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О ПРОБЛЕМЕ СЕПАРАБЕЛЬНОСТИ СОСТАВНЫХ КВАНТОВЫХ СИСТЕМ

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Данная статья посвящена изучению задачи "квантовой сепарабельности" – математической проблеме, которая лежит в основе квантовой информатики и теории квантовых коммуникаций. Проблема сепарабельности состоит в разработке эффективных алгоритмов определения возможности представления вектора состояния составной квантовой системы в виде произведения состояний, описывающих ее подсистемы. В работе обсуждаются теоретико-вероятностные аспекты данной проблемы и приведены результаты расчетов геометрической вероятности сепарабельности/перепутанности состояний квантовых систем, состоящих из двух кубитов и пары кубит-кутрита.

Ключевые слова: квантовая информация, перепутанность, случайные матрицы, статистическая мера.

ON THE SEPARABILITY PROBLEM FOR QUANTUM COMPOSITE SYSTEMS

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The present article addresses the so-called "quantum separability problem", the mathematical issue that lies in foundations of quantum information and communication theory. The separability problem consist in elaboration of efficient computational algorithms for determination of whether a given state of a composite quantum system admits representation in a product form, with factors corresponding to each subsystem. The measurement theoretical aspects of this problem are discussed and the geometric probability of the mixed separable/entangled states in quantum systems composed from 2-qubits and qubit-qutrit pairs are computed.

Key words: quantum information, entanglement, random matrices, statistical measures.

1. Introduction

The detailed estimation of "quantumness resource" for a given quantum system is the basic issue in theory of processing with quantum information and communications. This problem is in a strong interplay with the fundamental property of quantum word, existence for composite systems the so-called, entangled states [1][1]. It turns that, more the entangled states a system has, more quantum weirdness it exposes. In order to qualitatively describe the "quantumness" of a given system it is reasonable to "count" the entangled states. From the measurement theoretical point of view, (cf. [2-4]) "counting" of the entangled states corresponds to the determination of a relative volume of the entangled states with respect to all possible states. This number defines the geometric probability of the entangled states [5].

In other words, the geometric separability /entanglement probability is the probability for a positive definite Hermitian matrix, distributed in accordance with a measure defined on the space of states, to be the separable/entangled matrix. This is what we call the theoretical measurement formulation of quantum separability problem. The separability problem is a subject of intense current research. It is considered difficult, and computationally has been shown to be NP-hard (cf. [6-8]). In the present note we intend to discuss the measurement theoretical aspect of this problem only and to give the results of our calculations for the geometric probability of mixed separable/entangled states for systems of 2-qubits and qubit-qutrit pairs.

The article is organized a follows. From the beginning ingredients necessary for computations are introduced. In the section I the method of generation of random density matrices, distributed in accordance with the Hilbert-Schmidt and Bures metric defined on the space of states of a finite-dimensional quantum system, is sketched. Next section is devoted to the statement of algorithms for selecting the separable matrices among all randomly generated matrices. Concluding, the results of our numerical calculations for the geometric probabilities of binary systems composed from two qubits and qubit-qutrit pairs will be given and briefly commented.

2. Generating random matrices

Here, adopting the method of the so-called induced measures (cf. [9,10]), the generation of random density matrices distributed according to the Hilbert-Schmidt and Bures probability measures are described.

According to [9,10], the starting position for the generation procedure consist in the usage of the well-known Ginibre ensemble of random matrices [11]. The Ginibre ensemble is defined as follows. Let $\mathrm{M}(C,n)$ is the space of $n\times n$ matrices whose entries are complex numbers and the conventional linear measure on $\mathrm{M}(C,n)$ is fixed. Assume that the elements of an arbitrary matrix $Z\in\mathrm{M}(C,n)$ are independent identically distributed standard normal complex random variables

$$p(z_{ij})=\frac{1}{\pi}\exp(-|z_{ij}|),\qquad i,j=1,2,\ldots,n.$$

Using the joint probability distribution

[1] According to the definition a composite quantum system is in classically correlated/separable state if the later represents a convex combination of the product states. A state that is not separable is said to be the entangled.

$$P(Z) = \prod_{i,j=1}^{n} p(z_{ij}) = \frac{1}{\pi^{n^2}} \exp\left(-\mathrm{Tr}\left(Z^{\dagger} Z\right)\right) \tag{1}$$

the Ginibre's measure for probability distribution is given as:

$$d\mu_{\mathrm{G}}(Z) = P(z)\mathrm{Tr}\left(dZ^{\dagger} dZ\right). \tag{2}$$

Having the random Ginibre matrices the generation of elements from both the Hilbert-Schmidt and the Bures ensembles becomes easily achievable applying the following simple algorithms.

The Hilbert-Schmidt ensemble. In order to generate the Hilbert-Schmidt states,

$$P(\varrho)_{\mathrm{HS}} \approx \Theta(\varrho)\delta(1-\varrho), \tag{3}$$

consider a square $n \times n$ complex random matrix Z from the Ginibre ensemble. Introducing, for given Z, the matrix

$$\varrho_{\mathrm{HS}} = \frac{Z^{\dagger} Z}{\mathrm{Tr}\left(Z^{\dagger} Z\right)}, \tag{4}$$

it is easy to convinced that ϱ by construction is Hermitian, positive definite matrix with a unit norm. Furthermore, when matrices Z are generated in Ginibre form, the matrices ϱ represent elements from the Hilbert-Schmidt ensemble (3).

The Bures ensemble. The Bures measure [12] originates from the statistical distance between quantum states [13] and can be derived from the following metric corresponding to the infinitesimal form of the quantum fidelity between states:

$$ds_{\mathrm{B}}^2 = \frac{1}{2}\mathrm{Tr}\left(G \mathrm{d}\varrho\right), \tag{5}$$

where, unknown G is subject to the equation $d\varrho = G\varrho + \varrho G$. The density matrix distributed in accordance with the Bures measure can be generated as follows [10]. Consider the random matrix of the form:

$$\varrho_{\mathrm{B}} = \frac{(\mathbb{I}+U)ZZ^{+}(\mathbb{I}+U^{+})}{\mathrm{Tr}\left[(\mathbb{I}+U)ZZ^{+}(\mathbb{I}+U^{+})\right]}, \tag{6}$$

where complex matrix Z belongs to the Ginibre ensemble, while U is an unitary matrix, distributed according to the Haar measure on the unitary group $U(n)$. It can be shown that the probability distribution for matrices ϱ_{B} coincides with the Bures one [10].

3. Selecting the separable matrices

Now we address the question how to find the separable density matrices among elements from the generated ensembles. The complete answer to this question for a generic case of an arbitrary $n \times n$ matrices is unknown. However, for a binary system, composed from two qubits ($2 \otimes 2$) and qubit-qutrit pairs ($2 \otimes 3$), there exist well-established criterion.

The separability criterion. Perhaps the most useful tool for qualifying separability is the famous Peres-Horodecki criterion [14,15], which is based on the idea of the partial transposition. The partial transpose ρ^{T_B} of a density matrix ϱ for the binary system ($A \otimes B$) with respect to the second subsystem B is defined as

$$\rho^{T_B} = I \otimes T\varrho, \tag{7}$$

where T stands for the standard transposition operation in the subsystem B. According to the Peres-Horodecki a given state ϱ in dimensions $2 \otimes 2$ and $2 \otimes 3$ are separable if its partially transposed matrix is semi-positive and only then. Unfortunately, this criterion is not universal. For higher dimensions, there are entangled states with a positive partial transpose (PPT), e.g., even for binary $3 \otimes 3$ system one can find the counterexample for the Peres-Horodecki criterion. Having in mind that ,for systems we are interested in, the Peres-Horodecki criterion is applicable, a search for separable matrices among to the generated density matrices reduces to the checking of the positivity of their partial transpositions.

Positivity of the density matrices. Positive semi-definiteness of the Hermitian $n \times n$ matrix ϱ implies non-negativity of its eigenvalues:

$$x_k \geq 0, \qquad k = 1, 2, \ldots, n. \tag{8}$$

Since the eigenvalues $\{x\}$ are non-polynomial functions of elements of the density matrix ϱ the usage of inequalities (8) is not helpful computationally. Fortunately, for the Hermitian matrices the inequalities (8) are equivalent to the non-negativity of the first n-symmetric polynomials in eigenvalues, the coefficients of characteristic equation for the matrix ϱ

$$S_k \geq 0, \qquad k = 1, 2, \ldots, n. \tag{9}$$

The coefficients S_k, being the polynomial functions of density matrix, are expressible in terms of the traces of powers of the density matrix $t_k = \mathrm{Tr}(\rho^k)$,

$$S_k = \frac{1}{k!}\begin{pmatrix} t_1 & 1 & 0 & \cdots & 0 \\ t_2 & t_1 & 2 & \cdots & 1 \\ t_3 & t_2 & t_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & k-1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & t_1 \end{pmatrix}, \tag{10}$$

and therefore are more attractive from the computational point of view. For further details see e.g., [16,17] and references therein.

4. Geometric probability

Gathering all above preliminary information one can define the separability probabilit for a bipartite systems of 2-qubits or qubit-qutrit as

$$\mathcal{P}_{\text{sep}}(\mathfrak{P}_+ \cap \tilde{\mathfrak{P}}_+) = \frac{\int_{\mathfrak{P}_+ \cap \tilde{\mathfrak{P}}_+} d\mu}{\int_{\mathfrak{P}_+} d\mu}, \tag{11}$$

where the integrals in (11) are defined over the following spaces: $\mathfrak{P}_+$ is the total space of states, $\tilde{\mathfrak{P}}_+$ the image of $\mathfrak{P}_+$ under the partial transposition map $\mathrm{I} \otimes \mathrm{T}$. The intersection $\mathfrak{P}_+ \cap \tilde{\mathfrak{P}}_+$ represents the subset of $\mathfrak{P}_+$ invariant under the partial transposition map $\mathrm{I} \otimes \mathrm{T}$:

$$\mathfrak{P}_+ \cap \tilde{\mathfrak{P}}_+ = \{\rho \in \mathfrak{P}_+ | \mathrm{I} \otimes \mathrm{T}\rho \in \mathfrak{P}_+\} .$$

The measure $\mathrm{d}\mu$ in integrals (11) is determined by the Riemannian metrics defined on the space of density matrices. Noting, that the volume of space of states in terms of both Hilbert-Schmidt metric [18] and Bures metric [19] is known, the problem of determination of separability probability reduces to the evaluation of the integral over the set $\mathfrak{P}_+ \cap \tilde{\mathfrak{P}}_+$.

5. Results and concluding comments

Even from the first glance it is clear that a straightforward calculation of the multidimensional integral over the set $\mathfrak{P}_+ \cap \tilde{\mathfrak{P}}_+$ is not a simple task. To avoid very cumbersome computations it is instructive to proceed with numerical methods, adopting the Monte-Carlo ideology.

Generating random density matrices, distributing according a certain measure, and then counting the number of matrices satisfying the PPT conditions:

$$S_k^{T_B} \geq 0, \qquad k = 1, 2, \ldots, n, \tag{12}$$

we will determine the separability probability. The results of our numeric experiments are listed in the Table 1, where the fractional approximations for the probabilities are given in the last column.

Table 1. Probabilities for 2-qubits and qubit-qutrit pairs.

Quantum System	Separable	Entangled	Rational	Primes
Hilbert-Schmidt metric				
$2 \otimes 2$	24.24%	75.76%	$\frac{8}{33}$	$\frac{2^3}{3*11}$
$2 \otimes 3$	3.73%	96.27%	$\frac{16}{429}$	$\frac{2^4}{3*11*13}$
Bures metric				
$2 \otimes 2$	7.3%	92.7%	$\frac{799}{10843}$	$\frac{799}{7*1549}$
$2 \otimes 3$	0.1%	99.9%	$\frac{79}{63499}$	$\frac{79}{11*13*443}$

Concluding it is necessary to note, that our numerical computations strongly supports the fractional value $8/33$ for qubits pairs separability probability, that had been conjectured by P.B. Slater few years ago [20]. However, a rigorous analytical derivation of this result, and other simple rational values, given in the Table 1, remains an interesting unsolved yet problem.

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