyl kernel. It is shown that the moduli space of Stratonovich–Weyl kernels is given by the intersection of the coadjoint orbit space of the group $\mathrm{SU}(N)$ *and a unit* $(N-2)$*-dimensional sphere. The general considerations are exemplified by a detailed description of the moduli space of* 2, 3, *and* 4*-dimensional systems. Bibliography:* 30 *titles.*

1. Introduction

According to the postulates of quantum theory, the fundamental description of a physical system is provided by the density operator (see [1])

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \tag{1}$$

which represents a quantum statistical ensemble $\{p_k|\psi_k\rangle\}$, i.e., a set consisting of vectors $|\psi_k\rangle \in \mathcal{H}$ of a Hilbert space $\mathcal{H}$ and their probabilities p_k summing to one, $\sum\limits_k p_k = 1$. The density operator ϱ determines the expectation value $\mathbb{E}(\widehat{A})$ of a Hermitian operator $\widehat{A}$ acting on $\mathcal{H}$:

$$\mathbb{E}(\widehat{A}) = \mathrm{Tr}\left[\widehat{A}\varrho\right], \quad \text{with} \quad \mathrm{Tr}\left[\varrho\right] = 1. \tag{2}$$

The latter is assigned to the physical observable associated with the operator $\widehat{A}$. On the other hand, an ensemble of a classical mechanical system is characterized by a probability distribution function $\rho(q,p)$, i.e., the density of the probability to find the system in a state localized in the vicinity of a phase space point with coordinates q and p. Correspondingly, the statistical average, i.e., the expectation value $\mathbb{E}(A)$ of a physical quantity described by the function $A(q,p)$ on the phase space, is given by the following convolution:

$$\mathbb{E}(A) = \int \mathrm{d}\Omega\, A(q,p)\,\rho(q,p), \quad \text{with} \quad \int \mathrm{d}\Omega\, \rho(q,p) = 1, \tag{3}$$

where $\mathrm{d}\Omega$ denotes the normalized volume form of the classical phase space.

Aiming to collate two representations of observables, classical (3) and quantum (2), the so-called Weyl–Wigner invertible map between Hilbert space operators and functions on the phase space was introduced in the early stages of the development of quantum mechanics [2–6]. The primary elements of this map are two notions: the *symbol of an operator*, i.e., a function

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$A_W(q,p)$ corresponding to an operator A, and a *quasi-distribution function* $W(q,p)$ defined on the phase space. As a result, the quantum analog of the statistical average (3) reads

$$\mathbb{E}(\widehat{A}) = \int \mathrm{d}\Omega\, A_W(q,p)\, W(q,p), \quad \text{with} \quad \int \mathrm{d}\Omega\, W(q,p) = 1. \tag{4}$$

However, even a quick look at this attempt to build a bridge between the classical and quantum statistical pictures shows a lack of equivalence between them. Indeed, one can point out the following observations:

- Because of Heisenberg's uncertainty principle, the function $W(q,p)$ has negative values for certain quantum states. Hence, it is not a true probability density, and it is referred to as the quasiprobability distribution.
- Dirac's quantization rule based on the canonical commutation relations makes the interplay between operators and their symbols highly sophisticated. The replacement of canonical variables by their quantum counterparts in expressions for functions on the phase space faces the ambiguity of ordering the corresponding canonical operators.[1]

In spite of both flaws, Wigner functions or other suggested quasiprobability distributions, such as Husimi's [4] and Glauber and Sudarshan's [8, 9] representations, remain today an important tool for understanding the interrelations between quantum and classical statistical descriptions [10]. Moreover, nowadays one can see a growing interest to the phase space formulation of quantum mechanics based on the method of quasiprobability distributions for finite-dimensional systems (see, e.g., [11–18] and the references therein). The latter comes from the needs of diverse applications in quantum optics [19] and also in quantum information and communications [20]. Such an intense usage of quasi-distributions again raises the issue of understanding the above-mentioned shortcomings[2].

In the present note, the problem of constructing quasiprobability distribution functions for generic N-level systems is studied within a purely algebraic approach. The basic mathematical objects in this approach are: the special unitary group $G = \mathrm{SU}(N)$, its Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$,

$$\mathfrak{su}(N) = \{X \in M(N,\mathbb{C}) \mid X = -X^\dagger, \quad \mathrm{tr}X = 0\}, \tag{7}$$

[1] According to Weyl's quantization rule [2], any classical observable $A(\boldsymbol{p},\boldsymbol{q})$, i.e., a function on the phase space $\mathbb{R}^{2n}$ with the standard canonical symplectic structure, is associated with an operator $\widehat{A}_\omega$ on the Hilbert space $L^2(\mathbb{R}^n)$ constructed as the "*Weyl quantum Fourier transform*":

$$A \mapsto \widehat{A}_\omega = \int\limits_{\mathbb{R}^{2n}} \mathrm{d}\Omega(\omega)\, \widetilde{A}(\boldsymbol{u},\boldsymbol{v}) \exp\left(\frac{\imath}{\hbar}\left(\boldsymbol{u}\widehat{\boldsymbol{P}} + \boldsymbol{v}\widehat{\boldsymbol{Q}}\right)\right), \quad \mathrm{d}\Omega = \omega(\boldsymbol{u},\boldsymbol{v})\,\mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{v}, \tag{5}$$

where $\widehat{P}$ and $\widehat{Q}$ are operators on $L^2(\mathbb{R}^n)$ obeying the canonical commutation relations, $\widetilde{A}(\boldsymbol{u},\boldsymbol{v})$ is the Fourier transform of $A(\boldsymbol{u},\boldsymbol{v})$, and the integration measure $\mathrm{d}\Omega$ is defined by a weight function $\omega(\boldsymbol{u},\boldsymbol{v})$. Different choices of $\omega(\boldsymbol{u},\boldsymbol{v})$ are a source of various orderings of the noncommutative operators $\widehat{P}$ and $\widehat{Q}$. For example, the factor $\omega(\boldsymbol{u},\boldsymbol{v}) = \exp\left(-\frac{\imath}{2}\,\boldsymbol{u}\boldsymbol{v}\right)$ corresponds to the standard ordering of polynomials in mathematical literature, when one writes first the position coordinate Q, then the momentum P. The so-called normal ordering is related to the weight $\omega(\boldsymbol{u},\boldsymbol{v}) = \exp\left(-\frac{1}{4}\,(\boldsymbol{u}^2+\boldsymbol{v}^2)\right)$, while the original Weyl, or symmetric, ordering complies with $\omega(\boldsymbol{u},\boldsymbol{v}) = 1$. The inverse formula, which maps an operator to its symbol, is due to Wigner [3]. For the case of the unit weight factor, $\omega = 1$, the inverse formula reads

$$A(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{(2\pi\hbar)^n}\mathrm{tr}\left[\widehat{A}_1 \exp\left(-\frac{\imath}{\hbar}\left(\boldsymbol{u}\widehat{\boldsymbol{P}} + \boldsymbol{v}\widehat{\boldsymbol{Q}}\right)\right)\right]. \tag{6}$$

A further elaboration of Weyl's quantization scheme leads to a noncommutative formulation of mechanics [6] and, finally, to the development of the so-called deformation quantization, cf. [7].

[2] The history going back to Dirac's idea of negative energies teaches us to pay more attention to the "nonsense" of negative probabilities. In this context, it is best to quote the following words by R. Feynman: "*It is that a situation for which a negative probability is calculated is impossible, not in the sense that the chance for it happening is zero, but rather in the sense that the assumed conditions of preparation or verification are experimentally unattainable*" [21].

and its dual space $\mathfrak{g}^* = \mathfrak{su}(N)^*$.[3] It is well known that the universal covering algebra $\mathfrak{U}(\mathfrak{su}(N))$ of the Lie algebra $\mathfrak{su}(N)$ is an arena of the basic objects of an N-level quantum system. In particular, the state space is defined as the space of *positive semidefinite* $N \times N$ Hermitian matrices H_N with unit trace:

$$\mathfrak{P}_N = \{X \in H_N \mid X \geq 0, \quad \mathrm{tr}\,(X) = 1\}. \tag{9}$$

Every state described by a density matrix $\varrho \in \mathfrak{P}_N$ is in correspondence with some element of the Lie algebra $\mathfrak{su}(N)$:

$$\varrho = \frac{1}{N}\mathbb{I}_N + \frac{1}{N}\,\imath\,\mathfrak{su}(N). \tag{10}$$

In order to build the Wigner function, apart from the quantum state space $\mathfrak{P}_N$, we need the notion of its dual $\mathfrak{P}^*_N$. Every point of the dual space determines a Stratonovich–Weyl (SW) kernel [22, 23]. As it was shown recently in [24], the space $\mathfrak{P}^*_N$ can be defined as follows:

$$\mathfrak{P}^*_N = \{X \in H_N \| \ \mathrm{tr}\,(X) = 1, \quad \mathrm{tr}\left(X^2\right) = N\}. \tag{11}$$

It turns out that the dual pairing (8) of a density matrix $\varrho \in \mathfrak{P}_N$ and an SW kernel $\Delta(\Omega_N) \in \mathfrak{P}^*_N$,

$$W_\varrho(\Omega_N) = \mathrm{tr}\left[\varrho\,\Delta(\Omega_N)\right], \tag{12}$$

provides us with a proper Wigner function which satisfies all the Stratonovich–Weyl postulates [22,23]. Taking into account the unit trace condition, an SW kernel $\Delta(\Omega_N)$ can be related to the dual of $\mathfrak{su}(N)$:

$$\Delta(\Omega_N) = \frac{1}{N}\mathbb{I}_N + \kappa\,\frac{1}{N}\,\imath\,\mathfrak{su}(N)^*, \tag{13}$$

where $\kappa = \sqrt{N(N^2-1)/2}$ is a normalization constant. From the representations (10) and (13) it follows that all nontrivial information comes from the pairing between the traceless parts of a density matrix and an SW kernel. In the subsequent sections, after a short overview of the Stratonovich–Weyl postulates, algebraic and geometric aspects of the dual space $\mathfrak{P}^*_N$ are discussed. In particular, we establish an interrelation between the Wigner functions and the coadjoint orbits (see [25]) $\mathcal{O}_{\boldsymbol{r}}$ of $\mathrm{SU}(N)$:

$$\mathcal{O}_{\boldsymbol{r}} = \{UDU^\dagger\ : U \in \mathrm{SU}(N)\}, \tag{14}$$

where $\boldsymbol{r}$ denotes an N-tuple of real numbers $\boldsymbol{r} = r_1, r_2, \ldots, r_N$ which are the elements of a diagonal matrix $D = \mathrm{diag}||r_1, r_2, \ldots, r_N||$ ordered as $r_1 \geq r_2 \geq \cdots \geq r_N$ and summing to zero, $\sum\limits_{i=1}^{N} r_i = 0$. It is then proved that

$$W_\varrho(\Omega_N) - \frac{I}{N}\ :\ \mathfrak{P}_N \times \mathcal{O}_{\boldsymbol{r}}\big|_{\sum r_i^2 = N/(N-1)} \quad \to \quad \mathbb{R}. \tag{15}$$

Furthermore, in order to describe in a unitarily invariant way the ambiguity of the Wigner function, we introduce the *moduli space* $\mathcal{P}_N$ of an SW kernel as the following coset:

$$\mathcal{P}_N := \left.\frac{\mathcal{O}_{\boldsymbol{r}}}{\mathrm{SU}(N)}\right|_{\sum r_i^2 = N/(N-1)}. \tag{16}$$

The moduli space geometrically represents the intersections of the orbit space of the coadjoint action of the group $\mathrm{SU}(N)$ with an $(N-2)$-dimensional sphere. In conclusion, we give a few

[3]Since $\mathfrak{g}$ is a linear space over the real field $\mathbb{R}$, one can define a bilinear map $\langle\,.\,,\,.\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and identify the algebra with its dual. The conventional inner product on $\mathfrak{g}$,

$$\langle A,\, B\rangle := \mathrm{tr}\left(A^\dagger B\right), \quad A, B \in \mathfrak{su}(N), \tag{8}$$

enables one to set up a duality pairing and to realize an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$.

examples of moduli spaces of Wigner functions for low-level quantum systems, for a qubit ($N = 2$), qutrit ($N = 3$), and quatrit ($N = 4$).

2. Constructing the Wigner function

Below, we give a brief summary of the construction of the Wigner quasiprobability distribution starting from the basic Stratonovich–Weyl postulates and reformulating them into a set of algebraic constraints on the spectrum of SW kernels $\Delta(\Omega_N)$.

The Stratonovich–Weyl principles. Following Brif and Mann [23], the postulates known as the Stratonovich–Weyl correspondence can be written as the following constraints on the kernel $\Delta(\Omega_N)$.

(1) **Reconstruction**: a state ϱ can be reconstructed from the WF (12) via an integral over the phase space:

$$\varrho = \int\limits_{\Omega_N} \mathrm{d}\Omega_N \, \Delta(\Omega_N) W_\varrho(\Omega_N). \tag{17}$$

(2) **Hermiticity**:

$$\Delta(\Omega_N) = \Delta(\Omega_N)^\dagger. \tag{18}$$

(3) **Finite norm**: the norm of a state is given by the integral of the Wigner distribution:

$$\mathrm{tr}[\varrho] = \int\limits_{\Omega_N} \mathrm{d}\Omega_N W_\varrho(\Omega_N), \qquad \int\limits_{\Omega_N} \mathrm{d}\Omega_N \, \Delta(\Omega_N) = 1. \tag{19}$$

(4) **Covariance**: the unitary transformations $\varrho' = U(\alpha)\varrho U^\dagger(\alpha)$ induce a kernel change:

$$\Delta(\Omega_N') = U(\alpha)^\dagger \Delta(\Omega_N) U(\alpha). \tag{20}$$

The algebraic master equation for SW kernels. The above axioms allow one to derive algebraic equations for SW kernels of N-level quantum systems. With this goal, following the paper [24], we accomplish the following steps.

I. Identification of the phase space Ω_N with a complex flag manifold.
Hereafter, the phase space Ω_N will be identified with a complex flag manifold, $\Omega_N = \mathbb{F}^N_{d_1,d_2,\dots,d_s}$. The latter emerges as follows: assuming that the spectrum of an SW kernel $\Delta(\Omega_N)$ consists of real eigenvalues with algebraic multiplicities k_i, i.e., that the isotropy group H of the kernel is

$$H = U(k_1) \times U(k_2) \times U(k_{s+1}),$$

one can see that the phase space Ω_N can be realized as the coset space $U(N)/H$, the complex flag manifold $\mathbb{F}^N_{d_1,d_2,\dots,d_s}$, where $(d_1, d_2, \dots, d_s)$ is a sequence of positive integers with sum N such that $k_1 = d_1$ and $k_{i+1} = d_{i+1} - d_i$ with $d_{s+1} = N$. Furthermore, since the flag manifold represents a coadjoin orbit of $\mathrm{SU}(N)$, its symplectic structure is given by the corresponding Kirillov–Kostant–Souriau symplectic 2-form [25].

II. Enlarging the phase space Ω_N to the $\mathrm{SU}(N)$ group manifold.
Owing to the unitary symmetry of an N-dimensional quantum system, we can relate the measure $\mathrm{d}\Omega_N$ on the symplectic space Ω_N with the normalized Haar measure $\mathrm{d}\mu_{\mathrm{SU}(N)}$ on the $\mathrm{SU}(N)$ group manifold:

$$\mathrm{d}\Omega_N = C_N^{-1} \mathrm{d}\mu_{\mathrm{SU}(N)} / \mathrm{d}\mu_H.$$

Here C_N is a real normalization constant, $\mathrm{d}\mu_H$ is the Haar measure on the isotropy group H induced by the embedding $H \subset \mathrm{SU}(N)$. Observing that the integrand in (17) is a function of the coset variables only, the reconstruction integral can be extended to the whole group SU(N):

$$\varrho = Z_N^{-1} \int\limits_{\mathrm{SU}(N)} \mathrm{d}\mu_{\mathrm{SU}(N)}\, \Delta(\Omega_N) W_\varrho(\Omega_N), \tag{21}$$

where the normalization constant $Z_N^{-1} = C_N^{-1}/\mathrm{vol}(H)$ includes the factor vol(H) which is the volume of the isotropy group H.

III. Derivation of algebraic equations for an SW kernel.
Relations (12) and (21) imply the integral identity

$$\varrho = Z_N^{-1} \int\limits_{\mathrm{SU}(N)} \mathrm{d}\mu_{\mathrm{SU}(N)}\, \Delta(\Omega_N)\, \mathrm{tr}\left[\varrho \Delta(\Omega_N)\right]. \tag{22}$$

Substituting the singular value decomposition for an SW kernel into (22) and evaluating the integral using the Weingarten formula [26–28],

$$\begin{aligned}\int d\mu U_{i_1 j_1} U_{i_2 j_2} \bar{U}_{k_1 l_1} \bar{U}_{k_2 l_2} = {}& \frac{1}{N^2-1}(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_1}\delta_{j_2 l_2} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_2}\delta_{j_2 l_1}) \\ &- \frac{1}{N(N^2-1)}(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_2}\delta_{j_2 l_1} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_1}\delta_{j_2 l_2}),\end{aligned}$$

we derive the equations:

$$(\mathrm{tr}[\Delta(\Omega_N)])^2 = Z_N N, \quad \mathrm{tr}[\Delta(\Omega_N)^2] = Z_N N^2. \tag{23}$$

IV. Normalization of an SW kernel.
The constant Z_N in Eq. (21) can be determined with the aid of the so-called standardization condition

$$Z_N^{-1} \int \mathrm{d}\mu_{\mathrm{SU}(N)} W_A(\Omega_N) = \mathrm{tr}[A]. \tag{24}$$

Fixing the normalization constant Z_N, we finally arrive at the **"master equations"** for an SW kernel:

$$\mathrm{tr}\left[\Delta(\Omega_N)\right] = 1, \qquad \mathrm{tr}[\Delta(\Omega_N)^2] = N. \tag{25}$$

3. Moduli space: counting the solutions to the "master equations"

Classifying the solutions to the master equations (25), we arrive at the notion of "*moduli space*" as the space $\mathcal{P}_N$ whose points are associated with the unitary equivalent admissible SW kernels of N-dimensional quantum systems. The analysis of the solution space of Eq. (25) displays the following properties of the moduli space $\mathcal{P}_N$:

(1) $\dim(\mathcal{P}_N(\boldsymbol{\nu})) = N-2$, i.e., the maximum number of continuous parameters $\boldsymbol{\nu}$ characterizing the solution $\Delta(\Omega_N\,|\,\boldsymbol{\nu})$ is $N-2$;

(2) geometrically, $\mathcal{P}_N$ is represented as the intersection of an $(N-2)$-dimensional sphere $\mathbb{S}_{N-2}$ with the orbit space $\mathfrak{su}(N)^*/\,\mathrm{SU}(N)$ of the action of SU(N) on the dual space $\mathfrak{su}(N)^*$:

$$\mathcal{P}_N \cong \mathbb{S}_{N-2} \bigcap \frac{\mathfrak{su}(N)^*}{\mathrm{SU}(N)}. \tag{26}$$

In order to become convinced in the above statements, consider the singular value decomposition of an SW kernel and assume that the kernel is generic with all eigenvalues distinct.[4]

[4]In this case, the isotropy group of the SW kernel is isomorphic to the $(N-1)$-dimensional torus $\mathbb{T}^{N-1} = \{g \in \mathrm{SU}(N) : g \text{ is diagonal}\}$.

Using the orthonormal basis $\{\lambda_1, \lambda_2, \dots, \lambda_{N^2-1}\}$ of $\mathfrak{su}(N)$, the SVD decomposition reads

$$\Delta(\Omega_N|\boldsymbol{\nu}) = \frac{1}{N}U(\Omega_N)\left[I + \kappa\sum_{\lambda\in H}\mu_s(\boldsymbol{\nu})\lambda_s\right]U(\Omega_N)^\dagger, \tag{27}$$

where $\kappa = \sqrt{N(N^2-1)/2}$ and H is the Cartan subalgebra $H \subset \mathfrak{su}(N)$.

From the master equation (25) it follows that the coefficients $\mu_s(\boldsymbol{\nu})$ live on an $(N-2)$-dimensional sphere $\mathbb{S}_{N-2}(1)$ of radius 1:

$$\sum_{s=2}^{N}\mu^2_{s^2-1}(\boldsymbol{\nu}) = 1. \tag{28}$$

A generic SW kernel can be parametrized by $N-2$ spherical angles. The parameter $(\boldsymbol{\nu})$ introduced in order to label the members of the family of Wigner functions can be associated with a point on $\mathbb{S}_{N-2}(1)$. More precisely, a one-to-one correspondence between points on this sphere and unitary nonequivalent SW kernels holds only for a certain subspace of $\mathbb{S}_{N-2}(1)$. This subspace $\mathcal{P}_N(\boldsymbol{\nu}) \subset \mathbb{S}_{N-2}(1)$ represents the moduli space of SW kernels. Its geometry is determined by the ordering of the eigenvalues of $\Delta(\Omega_N\,|\,\boldsymbol{\nu})$. The chosen descending order restricts the range of spherical angles parametrizing (28) and cuts out the moduli space of $\Delta(\Omega_N\,|\,\boldsymbol{\nu})$ in the form of a spherical polyhedron. Details of the parametrization of SW kernels in terms of spherical angles are given in the appendix.

4. The Wigner function as a dual pairing between ϱ and Δ

As soon as the space of all possible SW kernels is known, the construction of the Wigner function reduces to the computation of the pairing (12). Using the $\mathfrak{su}(N)$ expansions (10) for a density matrix ϱ_ξ of an N-level system characterized by an (N^2-1)-dimensional Bloch vector $\boldsymbol{\xi}$,

$$\varrho_\xi = \frac{1}{N}\left(I + \sqrt{\frac{N\,(N-1)}{2}}\,(\boldsymbol{\xi}, \boldsymbol{\lambda})\right),$$

and the SW kernel decomposition (27), we arrive at a general representation for the WF:

$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\theta_1, \theta_2, \dots, \theta_d) = \frac{1}{N}\left[1 + \frac{N^2-1}{\sqrt{N+1}}\,(\boldsymbol{n}, \boldsymbol{\xi})\right], \tag{29}$$

where the (N^2-1)-dimensional vector $\boldsymbol{n}$ is the linear combination of $N-1$ orthonormal vectors $\boldsymbol{n}^{(s^2-1)}$ with coefficients $\mu_{s^2-1}(\boldsymbol{\nu})$, $s = 2, 3, \dots, N$:

$$\boldsymbol{n} = \mu_3\boldsymbol{n}^{(3)} + \mu_8\boldsymbol{n}^{(8)} + \cdots + \mu_{N^2-1}\boldsymbol{n}^{(N^2-1)}.$$

The vectors $\boldsymbol{n}^{(s^2-1)}$ are determined by elements λ_{s^2-1} of the Cartan subalgebra H:

$$\boldsymbol{n}^{(s^2-1)}_\mu = \frac{1}{2}\,\mathrm{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right), \quad s = 2, 3, \dots, N.$$

As mentioned in the introduction, the number of symplectic coordinates $\vartheta_1, \vartheta_2, \dots, \vartheta_d$ of the Wigner function depends on the isotropy group of the SW kernel (cf. details in [24]).

5. Examples

Below, we present an explicit parametrization for a moduli space of a few low-dimensional quantum systems, including a single qubit, qutrit, and quatrit.

5.1. The moduli space of SW kernels of a single qubit. For a two-level quantum system, a qubit, the master equations (25) determine the spectrum (up to a permutation) of a two-dimensional SW kernel uniquely:

$$\Delta^{(2)}(\Omega_2) = \frac{1}{2} U(\Omega_2) \begin{pmatrix} 1+\sqrt{3} & 0 \\ 0 & 1-\sqrt{3} \end{pmatrix} U(\Omega_2)^{\dagger}, \tag{30}$$

with $U(\Omega_2) \in \mathrm{SU}(2)/U(1)$. Its connection to the structure of coadjoint orbits of SU(2) is straightforward. There are two types of coadjoint orbits of SU(2):

(1) two-dimensional regular orbits $\mathcal{O}_{\boldsymbol{r}}$,

$$\mathcal{O}_{\{r,-r\}} = \left\{ U \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} U^{\dagger},\, U \in \mathrm{SU}(2) \right\},$$

defined for an ordered 2-tuple $\boldsymbol{r} = \{r, -r\}$, $r > 0$. They are isomorphic to a two-dimensional sphere $\mathbb{S}_2(r)$ with radius given by the value of the SU(2)-invariant:

$$r^2 = -\det\left(\mathcal{O}_{\boldsymbol{r}}\right); \tag{31}$$

(2) a zero-dimensional orbit, the point $r = 0$.

Identifying these orbits with the traceless part of the SW kernel $\Delta^{(2)} - \frac{1}{2}\mathbb{I}$ and taking into account the expression (30), we see that

$$r^2 = \frac{4}{3} \operatorname{tr}\left[\left(\Delta^{(2)} - \frac{1}{2}\mathbb{I} \right)^2 \right] = 2.$$

Thus, the moduli space of SW kernels of a qubit is a single point, $r^2 = 2$, from the set of equivalence classes of regular SU(2)-orbits, $[\mathcal{O}_{\boldsymbol{r}}] \cong \mathfrak{su}(2)/U(1)$.

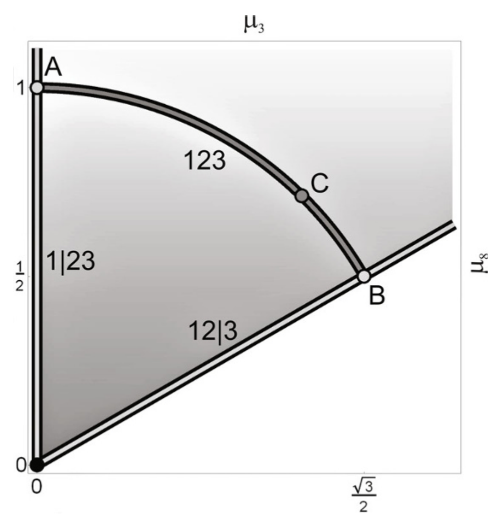

Fig. 1. The cone representing the orbit space of SU(3). The interior of the cone represents orbits with $\dim \mathcal{O} = 6$. The apex corresponds to a zero-dimensional orbit, while the other points on the ordinate axis ($\mu_8 = 0$) and on the positive ray $\mu_8 = \mu_3/\sqrt{3}$ determine orbits with $\dim \mathcal{O} = 4$. The intersection of the cone with the unit circle is an arc which is the moduli space of SW kernels of a qutrit. The point C with $\cos(\zeta_C) = (-1 + 3\sqrt{5})/8$ describes the singular SW kernel.

5.2. The moduli space of SW kernels of a single qutrit. The master equations (25) determine two lowest-degree polynomial SU(N)-invariants of an SW kernel $\Delta(\Omega_3)$, linear and quadratic ones. For the case of a three-dimensional quantum system, a qutrit, the third algebraically independent polynomial SU(3)-invariant remains unfixed, thus allowing a one-parameter family of SW kernels to exist.

Following the normalization convention (27), let us write the SVD decomposition of a qutrit SW kernel in the following form:

$$\Delta(\Omega_3) = U(\Omega_3)PU(\Omega_3)^\dagger = U(\Omega_3)\left[\frac{1}{3}\mathbb{I} + \frac{2}{\sqrt{3}}\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}\right]U(\Omega_3)^\dagger, \tag{32}$$

where $U(\Omega_3) \in \mathrm{SU}(3)$ and a 3-tuple $\boldsymbol{r} = \{r_1, r_2, r_3\}$ parametrizes the traceless diagonal part of the SVD decomposition of an SW kernel, $r_1 + r_2 + r_3 = 0$. Expanding P over the Gell-Mann basis elements $\lambda_3 = \mathrm{diag}\|1, -1, 0\|$ and $\lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}\|1, 1, -2\|$ of the Cartan subalgebra of SU(3),

$$P = \frac{1}{3}\mathbb{I} + \frac{2}{\sqrt{3}}\left(\mu_3\lambda_3 + \mu_8\lambda_8\right), \tag{33}$$

we find

$$r_1 = \mu_3 + \frac{1}{\sqrt{3}}\mu_8, \qquad r_2 = -\mu_3 + \frac{1}{\sqrt{3}}\mu_8, \qquad r_3 = -\frac{2}{\sqrt{3}}\mu_8. \tag{34}$$

From these relations it follows that the chosen decreasing order of parameters $r_1 \geq r_2 \geq r_3$ determines on the (μ_3, μ_8)-plane a two-dimensional polyhedral cone $C_2(\pi/3)$ with apex angle $\pi/3$:

$$C_2(\pi/3) = \left\{\boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 0 \\ \frac{-1}{\sqrt{3}} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0\right\}. \tag{35}$$

• **The** SU(3) **orbits** • The cone $C_2(\pi/3)$ represents the orbit space of the SU(3) group action on the $\mathfrak{su}(3)$ algebra. Next, we identify this algebra times $\imath$ with the traceless part of $\Delta(\Omega_3)$ and classify SW kernels in accordance to the corresponding coadjoint orbits. In order to realize this program, let us consider the tangent space to the SU(3)-orbits. It is spanned by linearly independent vectors built from the commutators $t_k = [\lambda_k, \Delta]$, $\lambda_k \in \mathfrak{su}(3)$. The number of independent vectors t_k determines the dimension of the orbits via the rank of the 8×8 Gram matrix

$$\mathcal{G}_{kl}(\Delta^{(3)}) = \frac{1}{2}\,\mathrm{tr}\,(t_k t_l), \qquad k, l = 1, 2, \ldots, 8. \tag{36}$$

Since the rank of the Gram matrix (36) is SU(3)-invariant, one can calculate it for the diagonal representative of the SW kernel (33). The straightforward computations give

$$\mathcal{G}(\Delta^{(3)}) = \frac{4}{3}\mathrm{diag}\,||\,g_1, g_1, 0, g_2, g_2, g_3, g_3, 0\,||, \tag{37}$$

where $g_1 = 4\mu_3^2$, $g_2 = \frac{1}{\sqrt{3}}(\mu_3 + \sqrt{3}\mu_8)^2$, $g_3 = \frac{1}{\sqrt{3}}(\mu_3 - \sqrt{3}\mu_8)^2$. From these expressions it follows that there are three types of SU(3)-orbits which can be classified according to their symmetry and dimensions:

(1) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 6.}$ These *regular orbits*, abbreviated as $\mathcal{O}(123)$ (or simply 123), are labeled by 3-tuples $\boldsymbol{r}$ with $r_1 > r_2 > r_3$ and have the isotropy group $H_{(123)}$ isomorphic to the two-dimensional torus, $H_{(123)} \cong \mathbb{T}^2$. They are in a one-to-one correspondence with the interior points of the cone $C_2(\pi/3)$ in Fig. 1.

(2) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 4.}$ These *degenerate orbits* represent two subfamilies with degenerate 3-tuples $\boldsymbol{r}$: either $r_1 = r_2 > r_3$ or $r_1 > r_2 = r_3$. Following V. I. Arnold [30], we denote them

by 1|23 and 12|3, respectively. Geometrically, the equivalence class $[\mathcal{O}]$ of degenerate orbits represents the boundary lines in the orbit space of SU(3):

$$\mathcal{O}(1|23) \mapsto 1|23: \quad \{\boldsymbol{x} \in C_2(\pi/3)|\ x_2 = 0\ \},$$
$$\mathcal{O}(12|3) \mapsto 12|3: \quad \{\boldsymbol{x} \in C_2(\pi/3)|\ x_2 = x_1/\sqrt{3}\ \}.$$

Both classes, up to conjugacy in SU(3), have the same isotropy group:

$$H_{(12|3)} \cong H_{(1|23)} = \left\{ h \in \left[\begin{array}{c|c} e^{i\alpha}g & 0 \\ \hline 0 & e^{-i\alpha} \end{array}\right] \;\middle|\; g \in \mathrm{SU}(2) \right\}. \tag{38}$$

(3) $\underline{\dim(\mathcal{O}_0) = 0}$. One orbit $\mathcal{O}_0$, the single point $(0,0)$ which is stationary under the group action of SU(3).

• **The parametrization of SW kernels of a qutrit** • We are now in a position to describe the moduli space of a qutrit as a certain subspace of the orbit space of SU(3). Indeed, taking into account that the second-order master equation (25) describes a circle of radius 1 centered at the origin of the (μ_3, μ_8)-plane, we see that the moduli space of SW kernels of a qutrit is the arc depicted in Fig. 1. More precisely, using the above classification of SU(3)-orbits, we treat the moduli space of a qutrit as the union of two strata:

- The regular stratum corresponding to the regular SU(3)-orbits. Geometrically, it is the arc $\widehat{AB}/\{A, B\}$ with its endpoints A and B excluded. The corresponding Wigner functions have a six-dimensional support and one-dimensional family of SW kernels, whose spectrum can be written as

$$\operatorname{spec}\left(\Delta^{(3)}(\nu)\right) = \left\{\frac{1-\nu+\delta}{2},\ \frac{1-\nu-\delta}{2},\ \nu\right\}, \tag{39}$$

where $\delta = \sqrt{(1+\nu)(5-3\nu)}$ and $\nu \in (-1, -1/3)$. The parameter ν is related to the apex angle[5] ζ of the cone $C_2(\pi/3)$:

$$\nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta), \quad \zeta \in [0,\ \pi/3]. \tag{40}$$

- The endpoints A and B of the arc $\widehat{AB}$ correspond to two degenerate SW kernels, with $\nu = -1$ and $\nu = -\frac{1}{3}$, respectively,

$$\operatorname{spec}\left(\Delta^{(3)}(-1)\right) = \{1, 1, -1\}, \quad \operatorname{spec}\left(\Delta^{(3)}\left(-\frac{1}{3}\right)\right) = \frac{1}{3}\{5, -1, -1\}.$$

It is necessary to point out that the kernel $\Delta^{(3)}(-1)$ was found by Luis [29].

• **The singular SW kernels of a qutrit** • Apart from the above categorization of SW kernels, we distinguish *singular kernels*, which have at least one zero eigenvalue. From the expression (39) it follows that in the case of a qutrit, among three zeros of the determinant

[5]The apex angle ζ determines the value of a third-order polynomial SU(3)-invariant:

$$\cos(3\zeta) = -\frac{27}{16}\det\left(\Delta^{(3)} - \frac{1}{3}\mathbb{I}\right) = -\frac{27}{16}\det\left(\Delta^{(3)}\right) - \frac{11}{16}.$$

$\det(\Delta^{(3)}) = \nu(\nu^2 - \nu - 1)$, only one, $\nu = (1 - \sqrt{5})/2$, is admissible[6]:

$$\operatorname{spec}\left(\Delta_{(103)}\right) = \left\{\frac{1+\sqrt{5}}{2},\, 0,\, \frac{1-\sqrt{5}}{2}\right\}.$$

6. The moduli space of SW kernels of a single quatrit

The master equations (25) for a four-level system, a quatrit, determine a two-parameter family of SW kernels. We start, as in the case of a qutrit, with the SVD decomposition of a quatrit SW kernel:

$$\Delta(\Omega_4) = U(\Omega_4) P U(\Omega_4)^\dagger = U(\Omega_4)\left[\frac{1}{4}\mathbb{I} + \frac{\sqrt{30}}{4}\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}\right] U(\Omega_4)^\dagger,$$

with $U(\Omega_4) \in \mathrm{SU}(4)$ and a 4-tuple $\boldsymbol{r} = \{r_1, r_2, r_3, r_4\}$ such that $r_1 + r_2 + r_3 + r_4 = 0$. These parameters can be expressed in terms of the expansion coefficients of P over the Gell-Mann basis elements $\lambda_3 = \mathrm{diag}||1,-1,0,0||$, $\lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}||1,1,-2,0||$, and $\lambda_{15} = \frac{1}{\sqrt{3}}\mathrm{diag}||1,1,1,-3||$ of the Cartan subalgebra of SU(3),

$$P = \frac{1}{4}\mathbb{I} + \frac{\sqrt{30}}{4}\left(\mu_3\lambda_3 + \mu_8\lambda_8 + \mu_{15}\lambda_{15}\right), \tag{41}$$

as follows:

$$r_1 = \mu_3 + \frac{1}{\sqrt{3}}\mu_8 + \frac{1}{\sqrt{6}}\mu_{15}, \qquad r_2 = -\mu_3 + \frac{1}{\sqrt{3}}\mu_8 + \frac{1}{\sqrt{6}}\mu_{15}, \tag{42}$$

$$r_3 = -\frac{2}{\sqrt{3}}\mu_8 + \frac{1}{\sqrt{6}}\mu_{15}, \qquad r_4 = -\frac{3}{\sqrt{6}}\mu_{15}. \tag{43}$$

Due to the order $r_1 \geq r_2 \geq r_3 \geq r_4$, the expansion coefficients μ_3, μ_8, and μ_{15} belong to a three-dimensional polyhedral cone $C_3(\pi/6)$ with apex angle $\pi/6$:

$$C_3(\pi/6) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 1 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq 0 \right\}. \tag{44}$$

• **The** SU(4)**-orbits** • The cone $C_3(\pi/6)$ represents the orbit space of SU(4). The Gram matrix $\mathcal{G}(\Delta^{(4)})_{15\times 15}$ calculated for the diagonal representative reads as follows:

$$\mathcal{G}(\Delta^{(4)}) = \frac{5}{2}\mathrm{diag}||g_1, g_1, 0, g_2, g_2, g_3, g_3, 0, g_4, g_4, g_5, g_5, g_6, g_6, 0||, \tag{45}$$

where

$$g_1 = 3\mu_3^2, \quad g_2 = \frac{3}{4}\left(\mu_3 + \sqrt{3}\mu_8\right)^2, \quad g_3 = \frac{3}{4}\left(\mu_3 - \sqrt{3}\mu_8\right)^2,$$

$$g_4 = \frac{1}{8}\left(\sqrt{6}\mu_3 + \sqrt{2}\mu_8 + 4\mu_{15}\right)^2, \quad g_5 = \frac{1}{8}\left(-\sqrt{6}\mu_3 + \sqrt{2}\mu_8 + 4\mu_{15}\right)^2, \quad g_6 = \left(\mu_8 - \sqrt{2}\mu_{15}\right)^2.$$

The analysis of the zeros of the Gram matrix (45) shows the following pattern of regular and degenerate SU(4)-orbits.

[6]The traces of powers of this "*golden ratio*" kernel are given by the so-called *Lucas numbers*:

$$\mathrm{tr}\left(\Delta_{(103)}\right)^2 = 3, \quad \mathrm{tr}\left(\Delta_{(103)}\right)^3 = 4, \quad \dots, \quad \mathrm{tr}\left(\Delta_{(103)}\right)^n = L_n.$$

- $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 12}$. The regular orbits have the maximum dimension owing to the smallest isotropy group: $H_{(1234)} = \mathbb{T}^3 \in \mathrm{SU}(4)$. The equivalence class of regular orbits represents the interior of the cone $C_3\,(\pi/6)$.
- The degenerate orbits are divided into subclasses:
  (1) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 10}$. The equivalence class of these orbits is one of the following faces of the cone $C_3\,(\pi/6)$:

$$\mathcal{O}(1|234) \mapsto 1|234 : \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = 0\ \}, \tag{46}$$

$$\mathcal{O}(12|34) \mapsto 12|34 : \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = -\sqrt{3}x_2\}, \tag{47}$$

$$\mathcal{O}(123|4) \mapsto 123|4 : \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_2 = +\sqrt{2}x_3\}. \tag{48}$$

  All the above orbits have the same isotropy group (up to SU(4)-conjugation):

$$H_{(1|234)} = \left\{ h \in \left[\begin{array}{c|c|c} e^{i\alpha}g & 0 & 0 \\ \hline 0 & e^{i\beta} & 0 \\ \hline 0 & 0 & e^{i\gamma} \end{array}\right] \ \middle|\ g \in \mathrm{SU}(2),\ \alpha+\beta+\gamma = 0 \right\}.$$

  The dimension of this stratum is in agreement with the dimension of the corresponding isotropy group:

$$\dim(\mathcal{O}_{\boldsymbol{r}}) = \dim(\mathrm{SU}(4)) - \dim(H_{\boldsymbol{r}}) = 15 - (3+2) = 10.$$

  (2) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 8}$. The equivalence class of these orbits is the following edge of the cone $C_3\,(\pi/6)$:

$$\mathcal{O}(1|23|4) \mapsto 1|23|4 : \quad \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = 0,\ x_2 = \sqrt{2}x_3\ \}. \tag{49}$$

  The 7-dimensional isotropy group is

$$H_{(1|23|4)} = \left\{\ h \in \left[\begin{array}{c|c} e^{i\alpha}g & 0 \\ \hline 0 & e^{-i\alpha}g' \end{array}\right] \ \middle|\ g, g' \in \mathrm{SU}(2)\ \right\}. \tag{50}$$

  (3) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 6}$. The equivalence class of these orbits is one of the following edges of the cone $C_3\,(\pi/6)$:

$$\mathcal{O}(1|2|34) \mapsto 1|2|34 : \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = 0,\ x_2 = 0\}, \tag{51}$$

$$\mathcal{O}(12|3|4) \mapsto 12|3|4 : \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = \sqrt{3}x_2,\ x_2 = \sqrt{2}x_3\}. \tag{52}$$

  Both classes have the same, up to conjugacy, 9-dimensional isotropy group:

$$H_{(1|2|34)} = \left\{\ h \in \left[\begin{array}{c|c} e^{i\alpha}g & 0 \\ \hline 0 & e^{-i\alpha} \end{array}\right] \ \middle|\ g \in \mathrm{SU}(3)\ \right\}. \tag{53}$$

  (4) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 0}$. The apex of the cone $C_3\,(\pi/6)$ with stability group SU(4).

• **The parametrization of SW kernels of a quatrit** • Now we are ready to enumerate all SW kernels for a quatrit according to the above classification of SU(4)-orbits:

(1) The regular two-dimensional family of SW kernels:

$$\mathrm{spec}\left(\Delta^{(4)}(\nu_1,\nu_2)\right) = \left\{\frac{1-\nu_1-\nu_2+\delta}{2}, \frac{1-\nu_1-\nu_2-\delta}{2}, \nu_1, \nu_2\right\}, \tag{54}$$

where $\delta = \sqrt{7 + 2\nu_1 - 3\nu_1^2 + 2\nu_2 - 2\nu_1\nu_2 - 3\nu_2^2}$.

(2) The degenerate one-dimensional family of SW kernels:

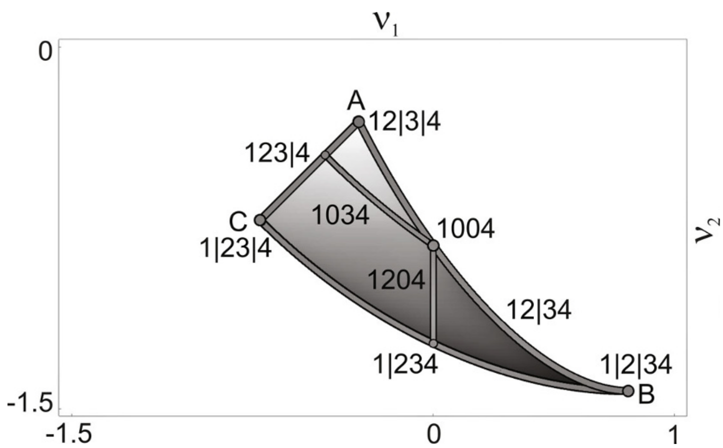

Fig. 2. The support of the SW kernels of a quatrit on the (ν_1, ν_2)-plane. The interior of the curvilinear triangle ABC corresponds to the regular SW kernels. The boundary lines describe the double degeneracy cases. The vertices A and B describe kernels with triple degeneracy, while the vertex C corresponds to a kernel with two double degeneracies.

(a) the family of SW kernels of type 1|234:

$$\operatorname{spec}\left(\Delta_{(1|234)}\right) = \left\{\frac{1-\nu}{3} + \frac{1}{6}\delta_1, \frac{1-\nu}{3} + \frac{1}{6}\delta_1, \nu, \frac{1-\nu-\delta_1}{3}\right\}, \tag{55}$$

where $\delta_1 = \sqrt{22+4\nu-8\nu^2}$ and $\nu \in \left(\frac{1}{4}(1-\sqrt{15}),\, \frac{1}{4}(1+\sqrt{5})\right)$;

(b) the family of SW kernels of type 12|34:

$$\operatorname{spec}\left(\Delta_{(12|34)}\right) = \left\{\frac{1-2\nu+\delta_2}{2}, \nu, \nu, \frac{1-2\nu-\delta_2}{2}\right\}, \tag{56}$$

where $\delta_2 = \sqrt{7+4\nu-8\nu^2}$ and $\nu \in \left(\frac{1}{4}(1-\sqrt{5}), \frac{1}{4}(1+\sqrt{5})\right)$;

(c) the family of SW kernels of type 123|4:

$$\operatorname{spec}\left(\Delta_{(123|4)}\right) = \left\{\frac{1-2\nu+\delta_2}{2}, \frac{1-2\nu-\delta_2}{2}, \nu, \nu\right\}, \tag{57}$$

where $\nu \in \left(\frac{1}{4}\left(1-\sqrt{15}\right), \frac{1}{4}\left(1-\sqrt{5}\right)\right)$.

(3) <u>The SW kernels with triple degeneracy</u>:

(a) the SW kernel of type 1|2|34:

$$\operatorname{spec}\left(\Delta_{(1|2|34)}\right) = \left\{\frac{1+\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}, \frac{1-3\sqrt{5}}{4}\right\}; \tag{58}$$

(b) the SW kernel of type 12|3|4:

$$\operatorname{spec}\left(\Delta_{(12|3|4)}\right) = \left\{\frac{1+3\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}\right\}. \tag{59}$$

(4) <u>The SW kernel with two double degeneracies</u>:

the SW kernel of type 1|23|4:

$$\operatorname{spec}\left(\Delta_{(1|23|4)}\right) = \left\{\frac{1+\sqrt{15}}{4}, \frac{1+\sqrt{15}}{4}, \frac{1-\sqrt{15}}{4}, \frac{1-\sqrt{15}}{4}\right\}. \tag{60}$$

All the above categories of SW kernels of a quatrit are depicted in Fig. 2. The interior of the curvilinear triangle ABC on the (ν_1, ν_2)-plane corresponds to the regular SW kernels. The boundary lines of the domain describe the double degeneracy cases:
(a) the SW kernel of type 12|34: the side $AB/\{A, B\}$ with both endpoints A and B excluded:

$$AB/\{A,B\}: \ \nu_2 = \frac{1}{2} - \nu_1 - \frac{1}{2}\sqrt{7 + 4\nu_1 - 8\nu_1^2}, \quad \nu_1 \in \left(\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}\right);$$

(b) the SW kernel of type 1|234: the side $CB/\{C, B\}$ without endpoints:

$$CB/\{C,B\}: \ \nu_2 = \frac{1}{3} - \frac{1}{3}\nu_1 - \frac{1}{3}\sqrt{22 + 4\nu_1 - 8\nu_1^2}, \quad \nu_1 \in \left(\frac{1-\sqrt{15}}{4}, \frac{1+\sqrt{5}}{4}\right);$$

(c) the SW kernel of type 123|4: the side $AC/\{A, C\}$ without endpoints:

$$\nu_2 = \nu_1, \quad \nu_1 \in \left(\frac{1-\sqrt{15}}{4}, \frac{1-\sqrt{5}}{4}\right).$$

The vertices A and B describe kernels with triple degeneracy:
(a) the SW kernel of type 12|3|4: the point A, $\nu_1 = \frac{1-\sqrt{5}}{4}$, $\nu_2 = \frac{1-\sqrt{5}}{4}$;
(b) the SW kernel of type 1|2|34: the point B, $\nu_1 = \frac{1+\sqrt{5}}{4}$, $\nu_2 = \frac{1-3\sqrt{5}}{4}$;
while the vertex C corresponds to a kernel with two double degeneracies of type 1|23|4: $\nu_1 = \nu_2 = \frac{1-\sqrt{15}}{4}$.

• **The singular SW kernels of a quatrit** • Among the SW kernels described above, one can distinguish a set of special elements with vanishing determinant. These singular quatrit kernels are listed below in accordance with the increasing singularity of the determinant:

- the SW kernels with a simple root of the determinant:
  (a) the one-parameter family of type 1204, $\frac{1}{3}(1-\sqrt{22}) \leq \nu < \frac{1}{2}(1-\sqrt{7})$,

$$\operatorname{spec}\left(\Delta_{(1204)}\right) = \left\{\frac{1-\nu+\sqrt{7+2\nu-3\nu^2}}{2}, \frac{1-\nu-\sqrt{7+2\nu-3\nu^2}}{2}, 0, \nu\right\}, \tag{61}$$

  (b) the one-parameter family of type 1034, $\frac{1}{6}\left(2-\sqrt{22}\right) \leq \nu < 0$,

$$\operatorname{spec}\left(\Delta_{(1034)}\right) = \left\{\frac{1-\nu+\sqrt{7+2\nu-3\nu^2}}{2}, 0, \nu, \frac{1-\nu-\sqrt{7+2\nu-3\nu^2}}{2}\right\}, \tag{62}$$

- the SW kernel with a double zero of the determinant:

$$\operatorname{spec}\left(\Delta_{(1004)}\right) = \left\{\frac{1+\sqrt{7}}{2}, 0, 0, \frac{1-\sqrt{7}}{2}\right\}. \tag{63}$$

• **The moduli space of a quatrit as the Möbius spherical triangle** • As it was mentioned above, the spectrum of $\Delta^{(4)}(\nu_1, \nu_2)$ is in a correspondence with the points on the unit 2-sphere associated with the expansion coefficients μ_3, μ_8, and μ_{15},

$$\mu_3^2(\boldsymbol{\nu}) + \mu_8^2(\boldsymbol{\nu}) + \mu_{15}^2(\boldsymbol{\nu}) = 1,$$

that satisfy the inequalities

$$\mu_3 \geq 0, \quad \mu_8 \geq \frac{\mu_3}{\sqrt{3}}, \quad \mu_{15} \geq \frac{\mu_8}{\sqrt{2}}.$$

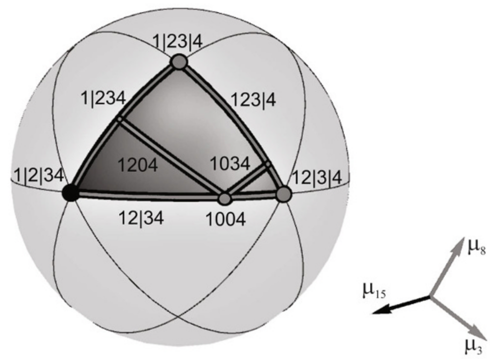

Fig. 3. The moduli space of a quatrit represented by the Möbius spherical triangle $(2, 3, 3)$ on the unit sphere.

Geometrically, these constraints define one out of 24 possible spherical triangles with angles $(\pi/2, \pi/3, \pi/3)$ that tessellate the unit sphere. The repeated reflections in the sides of the triangles tile the sphere exactly once. In accordance with Girard's theorem, the spherical excess of a triangle determines a solid angle: $\pi/2 + \pi/3 + \pi/3 - \pi = 4\pi/24$. The relation between the "flat" representation of the quatrit moduli space (Fig. 2) and its spherical realization (Fig. 3) is demonstrated by the projection pattern in Fig. 4.

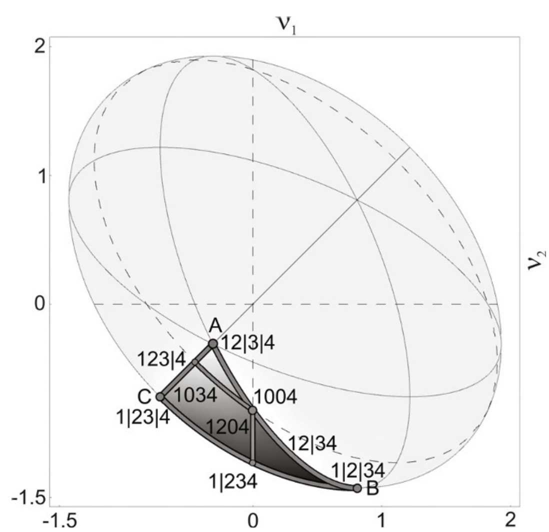

Fig. 4. The mapping of the tiling of $\mathbb{S}_2(1)$ by the Möbius triangles $(2, 3, 3)$ onto a subset of the plane (ν_1, ν_2). The dashed lines represent the degeneracies of the spectrum.

Concluding remark

The master equations (25) for kernels of Wigner functions determine the first- and second-degree polynomial SU(N)-invariants of an N-dimensional system. The remaining $N-2$ algebraically independent invariants parametrize the moduli space of SW kernels. In the present article, we establish a relation between this moduli space and the orbit space of the group SU(N). The next important issue is to clarify the role these unitary invariant moduli parameters play in the dynamics of classical and quantum systems. With this aim, in a forthcoming publication, a detailed analysis of the Kirillov–Kostant–Souriau symplectic 2-form for the whole family of Wigner functions will be given.

Appendix. Parametrization of the moduli space $\mathcal{P}_N(\boldsymbol{\nu})$

As it was mentioned in the main text, the Stratonovich–Weyl kernel can be parametrized by $N-2$ spherical angles. Each member of the family of Wigner functions can be associated with a point of the subspace $\mathcal{P}_N(\boldsymbol{\nu}) \subset \mathbb{S}_{N-2}(1)$, which is determined by an ordering of the eigenvalues of the Stratonovich–Weyl kernel. In order to define the subspace $\mathcal{P}_N(\boldsymbol{\nu})$ corresponding to this descending ordering, by means of using the kernel decomposition in Gell-Mann bases, let us represent the spectrum of the Stratonovich–Weyl kernel in the following form:

$$
\begin{aligned}
\pi_1 &= \frac{1}{N}\left(1+\sqrt{2}\,\kappa\sum_{s=2}^{N}\frac{\mu_{s^2-1}}{\sqrt{s\,(s-1)}}\right),\\
&\vdots\\
\pi_i &= \frac{1}{N}\left(1+\sqrt{2}\,\kappa\sum_{s=i+1}^{N}\frac{\mu_{s^2-1}}{\sqrt{s\,(s-1)}}-\kappa\sqrt{\frac{2\,(i-1)}{i}}\mu_{i^2-1}\right),\\
&\vdots\\
\pi_N &= \frac{1}{N}\left(1-\frac{N^2-1}{\sqrt{N+1}}\mu_{N^2-1}\right).
\end{aligned}
$$

Introducing the conventional parametrization for the unit sphere $\mathbb{S}_{N-2}(1)$ in terms of $N-2$ spherical angles,

$$
\begin{aligned}
\mu_3 &= \sin\psi_1\cdots\sin\psi_{N-2},\\
\mu_8 &= \sin\psi_1\cdots\sin\psi_{N-3}\cos\psi_{N-2},\\
&\vdots\\
\mu_{i^2-1} &= \sin\psi_1\cdots\sin\psi_{N-i}\cos\psi_{N-i+1},\\
&\vdots\\
\mu_{N^2-1} &= \cos\psi_1,
\end{aligned}
\tag{64}
$$

with $\psi_i \in [0,\pi]$, $i=\overline{1,N-3}$, and $\psi_{N-2}\in[0,2\pi)$, and requiring the descending order of the eigenvalues, we obtain the following constraints on the coefficients μ_i:

$$\mu_3 \geq 0, \tag{65}$$

$$\mu_{(i+1)^2-1} \geq \sqrt{\frac{i-1}{i+1}}\,\mu_{i^2-1}, \quad i=\overline{2,N-1}. \tag{66}$$

Let us introduce the following notation:

$$\mathcal{P}_1 = \{\psi_1 = 0\},$$

$$\mathcal{P}_2^{(k)} = \begin{cases} \sin\psi_{N-k} = 0, \\ \sin\psi_{N-(k+1)}\cos\psi_{N-k} > 0, \\ \cot\psi_{N-i} \geq \sqrt{\frac{i-1}{i+1}}\cos\psi_{N-i+1}, \\ 0 < \psi_{i-k} < \pi, \quad i = \overline{k+1,\, N-1}, \end{cases} \tag{67}$$

$$\mathcal{P}_3 = \begin{cases} \sin\psi_{N-2} > 0, \\ \cos\psi_{N-2} \geq \frac{1}{\sqrt{3}}\sin\psi_{N-2}, \\ \cot\psi_{N-i} \geq \sqrt{\frac{i-1}{i+1}}\cos\psi_{N-i+1}, \\ 0 < \psi_{i-2} < \pi, \quad i = \overline{3,\, N-1}, \\ 0 < \psi_{N-2} < 2\pi. \end{cases}$$

In this notation, substituting the expressions for μ_i in terms of the spherical angles ψ_i into (65) and (66) shows that if $k = 2, \ldots N-2$ is the greatest positive integer for which $\sin\psi_{N-k} = 0$ (if there are any), then the simplex is described by the restrictions

$$\mathcal{P}_2^{(k)} \subset \mathbb{S}_{N-(k+1)}(1)$$

(these are some of the $(N-(k+1))$-dimensional boundaries of the simplex); otherwise, if there is no such k, then the restrictions are $\mathcal{P}_3$. Hence, the simplex will be completely defined by

$$\mathcal{P} = \mathcal{P}_1 \cup \left(\bigcup_{k=2}^{N-2} \mathcal{P}_2^{(k)}\right) \cup \mathcal{P}_3. \tag{68}$$

Partially reducing the set of inequalities for $\mathcal{P}_3$, we get

$$\mathcal{P}_3 = \begin{cases} 0 < \psi_{N-2} \leq \frac{\pi}{3}, \\ 0 < \psi_{i-2} < \pi, \quad i = \overline{3,\, N-1}, \\ \cot\psi_{N-i} \geq \sqrt{\frac{i-1}{i+1}}\cos\psi_{N-i+1}. \end{cases} \tag{69}$$

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