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CONSTRUCTING THE $\mathrm{SU}(2) \times \mathrm{U}(1)$ ORBIT SPACE FOR QUTRIT MIXED STATES

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The orbit space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{G}$ of the group $\mathrm{G} := \mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{U}(3)$ acting adjointly on the state space $\mathfrak{P}(\mathbb{R}^8)$ of a three-level quantum system is discussed. The semialgebraic structure of $\mathfrak{P}(\mathbb{R}^8)/\mathrm{G}$ is determined within the Procesi–Schwarz method. Using an integrity basis for the ring of G-invariant polynomials $\mathbb{R}[\mathfrak{P}(\mathbb{R}^8)]^{\mathrm{G}}$, the set of constraints on the Casimir invariants of the group $\mathrm{U}(3)$ coming from the positivity requirement $\mathrm{Grad}(z) \geq 0$ for the Procesi–Schwarz gradient matrix is analyzed in detail. Bibliography: 9 titles.

1. Introduction

Since the very beginning of quantum mechanics, a highly nontrivial interplay between quantities describing a composite quantum system as a "single whole" and "local characteristics" of its constituents became the subject of intensive studies (holistic vs. reductionistic views). The present note aims to discuss a mathematical aspect of "the whole and the parts" problem in quantum theory considering a model of a three-dimensional quantum system, qutrit. Putting aside the physical motivation, these mathematical issues can be formulated as follows.

Consider a compact Lie group G acting on a real n-dimensional space V, and let $\mathrm{H} \subset \mathrm{G}$ be a compact subgroup of G. Assume that the corresponding orbit spaces V/G and V/H admit a realization as semialgebraic subsets, $Z(V/\mathrm{G})$ and $Z(V/\mathrm{H})$, of $\mathbb{R}^q$ for some q. A mathematical version of "the whole and the parts" dilemma can be formulated as the problem of finding a correspondence between the sets $Z(V/\mathrm{H})$ and $Z(V/\mathrm{G})$.

In applications to quantum theory, the role of V is played by the space $\mathfrak{P}(\mathbb{R}^{n^2-1})$ of mixed states of an n-dimensional binary quantum system. The groups G and H are associated with the unitary group $\mathrm{U}(n)$ and its subgroup[1] $\mathrm{U}(n_1) \times \mathrm{U}(n_2) \subset \mathrm{U}(n)$ with the adjoint action

$$\mathrm{Ad}\,(g)\,\varrho = g\varrho g^{-1}, \qquad g \in \mathrm{U}(n), \tag{1}$$

on density matrices $\varrho \in \mathfrak{P}(\mathbb{R}^{n^2-1})$. The action (1) determines the "global orbit space" $\mathfrak{P}(\mathbb{R}^{n^2-1})\,|\mathrm{U}(n)$ and the so-called *entanglement space* $\mathfrak{P}(\mathbb{R}^{n^2-1})\,|\mathrm{U}(n_1) \times \mathrm{U}(n_2)$ of a binary $n_1 \times n_2$ system.

The semialgebraic structure of both orbit spaces admits a description in terms of the corresponding rings of G-invariant polynomials, $\mathbb{R}[\mathfrak{P}]^{\mathrm{U}(n)}$ and $\mathbb{R}[\mathfrak{P}]^{\mathrm{U}(n_1)\times\mathrm{U}(n_2)}$. According to the Procesi–Schwarz method [1, 2], these semialgebraic varieties in $\mathbb{R}^q$ are determined by the syzygy ideal for the corresponding integrity basis and the semipositivity of the so-called gradient matrix, $\mathrm{Grad}(z) \geq 0$. As discussed recently in [3], the representation of the orbit space $\mathfrak{P}(\mathbb{R}^{n^2-1})\,|\mathrm{U}(n)$ in terms of an integrity basis for the ring of $\mathrm{U}(n)$-invariant polynomials is

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[1] The subgroup H is determined by a fixed decomposition of the system into n_1- and n_2-dimensional subsystems such that $n = n_1 \times n_2$.

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completely determined from the physical requirements formulated as the semipositivity and Hermiticity of density matrices. The conditions $\mathrm{Grad}(z) \geq 0$ do not impose any new constraint on the elements of the integrity basis for $\mathbb{R}[\mathfrak{P}]^{\mathrm{U}(n)}$. In contrast to that case, the algebraic and geometric properties of the entanglement space are more subtle. It turns out that in order to determine the local orbit space $\mathfrak{P}(\mathbb{R}^{n^2-1})\,|\mathrm{U}(n_1) \times \mathrm{U}(n_2)$, additional constraints arising from the semipositivity of the Grad-matrix should be taken into account. Moreover, additional inequalities in the elements of the integrity basis for $\mathbb{R}[\mathfrak{P}]^{\mathrm{U}(n_1)\times \mathrm{U}(n_2)}$ provide constraints on the $\mathrm{U}(n)$-invariants. Below, aiming to exemplify this statement, we will study a toy model that mimics the generic case of a binary composite system. Namely, we consider the three-dimensional quantum system defined by the state space $\mathfrak{P}(\mathbb{R}^8)$ that is acted upon by the symmetry group $\mathrm{U}(3)$ and its $\mathrm{U}(2)$ subgroup $\mathrm{SU}(2) \times \mathrm{U}(1)$.

2. Qutrit

• **Parametrization of the qutrit states.** Consider the quantum three-level system called the qutrit. A state of this system, a semipositive Hermitian matrix ϱ with unit trace, can be parameterized as follows:

$$\varrho = \frac{1}{3}\left(\mathbb{I}_3 + \sqrt{3}\sum_{a=1}^{8}\xi_a\lambda_a\right). \tag{2}$$

Here the real parameters $\{\xi_a\}_{a=1,\dots,8}$ are the components of the eight-dimensional Bloch vector $\boldsymbol{\xi}$, and $\{\lambda_a\}_{a=1,\dots,8}$ are the Gell-Mann matrices generating the Hermitian basis of the Lie algebra $\mathfrak{su}(3)$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The product of a pair of Gell-Mann matrices involves two basic sets of structure constants of the algebra $\mathfrak{su}(3)$:

$$\lambda_a\lambda_b = \frac{2}{3}\delta_{ab} + \left(d_{abc} + i f_{abc}\right)\lambda_c, \tag{3}$$

where d_{abc} and f_{abc} are the components of the completely symmetric and skew-symmetric symbols defined via the anti-commutators $\{,\}$ and commutators $[,]$ of the Gell-Mann matrices:

$$d_{abc} = \tfrac{1}{4}\mathrm{Tr}(\{\lambda_a, \lambda_b\}\lambda_c), \quad f_{abc} = \tfrac{1}{4}\mathrm{Tr}([\lambda_a, \lambda_b]\lambda_c).$$

The matrix ϱ from (2) represents a physical state of the qutrit if and only if the Bloch vector $\boldsymbol{\xi}$ is subject to the following polynomial constraints:[2]

$$\xi_a\xi_a \leq 1\,, \tag{4}$$

$$0 \leq \xi_a\xi_a - \frac{2}{\sqrt{3}}d_{abc}\xi_a\xi_b\xi_c \leq \frac{1}{3}\,. \tag{5}$$

[2] Inequalities (4) and (5) reflect the semipositivity $\varrho \geq 0$ of the qutrit density matrices.

• **The unitary symmetry of the qutrit.** As mentioned above, we consider the adjoint action of the unitary group U(3) on $\mathfrak{P}(\mathbb{R}^8)$. The Bloch vector $\boldsymbol{\xi}$ transforms under the Ad-action as an eight-dimensional vector:

$$\xi'_a = O_{ab}\xi_b, \qquad O \in \mathrm{SO}(8),$$

where O is an element of the eight-parameter subgroup of SO(8).[3]

• **The "local symmetry" SU(2) × U(1).** Consider the U(2) subgroup of U(3) identified (up to conjugation) by the standard embedding

$$U(2) = \left\{ g(u) = \left(\begin{array}{c|c} u & \\ \hline & (\det u)^{-1} \end{array} \right) \mid u \in \mathrm{U}(2) \right\} \subset \mathrm{SU}(3). \tag{6}$$

According to the embedding (6) and to the choice of the Gell-Mann basis, the U(2) subgroup is generated by $\lambda_1, \lambda_2, \lambda_3$ (the generators of the SU(2) subgroup) and λ_8 (the generator of the U(1) subgroup). An element of the U(2) subgroup can be written as

$$g = \exp(i\lambda_1\alpha)\exp(i\lambda_2\beta)\exp(i\lambda_3\gamma)\exp(i\theta\lambda_8), \tag{7}$$

where the Euler angles α, β, γ parameterize the SU(2) group and the angle θ corresponds to the U(1) phase, $\det u = \exp(i\frac{2}{\sqrt{3}}\theta)$.

3. A sketch of the Procesi–Schwarz method

The classical theory of invariants is the cornerstone for the description of orbit spaces. Using this theory (see, e.g., [6]), the basic ingredients of the description can be formulated as follows.

Consider a compact Lie group G acting linearly on a real d-dimensional vector space V. Let $\mathbb{R}[V]^{\mathrm{G}}$ be the corresponding ring of G-invariant polynomials on V. Let $\mathcal{P} = (p_1, p_2, \ldots, p_q)$ be a set of homogeneous polynomials that form an integrity basis:

$$\mathbb{R}[x_1, x_2, \ldots, x_d]^{\mathrm{G}} = \mathbb{R}[p_1, p_2, \ldots, p_q].$$

The elements of the integrity basis determine a polynomial mapping

$$p\,:\, V \to \mathbb{R}^q\,; \qquad (x_1, x_2, \ldots, x_d) \to (p_1, p_2, \ldots, p_q). \tag{8}$$

Since p is constant on the orbits of G, it induces a homeomorphism $V/\mathrm{G} \simeq X$ of the orbit space V/G and the image X of p, see [7]. In order to describe X in terms of $\mathcal{P}$ uniquely, it is necessary to take into account the *syzygy ideal*

$$I_{\mathcal{P}} = \{h \in \mathbb{R}[y_1, y_2, \ldots, y_q] : h(p_1, p_2, \ldots, p_q) = 0 \text{ in } \mathbb{R}[V]\,\}.$$

Let $Z \subseteq \mathbb{R}^q$ denote the locus of common zeros of all elements of $I_{\mathcal{P}}$. Then Z is an algebraic subset of $\mathbb{R}^q$ such that $X \subseteq Z$. Denoting by $\mathbb{R}[Z]$ the restriction of $\mathbb{R}[y_1, y_2, \ldots, y_q]$ to Z, one can easily verify that $\mathbb{R}[Z]$ is isomorphic to the quotient $\mathbb{R}[y_1, y_2, \ldots, y_q]/I_{\mathcal{P}}$, and thus $\mathbb{R}[Z] \simeq \mathbb{R}[V]^{\mathrm{G}}$. Therefore, the subset Z is essentially determined by $\mathbb{R}[V]^{\mathrm{G}}$, but in order to describe X, further steps are required. According to [1, 2], the necessary information on X is encoded in the structure of the $q \times q$ matrix whose elements are the inner products of the gradients grad(p_i):

$$\|\mathrm{Grad}\|_{ij} = (\mathrm{grad}\,(p_i)\,, \mathrm{grad}\,(p_j))\ . \tag{9}$$

Thus, summarizing all the above observations, the orbit space can be identified with the semialgebraic variety defined as the set of points satisfying two conditions:

[3] More details on the algebraic and geometric structures of the group SU(3) can be found in the classical paper [8].

(a) $z \in Z$, where Z is the surface determined by the syzygy ideal for the integrity basis in $\mathbb{R}[V]^{\mathrm{G}}$;
(b) $\mathrm{Grad}(z) \geq 0$.

4. Constructing the G-invariant polynomials

Let $\mathrm{GL}(n, \mathbb{C})$ be the general linear group of degree n over the field $\mathbb{C}$. Assume that $\mathrm{GL}(n, \mathbb{C})$ acts on polynomials $p(x_1, x_2, \dots, x_n) \in \mathbb{C}[x_1, x_2, \dots, x_n]$ as follows:

$$(gp)\,(x_1, x_2, \dots, x_n) := p\left(x_1', x_2', \dots, x_n'\right), \quad g \in \mathrm{GL}(n, \mathbb{C}), \tag{10}$$

where

$$x_i' = g_{ij}^{-1} x_j. \tag{11}$$

A polynomial $p(x_1, x_2, \dots, x_n)$ is called G-invariant if it is a fixed point of the transformation (10):

$$(gp)\,(x_1, x_2, \dots, x_n) := p\,(x_1, x_2, \dots, x_n)\,. \tag{12}$$

Here we deal with polynomials in the n^2 complex entries of density matrices $p(\varrho) = p(\varrho_{11}, \varrho_{12}, \dots, \varrho_{nn})$. To reduce the adjoint action (1) to a linear transformation of the form (11), one can identify a Hermitian density matrix ϱ with a complex vector V of length n^2 and consider the linear representation of the subgroup $L \subset \mathrm{GL}(n, \mathbb{C})$ defined via the tensor product of a unitary matrix with its complex conjugate:

$$L := \mathrm{U}(n) \otimes \overline{\mathrm{U}(n)}. \tag{13}$$

The invariant polynomials satisfying (12) form an algebra over $\mathbb{C}$, and any such invariant can be expressed as a polynomial in the so-called fundamental invariants, homogeneous polynomials of fixed degrees. Since the homogeneous invariants of a fixed degree form a vector space, it is sufficient to find a maximal linearly independent set of homogeneous invariants, i.e., a basis for that vector space. The dimension of this vector space can be extracted from the power series (Poincare series [4]) expansion of the Molien function [5]. In fact, given a compact Lie group G and a representation π of G, the Molien function can be directly defined by the power series (cf. [5])

$$M_\pi(\mathbb{C}[V]^{\mathrm{G}}, q) = \sum_{k=0}^{\infty} c_k(\pi) q^k. \tag{14}$$

Here $c_k(\pi)$ is the number of linearly independent G-invariant polynomials of degree k on V.

4.1. The Molien function. The Molien function (14) associated with a representation $\pi(g)$ of a compact Lie group G on V admits an integral representation [5,6] (Molien's formula):

$$M_\pi(\mathbb{C}[V]^{\mathrm{G}}, q) = \int_G \frac{d\mu(g)}{\det(\mathbb{I} - q\pi(g))}, \quad |q| < 1, \tag{15}$$

where $d\mu(g)$ is the Haar measure for the Lie group G. According to Weyl's integration formula [6], an integral over a compact Lie group G can be decomposed into a double integral over a maximal torus T and the quotient G/T of the group by this torus. If the integrand is a function invariant under the conjugation in the group, then the latter integral is "q-independent" and the total integral reduces to an integral over the maximal torus with coordinate x and the additional Weyl factor $A(x)$:

$$M_\pi(\mathbb{C}[V]^{\mathrm{G}}, q) = \int_T \frac{d\mu[x]\, A(x)}{\det(\mathbb{I} - q\pi(x))}. \tag{16}$$

The resulting integral is transformed into a complex path integral which can be evaluated by the residue theorem.

In what follows, we present the Molien functions for the group U(3) and its U(2) subgroup acting linearly according to (13) on the complex nine-dimensional space.

• **The Molien function for U(3)**. For the group U(3), the Weyl factor $A(x)$ is the squared Vandermonde determinant calculated for the torus coordinates divided by the order of the corresponding Weyl group:

$$A_{\mathrm{SU}(3)}(x_1, x_2, x_3) = \frac{1}{3!}\prod_{i<j}^{3}(x_i - x_j)\overline{(x_i - x_j)},$$

and the Molien function is given by

$$M^{(d=9)}_{\mathrm{U}(3)}(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)}. \tag{17}$$

• **The Molien function for SU(2)×U(1).** In this case, the $\pi \otimes \bar{\pi}$ representation for the maximal torus is

$$\pi \otimes \bar{\pi} = (x, x^{-1}, y) \otimes (x^{-1}, x, y^{-1}) = (1, x^2, xy^{-1}, x^{-2}, 1, x^{-1}y^{-1}, yx^{-1}, xy, 1),$$

where x is the coordinate on the torus of SU(2) and y is the coordinate on U(1). The Weyl factor for SU(2) is

$$A_{\mathrm{SU}(2)}(x) := 1 - \frac{x^2 - x^{-2}}{2},$$

so that the integral in (16) reduces to the double path integral

$$M^{(d=9)}_{\mathrm{SU}(2)\times\mathrm{U}(1)}(q) = \int \frac{d\,\mu_{\mathrm{SU}(2)} d\,\mu_{\mathrm{U}(1)}}{\det|1 - q\,\pi \otimes \bar{\pi}|}$$
$$= \frac{1}{8\pi^2}\frac{1}{(1-q)^3}\oint\limits_{|x|=1}\oint\limits_{|y|=1}\frac{(1-x^2)^2\,xdx\,ydy}{(1-qx^2)(1-qxy)(y-qx)(x-qy)(xy-q)(x^2-q)}.$$

The subsequent calculation of the residues of the integrand first with respect to y at the poles $P_y = \{qx,\ q/x\}$ and then with respect to x at the poles $P_x = \{\pm\sqrt{q},\ \pm q\}$ finally gives a rational expression for the Molien function:

$$M^{(d=9)}_{\mathrm{SU}(2)\times\mathrm{U}(1)}(q) = \frac{1}{(1-q)^2(1-q^2)^2(1-q^3)}. \tag{18}$$

4.2. U(3)- and SU(2)×U(1)-invariant polynomials. Expressions (17) and (18) for the Molien functions indicate that the set of fundamental homogeneous polynomials for the ring $\mathbb{C}[x]^{\mathrm{SU}(3)}$ consists of three polynomials of degrees 1, 2, and 3, while there are five SU(2) × U(1)-invariant homogeneous polynomials forming an integrity basis for the ring $\mathbb{C}[x]^{\mathrm{SU}(2)\times\mathrm{U}(1)}$. The latter basis includes one polynomial of degree 1, two polynomials of degree 2, and one polynomial of degree 3.

An integrity basis for the ring $\mathbb{C}[x]^{\mathrm{SU}(3)}$ can be composed either of the trace invariants $t_k = \mathrm{tr}\left(\varrho^k\right)$, $k = 1, 2, 3$, or of the SU(3) Casimir invariants constructed via the correspondence with the elements of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{su}(3))$.

• **Casimir invariants.** Using the Bloch parametrization for the density matrix (2) of the qutrit, we see that the quadratic and qubic Casimir invariants are the following polynomials:

$$\mathfrak{C}_2 = \xi_i\xi_i\,, \tag{19}$$

$$\mathfrak{C}_3 = \sqrt{3}\,d_{ijk}\xi_i\xi_j\xi_k. \tag{20}$$

• **SU(2)×U(1)-invariants.** Due to the graded structure of the ring of invariants, their construction reduces to constructing the basic homogeneous invariant polynomials. These homogeneous G-invariant polynomials of a given degree are subject to the system of linear homogeneous equations (12). Actually, these equations reduce to an infinitesimal version of the following form [4]:

$$e_i f = 0\,, \quad i = 1, \dots, m\,,$$
$$g_j f = f\,, \quad i = 1, \dots, s\,,$$

where $e_1, \dots, e_m$ form a basis of the Borel subgroup $B \subset \mathrm{G}$ and $g_1, \dots, g_s$ is a system of representatives of the conjugacy classes of the group G with respect to its maximal connected subgroup. Applying this general scheme, one can derive the following set of $\mathrm{SU}(2) \times \mathrm{U}(1)$-invariants:

$$f_1 = \xi_8\,, \tag{21}$$
$$f_2 = \xi_1^2 + \xi_2^2 + \xi_3^2\,, \tag{22}$$
$$f_3 = \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2\,, \tag{23}$$
$$f_4 = 2(-\xi_1(\xi_4\xi_6 + \xi_5\xi_7) + \xi_2(\xi_4\xi_7 - \xi_5\xi_6)) + \xi_3(-\xi_4^2 - \xi_5^2 + \xi_6^2 + \xi_7^2). \tag{24}$$

5. The orbit spaces of the qutrit

Before applying the above-mentioned method by Procesi and Schwarz [1,2] to the construction of the orbit space, let us reformulate the semialgebraic description of the qutrit state space $\mathfrak{P}(\mathbb{R}^8)$ in terms of the SU(3) Casimir invariants. In doing so, we mainly follow the ideology presented in [9].

5.1. The global orbit space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$. Let us start with the semialgebraic structure of the state space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$.

• **The semipositivity of the density matrix.** Equations (4) and (5) defining the semipositivity of the qutrit density matrix in terms of the Bloch vector $\boldsymbol{\xi}$ can be rewritten via two SU(3) Casimir invariants $\mathfrak{C}_2$ and $\mathfrak{C}_3$ as follows:

$$0 \le \mathfrak{C}_2 \le 1\,, \tag{25}$$
$$0 \le 3\mathfrak{C}_2 - 2\mathfrak{C}_3 \le 1. \tag{26}$$

• **The Hermiticity of the density matrix.** Inequalities (25) and (26) should be completed by the condition of being real for the eigenvalues of the qutrit density matrix. This condition can also be expressed as a polynomial inequality in two Casimirs. This inequality is the nonnegativity requirement for the discriminant of the characteristic equation $\det(\lambda - \varrho) = 0$ for the qutrit density matrix ϱ:

$$\mathrm{Disc} := \mathfrak{C}_2^3 - \mathfrak{C}_3^2 \ge 0. \tag{27}$$

Thus the intersection of the strip determined by the linear inequalities (25) and (26) with the domain (27) is the image of the qutrit state space under the polynomial mapping. This intersection is the curvilinear triangle ABC on the $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane depicted in Fig. 1.

Now we will show that the triangle ABC represents the coset space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$ for the qutrit state space. Indeed, since the determinant of the Procesi–Schwarz $\mathrm{Grad}_{\mathrm{SU}(3)}$-matrix

$$\mathrm{Grad}_{\mathrm{SU}(3)} = \begin{pmatrix} 4\mathfrak{C}_2 & 6\mathfrak{C}_3 \\ 6\mathfrak{C}_3 & 9\mathfrak{C}_2^2 \end{pmatrix} \tag{28}$$

is proportional to the discriminant (27)

$$\det \|\mathrm{Grad}_{\mathrm{SU}(3)}\| = 36(\mathfrak{C}_2^3 - \mathfrak{C}_3^2),$$

the semipositivity of the Grad-matrix that determines the orbit space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$ coincides with the Hermiticity requirement for the qutrit density matrix.

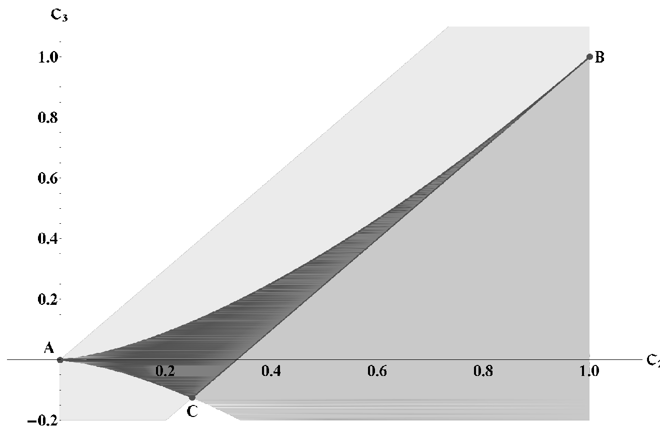


Fig. 1. The triangle ABC as the global orbit space of the qutrit on the Casimir $(\mathfrak{C}_2\,,\mathfrak{C}_3)$-plane.

5.2. The orbit space $\mathfrak{P}/\mathrm{SU}(2)\times\mathrm{U}(1)$. Let us start with the observation that the SU(3) Casimir invariants can be expressed in terms of the four $\mathrm{SU}(2)\times\mathrm{U}(1)$-invariants (21)–(24) as

$$\mathfrak{C}_2 = f_1^2 + f_2 + f_3\,, \quad \mathfrak{C}_3 = 3f_1(f_2 - \frac{1}{2}f_3) - \frac{3\sqrt{3}}{4}f_4 - f_1^3. \tag{29}$$

Since we are interested in the projection of the orbit space $\mathfrak{P}/\mathrm{SU}(2)\times\mathrm{U}(1)$ to the space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$, it is constructive to use relations (29) and to build an integrity basis that contains $\mathfrak{C}_2$ and $\mathfrak{C}_3$ as elements of the second and third degree:

$$\mathcal{P}^{\mathrm{SU}(2)\times\mathrm{U}(1)} := \{f_1,\ f_2,\ \mathfrak{C}_2,\ \mathfrak{C}_3\}\,.$$

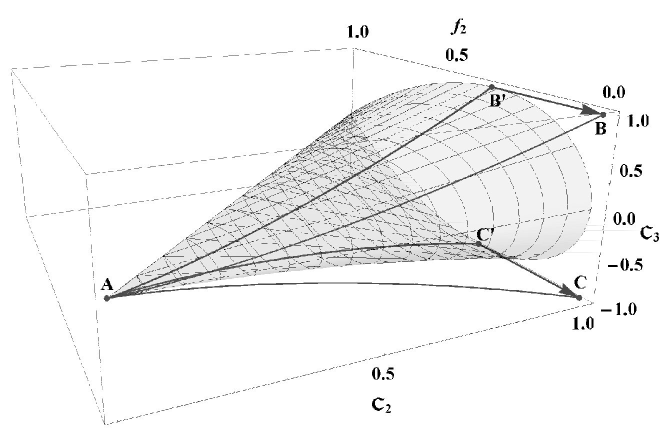


Fig. 2. The domain $\mathrm{Grad}_{\mathrm{SU}(2)\times\mathrm{U}(1)} \geq 0$ and its projection to the $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane for $f_1 = 0$.

Let us write the 4×4 Grad-matrix for the integrity basis $\{f_1, f_2, \mathfrak{C}_2, \mathfrak{C}_3\}$ in block form:

$$\mathrm{Grad}_{\mathrm{SU}(2)\times \mathrm{U}(1)} = \begin{pmatrix} \mathcal{A}, & \mathcal{B} \\ \mathcal{B}^T, & \mathcal{D} \end{pmatrix}. \tag{30}$$

Here $\mathcal{A} := \mathrm{diag}(1, 4f_2)$, $\mathcal{D}$ is the 2×2 diagonal matrix that coincides with the Grad-matrix (28), and

$$\mathcal{B} := \begin{pmatrix} 2f_1, & \frac{3}{2}(3f_2 - f_1^2 - \mathfrak{C}_2) \\ 4f_2, & 3f_1(f_2 + \mathfrak{C}_2) + 2\mathfrak{C}_3 - f_1^3 \end{pmatrix}. \tag{31}$$

It is easy to see that the semipositivity of the matrix (30) reduces to the nonnegativity condition for its determinant:

$$\det \|\mathrm{Grad}_{\mathrm{SU}(2)\times \mathrm{U}(1)}\| \geq 0. \tag{32}$$

Furthermore, from the expression

$$\begin{aligned} \det \|\mathrm{Grad}_{\mathrm{SU}(2)\times \mathrm{U}(1)}\| &= 4\left(\mathfrak{C}_2 + 3f_2 - f_1^2\right) \\ &\times \Big[- 9f_1^2\left(\mathfrak{C}_2^2 + 3f_2^2\right) - 12\mathfrak{C}_3 f_1(\mathfrak{C}_2 - 3f_2) + 3f_1^4(2\mathfrak{C}_2 + 3f_2) \\ &\quad + 27f_2(\mathfrak{C}_2 - f_2)^2 - 4\mathfrak{C}_3^2 + 4\mathfrak{C}_3 f_1^3 - f_1^6\Big]. \end{aligned} \tag{33}$$

it follows that the nonnegativity domain of the Grad-matrix is the four-dimensional body bounded by two three-dimensional hypersurfaces that we denote by Σ_+ and Σ_-. An explicit parametrization of $\Sigma_\pm$ can be found by solving the equation

$$\begin{aligned} &- 9f_1^2\left(\mathfrak{C}_2^2 + 3f_2^2\right) - 12\mathfrak{C}_3 f_1\big(\mathfrak{C}_2 - 3f_2 - \frac{1}{3}f_1^2\big) \\ &+ 3f_1^4(2\mathfrak{C}_2 + 3f_2) + 27f_2(\mathfrak{C}_2 - f_2)^2 - 4\mathfrak{C}_3^2 - f_1^6 = 0 \end{aligned} \tag{34}$$

with respect to $\mathfrak{C}_3$. Therefore, the hypersurfaces $\Sigma_\pm$ are given by the equations

$$\mathfrak{C}_3 = \frac{3}{2}\left(f_1(3f_2 - \mathfrak{C}_2) + \frac{f_1^3}{3} \mp \sqrt{3f_2}\left(-\mathfrak{C}_2 + f_2 + f_1^2\right)\right). \tag{35}$$

According to (35), the hyperfaces Σ_+ and Σ_- intersect if

$$\sqrt{3f_2}\left(f_2 + f_1^2 - \mathfrak{C}_2\right) = 0. \tag{36}$$

Thus the hypersurfaces $\Sigma_\pm$ intersect along the following two-dimensional surfaces Δ_1 and Δ_2:

(1) the surface Δ_1:

$$f_2 = 0\,, \qquad \mathfrak{C}_3 = \frac{3}{2}f_1\left(\frac{f_1^2}{3} - \mathfrak{C}_2\right), \tag{37}$$

(2) the surface Δ_2:

$$f_2 + f_1^2 - \mathfrak{C}_2 = 0\,, \qquad \mathfrak{C}_3 = 3f_1\left(\mathfrak{C}_2 - \frac{4}{3}f_1^2\right). \tag{38}$$

To make the description of the orbit space more transparent, consider its three-dimensional cross sections for different values of the "local" invariant f_1.

• $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ **for** $f_1 = 0$**.** Figure 2 shows the semipositivity domain for the Grad-matrix in the space $(f_2, \mathfrak{C}_2, \mathfrak{C}_3)$. The three-dimensional slice of the "local" orbit space fixed by the local invariant $f_1 = 0$ is shown in Fig. 7 (middle). From this picture one can see that the projection of the "semipositivity cone" of the Grad-matrix to the $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane exactly reproduces the ABC triangle, the orbit space $\mathfrak{P}(\mathbb{R}^8)/\mathrm{SU}(3)$ depicted in Fig. 1.

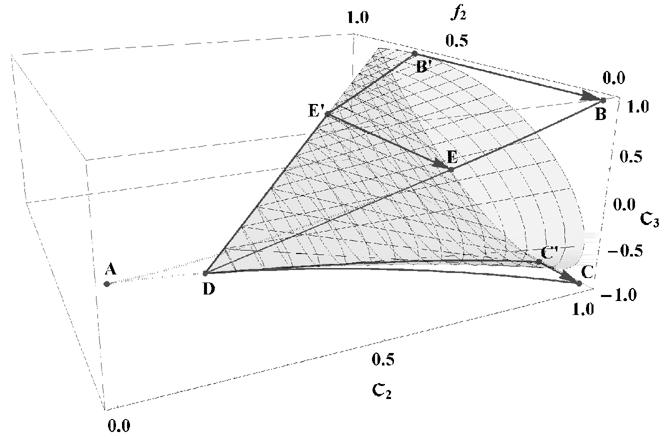


Fig. 3. The domain $\mathrm{Grad}_{\mathrm{SU}(2)\times\mathrm{U}(1)} \geq 0$ and its projection to the $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane for $f_1 = 2/5$.

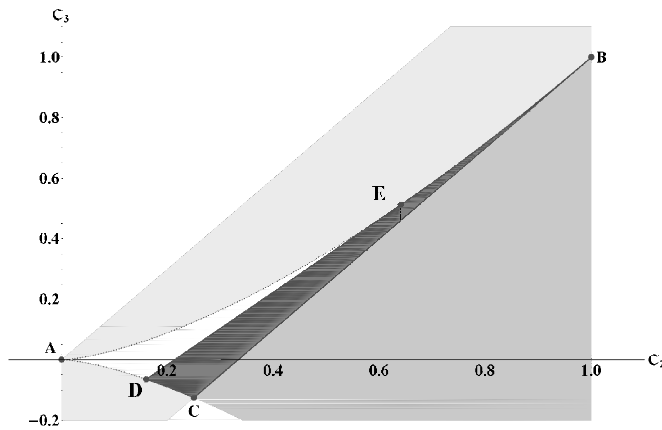


Fig. 4. DCBE is the image of $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ on the SU(3) orbit space for $f_1 = 2/5$.

For nonvanishing values of f_1, the attainable area of the Casimir invariants $(\mathfrak{C}_2,\ \mathfrak{C}_3)$ is shrinking. To illustrate this effect, below we give the corresponding pictures for a positive, $f_1 = 2/5$, and a negative, $f_1 = -2/5$, value of the invariant f_1.

• $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ **for** $f_1 = 2/5$**.** For this value, the "semipositivity cone" is shown in Fig. 3. For nonzero values of f_1, the vertex of the "semipositivity cone" intersects the $(\mathfrak{C}_2,\ \mathfrak{C}_3)$-plane of Casimir invariants at the point D which is different from A. The line DE is the projection of the surface Δ_2 with $f_1 = 2/5$. As f_1 grows, the line DE moves towards BC, and for $f_1 = 1/2$ covers it. To make a more vivid illustration, the shrinking area of allowed SU(3) Casimir invariants is shown in Fig. 4.

When the "local" invariant f_1 lies in the interval $(0, -1]$, an alternative mechanism of shrinking of the triangle ABC works.

• $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ **for** $f_1 = -2/5$**.** For this case, the "semipositivity cone" is depicted in Fig. 5. When f_1 takes negative values, the points D and E move toward the point B, and all of them coincide for $f_1 = -1$.

Figure 6 exemplifies the effect of shrinking of the domain of allowed SU(3) Casimir invariants for the negative value $f = -2/5$.

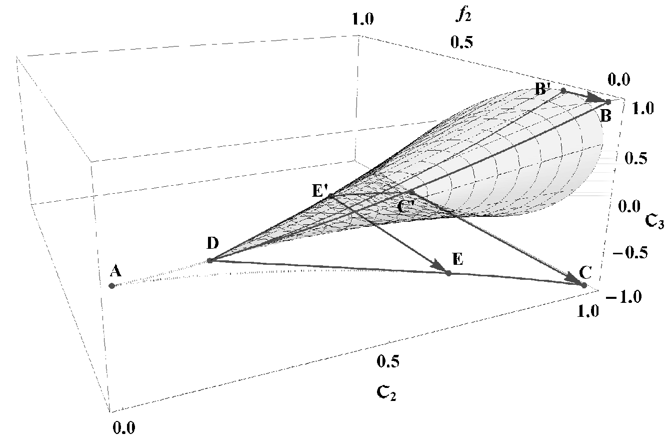


Fig. 5. The domain $\mathrm{Grad}_{\mathrm{SU}(2)\times\mathrm{U}(1)} \geq 0$ and its projection to the $(\mathfrak{C}_2, \mathfrak{C}_3)$-plane for $f_1 = -2/5$.

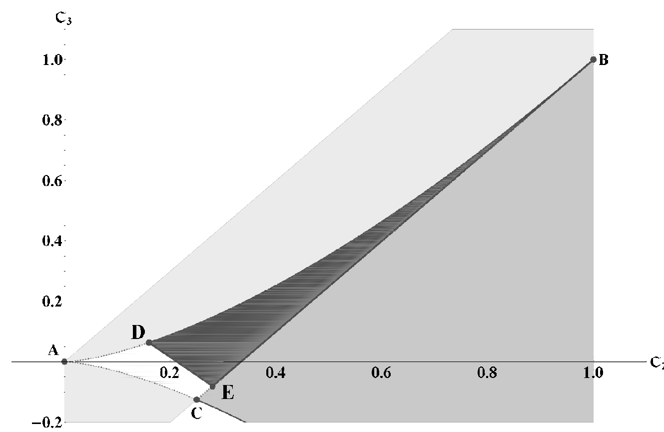
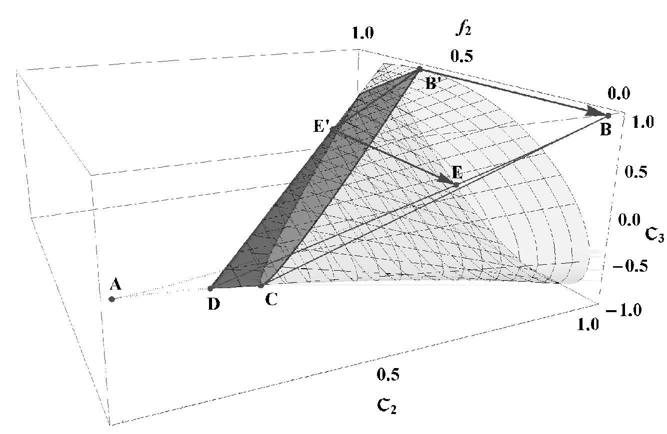


Fig. 6. DCBE is the image of $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ on the SU(3) orbit space for $f_1 = -2/5$.

Finally, the three-dimensional slices of the orbit space $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ for different values of f_1 are presented in Fig. 7.



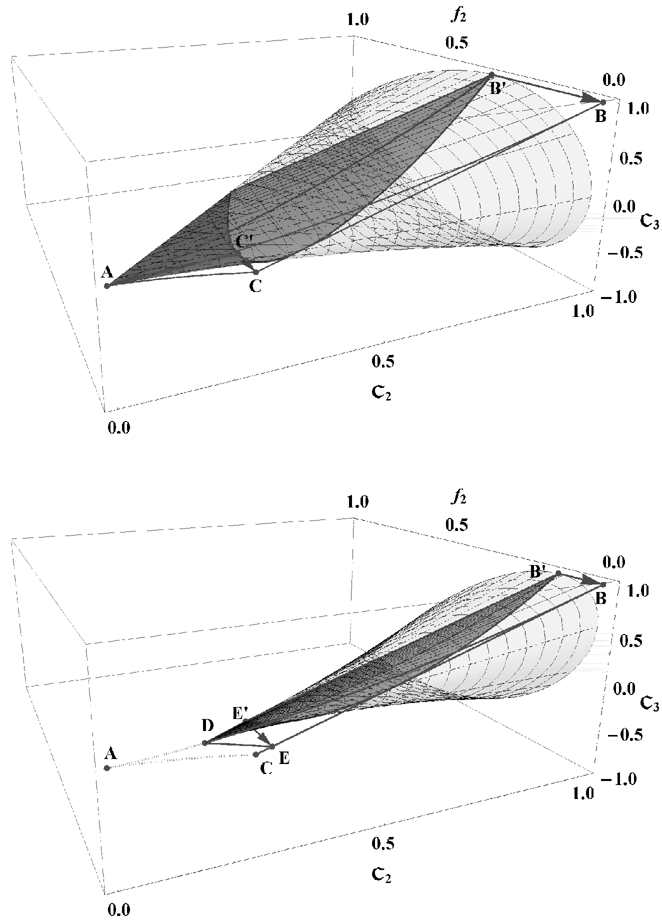


Fig. 7. $\mathfrak{P}/\mathrm{SU}(2) \times \mathrm{U}(1)$ slices for $f_1 = 2/5$ (top), $f_1 = 0$, and $f_1 = -2/5$ (bottom).

6. Conclusion

In the present note, we analyze the SU(2)×U(1) orbit space of the qutrit treating it as a simplified analog of the entanglement space of a composite system. The orbit space is described as a semialgebraic variety in $\mathbb{R}^4$ defined by a set of polynomial inequalities in SU(2)×U(1) adjoint invariants. These inequalities follow from the simultaneous semipositivity of two matrices, the qutrit density matrix and the Procesi–Schwarz Grad-matrix, constructed from a fundamental set of SU(2)× U(1) invariants. We discuss in detail how the semipositivity of the Grad-matrix for SU(2)× U(1) invariants provides new constraints on the geometry of the orbit space, in contrast to the case of the SU(3) orbit space.

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REFERENCES

1. C. Procesi and G. Schwarz, “The geometry of orbit spaces and gauge symmetry breaking in supersymmetric gauge theories,” *Phys. Lett. B*, **161**, 117–121 (1985).

2. C. Procesi and G. Schwarz, "Inequalities defining orbit spaces," *Invent. Math.*, **81**, 539–554 (1985).
3. V. Gerdt, A. Khvedelidze, and Yu. Palii, "Describing the orbit space of the global unitary actions for mixed qudit states," *J. Math. Sci.*, **200**, No. 6 (2014), 682–689; `arXiv:1311.4649`.
4. E. Vinberg and V. Popov, "Invariant theory," in: *Algebraic Geometry*-4, Itogi Nauki i Tekhniki, Ser. Sovrem. Probl. Mat. Fund. Napr., **55**, VINITI, Moscow (1989), pp. 137–309.
5. M. Forger, "Invariant polynomials and Molien functions," *J. Math. Phys.*, **39**, No. 2, 1107–1141 (1998).
6. C. Procesi, *Lie Groups. An Approach Through Invariants and Representations*, Springer, New York (2007).
7. D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, 3rd edition, Springer, New York (2007).
8. L. Michel and L. A. Radicati, "The geometry of the octet," *Ann. Inst. Henri Poincaré Nouv. Ser. Sect. A*, **18**, 185–214 (1973).
9. V. Gerdt, A. Khvedelidze, and Yu. Palii, "On the ring of local polynomial invariants for a pair of entangled qubits," *J. Math. Sci.*, **168**, No. 3 (2010), 368–378; `arXiv:1007.0968`.