Journal of Mathematical Sciences, Vol. 200, No. 6, August, 2014

DESCRIBING THE ORBIT SPACE OF THE GLOBAL UNITARY ACTIONS FOR MIXED QUDIT STATES

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The unitary $\mathrm{U}(d)$*-equivalence relation on the space* $\mathfrak{P}_+$ *of mixed states of a* d*-dimensional quantum system defines the orbit space* $\mathfrak{P}_+/\mathrm{U}(d)$ *and provides its description in terms of the ring* $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(d)}$ *of* $\mathrm{U}(d)$*-invariant polynomials. We prove that the semi-algebraic structure of* $\mathfrak{P}_+/\mathrm{U}(d)$ *is completely determined by two basic properties of density matrices, their semi-positivity and Hermiticity. In particular, it is shown that the Procesi–Schwarz inequalities in the elements of the integrity basis for* $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(d)}$ *defining the orbit space are identically satisfied for all elements of* $\mathfrak{P}_+$*. Bibliography: 9 titles.*

1. Introduction

The basic symmetry of isolated quantum systems is the unitary invariance. It determines equivalence relations between the states and defines the physically relevant quotient space. For composite systems, the implementation of this symmetry has very specific features leading to such a nontivial phenomenon as the entanglement of quantum states.

The space $\mathfrak{P}_+$ of mixed states of a d-dimensional binary quantum system is locus in quo for two unitary actions of groups: the group $\mathrm{U}(d)$ and the tensor product group $\mathrm{U}(d_1) \otimes \mathrm{U}(d_2)$, where d_1, d_2 stand for the dimensions of subsystems, $d = d_1 d_2$. Both groups act on a state $\varrho \in \mathfrak{P}_+$ in the adjoint manner:

$$(\mathrm{Ad}\, g\,)\varrho = g\,\varrho\, g^{-1}. \tag{1}$$

As a result of this action, one can consider two equivalence classes of ϱ: the *global* $\mathrm{U}(d)$-orbit and the *local* $\mathrm{U}(d_1) \otimes \mathrm{U}(d_2)$-orbit. The collection of all $\mathrm{U}(d)$-orbits, together with the quotient topology and differentiable structure, defines the "global orbit space," $\mathfrak{P}_+/\mathrm{U}(d)$, while the orbit space $\mathfrak{P}_+/\mathrm{U}(d_1) \otimes \mathrm{U}(d_2)$ represents the "local orbit space," or the so-called *entanglement space* $\mathcal{E}_{d_1 \times d_2}$. The latter space is a stage for manifestations of the intriguing effects occurring in quantum information processing and communications.

Both orbit spaces admit representations in terms of the elements of an integrity basis for the corresponding ring of G-invariant polynomials, where G is either $\mathrm{U}(d)$ or $\mathrm{U}(d_1)\otimes\mathrm{U}(d_2)$. They can be obtained by implementing the Procesi–Schwarz method, introduced in the 80s of the last century for describing the orbit space of an action of a compact Lie group on a linear space [1,2]. According to Procesi and Schwarz, the orbit space is identified with the semi-algebraic variety defined by the syzygy ideal for the integrity basis and the semi-positivity condition $\mathrm{Grad}(z) \geq 0$ for a certain matrix, the so-called "gradient matrix," which is constructed from the elements of the integrity basis. In the present note, we address the problem of applying this generic approach to the construction of $\mathfrak{P}_+/\mathrm{U}(d)$ and $\mathfrak{P}_+/\mathrm{U}(d_1)\otimes\mathrm{U}(d_2)$. Namely, we study whether the semi-positivity of the Grad matrix introduces new conditions on the elements of the integrity basis for the corresponding ring $\mathbb{R}[\mathfrak{P}_+]^G$. Below it will be shown that for the global unitary invariance, $G = \mathrm{U}(d)$, the semi-algebraic structure of the orbit space is determined solely from the physical conditions on density matrices, their semi-positivity and Hermiticity. The conditions $\mathrm{Grad}(z) \geq 0$ do not bring new restrictions on the elements of the integrity basis for $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(d)}$. In contrast to this case, for the local symmetries, the Procesi–Schwarz inequalities affect the algebraic and geometric properties of the entanglement space.

Our presentation is organized as follows. In Sec. 2, the Procesi–Schwarz method is briefly stated in the form applicable to the analysis of the adjoint unitary action on the space of states. In Sec. 3, the semi-algebraic structure of $\mathfrak{P}_+/\mathrm{U}(d)$ is discussed. The final section is devoted to a detailed consideration of two examples, the orbit space of a qutrit ($d = 3$) and the global orbit space of a four-level quantum system ($d = 4$).

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Published in *Zapiski Nauchnykh Seminarov POMI*, Vol. 421, 2014, pp. 68–80. Original article submitted November 12, 2013.

 1072-3374/14/2006-0682

2. The Procesi–Schwarz method

Here we briefly state the above-mentioned method for constructing the orbit space elaborated by Procesi and Schwarz for the case of an action of a compact Lie group on a linear space [1,2].

Consider a compact Lie group G acting linearly on the real d-dimensional vector space V. Let $\mathbb{R}[V]^G$ be the corresponding ring of G-invariant polynomials on V. Let $\mathcal{P} = (p_1, p_2, \ldots, p_q)$ be the set of homogeneous polynomials that form an integrity basis:

$$\mathbb{R}[x_1, x_2, \ldots, x_d]^G = \mathbb{R}[p_1, p_2, \ldots, p_q].$$

The elements of the integrity basis define the polynomial mapping

$$p: \quad V \to \mathbb{R}^q; \quad (x_1, x_2, \ldots, x_d) \to (p_1, p_2, \ldots, p_q). \tag{2}$$

Since p is constant on the orbits of G, it induces a homeomorphism of the orbit space V/G and the image X of the mapping p: $V/G \simeq X$, see [1,2]. In order to describe X in terms of $\mathcal{P}$ uniquely, it is necessary to take into account the *syzygy ideal* of $\mathcal{P}$, i.e.,

$$I_{\mathcal{P}} = \big\{h \in \mathbb{R}[y_1, y_2, \ldots, y_q] : h(p_1, p_2, \ldots, p_q) = 0\big\} \subseteq \mathbb{R}[V].$$

Let $Z \subseteq \mathbb{R}^q$ denote the locus of common zeros of all elements of $I_{\mathcal{P}}$; then Z is an affine variety in $\mathbb{R}^q$ such that $X \subseteq Z$. Denote by $\mathbb{R}[Z]$ the *coordinate ring* of Z, that is, the ring of polynomial functions on Z. Then the following isomorphism takes place [3]:

$$\mathbb{R}[Z] \simeq \mathbb{R}[y_1, y_2, \ldots, y_q]/I_{\mathcal{P}} \simeq \mathbb{R}[V]^G.$$

Therefore, the subset Z is essentially determined by $\mathbb{R}[V]^G$, but in order to describe X, further steps are required. According to [1,2], the necessary information about X is encoded in the semi-positivity of the $q \times q$ matrix with elements given by the inner products of the gradients $\operatorname{grad}(p_i)$:

$$\|\mathrm{Grad}\|_{ij} = \big(\operatorname{grad}(p_i), \operatorname{grad}(p_j)\big).$$

Briefly summarizing all the above, the G-orbit space can be identified with the semi-algebraic variety defined as the set of points satisfying the following two conditions:

(a) $z \in Z$, where Z is the surface defined by the syzygy ideal for the integrity basis of $\mathbb{R}[V]^{\mathrm{G}}$;
(b) $\mathrm{Grad}(z) \geq 0$.

With these basic facts in mind, one can pass to the construction of the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$. First we describe the generic semi-algebraic structure and then exemplify it by considering two simple three- and four-level quantum systems.

3. The semi-algebraic structure of $\mathfrak{P}_+/\mathrm{U}(d)$

The first step making the Procesi–Schwarz method applicable to the case we are interested in consists in the linearization of the adjoint $\mathrm{U}(d)$-action (1). For a unitary action, one can achieve this as follows. Consider the space $\mathcal{H}_{d\times d}$ of $d \times d$ Hermitian matrices and the mapping

$$\mathcal{H}_{d\times d} \to \mathbb{R}^{d^2};$$
$$\varrho_{11} = v_1, \quad \varrho_{12} = v_2, \ldots, \varrho_{1d} = v_d, \quad \varrho_{21} = v_{d+1} \ldots, \quad \varrho_{dd} = v_{d^2}.$$

Then it can easily be verified that the linear representation on $\mathbb{R}^{d^2}$ defined by

$$\boldsymbol{v}' = L\boldsymbol{v}\,, \quad L \in \mathrm{U}(d) \otimes \overline{\mathrm{U}(d)}\,,$$

where the bar stands for the complex conjugation, is isomorphic to the initial adjoint $\mathrm{U}(d)$-action (1).

Now, for the mapping (2), we need a corresponding integrity basis $\mathcal{P} = (p_1, p_2, \ldots, p_q)$ for the ring of invariant polynomials. To construct it, the following observation is in order. Starting from the center $\mathcal{Z}(\mathfrak{su}(d))$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{su}(d))$, according to the well-known Gelfand theorem, one can define an isomorphic commutative symmetrized algebra of invariants $S(\mathfrak{su}(d))$, which is in turn isomorphic to the algebra of invariant polynomials over $\mathfrak{su}(d)$, see [4]. The latter provides a needed source of coordinates that can be used to parameterize the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$. For our purposes, it is convenient to choose the integrity basis formed by the so-called trace invariants. Namely, below we use the polynomial ring $\mathbb{R}[v_1, v_2, \ldots, v_{d^2}]^{\mathrm{U}(d)} = \mathbb{R}[t_1, t_2, \ldots, t_d]$ with n basis elements

$$t_k = \operatorname{tr}(\varrho^k), \quad k = 1, 2. \ldots, d. \tag{3}$$

In terms of the integrity basis (3), the Grad matrix reads

$$\mathrm{Grad}(t_1, t_2, \ldots, t_d) = \begin{pmatrix} d & 2t_1 & 3t_2 & \cdots & dt_{d-1} \\ 2t_1 & 2^2 t_2 & 2\cdot 3t_3 & \cdots & 2\cdot dt_d \\ 3t_2 & 2\cdot 3t_3 & 3^2 t_4 & \cdots & 3\cdot dt_{d+1} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \\ dt_{d-1} & 2\cdot dt_d & 3\cdot dt_{d+1} & \cdots & d^2 t_{2d-2} \end{pmatrix}. \tag{4}$$

In (4), the polynomials t_k with $k > d$ are expressed as polynomials in $(t_1, t_2, \ldots, t_d)$. From (4) one can easily obtain that

$$\mathrm{Grad}(t_1, t_2, \ldots, t_d) = \chi \mathrm{Disc}(t_1, t_2, \ldots, t_d)\chi^T, \tag{5}$$

where $\chi = (1, 2, \ldots, d)$ and $\mathrm{Disc}(t_1, t_2, \ldots, t_d)$ denotes the matrix

$$\mathrm{Disc}(t_1, t_2, \ldots, t_d) = \begin{pmatrix} d & t_1 & t_2 & \cdots & t_{d-1} \\ t_1 & t_2 & t_3 & \cdots & t_d \\ t_2 & t_3 & t_4 & \cdots & t_{d+1} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \\ t_{d-1} & t_d & t_{d+1} & \cdots & t_{2d-2} \end{pmatrix}. \tag{6}$$

In turn, the matrix (6) can be written as the "square" $\mathrm{Disc}\,(t_1, t_2, \ldots, t_d) = \Delta\Delta^T$ of the Vandermonde matrix

$$\Delta(x_1, \ldots, x_d) = \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{d-1} \\ 1 & x_3 & x_3^2 & \ldots & x_3^{d-1} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \\ 1 & x_d & x_d^2 & \ldots & x_d^{d-1} \end{pmatrix}, \tag{7}$$

whose columns are determined by the powers of the roots $(x_1, x_2, \ldots, x_d)$ of the characteristic equation

$$\det \|x - \varrho\| = x^d - S_1 x^{d-1} + S_2 x^{d-2} - \cdots + (-1)^d S_d = 0. \tag{8}$$

The semi-positivity condition for the matrix (6) guarantees that the roots of (8) are real. Thus the semi-positivity of the Grad matrix is equivalent to the condition that the eigenvalues of the density matrix ϱ written in terms of the U(d) polynomial scalars are real. Finally, noting that, by construction, the density matrices are Hermitian, we see that the Procesi–Schwarz inequalities are satisfied identically on $\mathfrak{P}_+$.

Summarizing, the algebraic structure of the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$ is completely determined by the inequalities in the elements of the integrity basis for the polynomial ring $\mathbb{R}[t_1, t_2, \ldots, t_d]$ originating from the Hermiticity and semi-positivity conditions on the density matrices.

4. Two examples

The algebraic structure of the orbit space of a quantum system is highly intricate. The examples of $d = 3$ (qutrit) and $d = 4$ considered below demonstrate the degree of its complexity even for low-dimensional systems.

4.1. The orbit space of a qutrit. A qutrit is a 3-dimensional quantum system, and an integrity basis for the ring of U(3)-invariant polynomials consists of the first-, second-, and third-order trace polynomials t_1, t_2, t_3. For illustrative purposes, below we consider the case of normalized density matrices, assuming that $t_1 = 1$.[1]

The condition for the eigenvalues to be real is

$$0 \leq \frac{1}{6}\left(3t_2^3 - 21t_2^2 + 36t_3t_2 + 9t_2 - 18t_3^2 - 8t_3 - 1\right), \tag{9}$$

[1]It is worth noting that the description of the qutrit orbit space is similar to the studies of the flavor symmetries of hadrons performed more than forty years ago by Michel and Radicati [5] (cf. the adaptation of the method to the analysis of the space of quantum states [6–8]).

while the semi-positivity of the density matrices, stated as the nonnegativity of the coefficients of the characteristic equation (8), reads

$$0 \le \frac{1}{2}(1 - t_2) \le \frac{1}{3}\,,$$
$$0 \le \frac{1}{6}(1 - 3t_2 + 2t_3) \le \frac{1}{9}\,.$$

Solving the inequalities

$$\frac{1}{3} \le t_2 \le 1,$$
$$\frac{3}{2}t_2 - \frac{1}{2} \le t_3 \le \frac{3}{2}t_2 - \frac{1}{6},$$
$$-4 + 18t_2 - \sqrt{2}(3t_2 - 1)^{3/2} \le 18t_3 \le -4 + 18t_2 + \sqrt{2}(3t_2 - 1)^{3/2},$$

we get the intersection domain shown in Fig. 1. The triangular domain ABC bounded by the lines

$$AB: \quad t_3 = \frac{1}{18}(-4 + 18t_2 + \sqrt{2}(3t_2 - 1)^{3/2}),$$
$$AC: \quad t_3 = \frac{1}{18}(-4 + 18t_2 - \sqrt{2}(3t_2 - 1)^{3/2}),$$
$$BC: \quad t_3 = \frac{3}{2}t_2 - \frac{1}{2},$$

with vertices[2] $A(\frac{1}{3}, \frac{1}{9})$, $B(1, 1)$, and $C(\frac{1}{2}, \frac{1}{4})$, represents the orbit space of the qutrit in the parametrization by the trace polynomial coordinates.

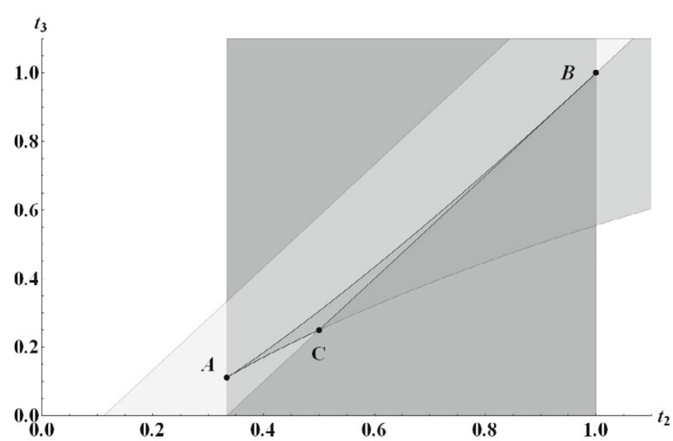

Fig. 1. The triangular domain ABC as the orbit space of the qutrit.

Now it is in order to discuss the correspondence between the above algebraic results and the known classification of orbits with respect to their stability group. With this issue in mind, consider the Bloch parametrization for the qutrit:

$$\rho = \frac{1}{3}\left(\mathbb{I}_3 + \sqrt{3}\,\boldsymbol{\xi} \cdot \boldsymbol{\lambda}\right), \tag{10}$$

[2]Note that the straight line BC is tangent to the curve AB at the point B:

$$\frac{dt_3}{dt_2} = 1 + \frac{1}{2\sqrt{2}}(3t_2 - 1)^{1/2}, \quad \frac{dt_3}{dt_2}\Big|_{t_2=1} = \frac{3}{2}\,.$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_8) \in \mathbb{R}^8$ denote the Bloch vector and $\boldsymbol{\lambda}$ is the vector whose components are the elements $(\lambda_1, \lambda_2, \cdots, \lambda_8)$ of a basis of the algebra $\mathfrak{su}(3)$, say the Gell–Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{11}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

obeying the relations

$$[\lambda_i, \lambda_j] = 2\imath f_{ijk} \lambda_k, \quad \mathrm{tr}\,(\lambda_i \lambda_j) = 2\delta_{ij}\,, \tag{12}$$

with nonvanishing structure constants

$$f_{123} = 2f_{147} = 2f_{246} = 2f_{257} = 2f_{345} = -2f_{156} = -2f_{367} = \frac{2}{\sqrt{3}} f_{458} = \frac{2}{\sqrt{3}} f_{678} = 1. \tag{13}$$

To analyze the adjoint orbit $\mathcal{O}_\varrho$ that passes through the point ϱ, we define the set of tangent vectors

$$l_i = \lim_{\theta_1, \theta_2, \ldots \theta_8 \to 0} \frac{\partial}{\partial \theta_i} \big[U\big(\theta_1, \theta_2, \ldots \theta_8\big)\, \varrho\, U\big(\theta_1, \theta_2, \ldots \theta_8\big) \big] = \imath [\lambda_i, \varrho]. \tag{14}$$

By definition, the dimension of the orbit $\dim(\mathcal{O}_\varrho)$ is given by the dimension of the tangent space to the orbit $T_{\mathcal{O}_\varrho}$ and, therefore, equals the number of linearly independent vectors among the eight tangent vectors $l_1, l_2, \ldots, l_8$. This number depends on the point ϱ and, according to a well-known theorem from linear algebra, is given by the rank of the so-called Gram matrix

$$A_{ij} = \frac{1}{2} \, \| \mathrm{tr}(l_i l_j) \|. \tag{15}$$

In the Bloch parametrization (10), we easily find that

$$A_{ij} = \frac{4}{3} \, f_{ims} f_{jns} \xi_m \xi_n\,. \tag{16}$$

To estimate the rank of the matrix (15), it is convenient to pass to the diagonal representative of the matrix ϱ:

$$\varrho = W \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} W^+\,, \tag{17}$$

where $W \in \mathrm{SU}(3)/S_3$ and the descending order of the eigenvalues is chosen:

$$1 \geq x_1 \geq x_2 \geq x_3 \geq 0.$$

The latter constraints allow one to avoid double counting due to the $S_3 \subset U(3)$ symmetry of permutations of eigenvalues of the density matrix. Using the principal axis transformation (17) and taking into account the adjoint properties of the Gell–Mann matrices $W^+ \lambda_i W = O_{ij} \lambda_j$ with $O \in SO(8)$, the matrix A_{ij} can be written as

$$A_{ij} = O_{ik} A^{\mathrm{diag}}_{kl} O^T_{lj}. \tag{18}$$

The matrix A^{diag}_{kl} in (18) is the matrix (15) constructed from the vectors $l^{\mathrm{diag}}_i = i[\lambda_i, \varrho_{\mathrm{diag}}]$ tangent to the orbit of the diagonal matrix $\varrho_{\mathrm{diag}} := \mathrm{diag}(x_1, x_2, x_3)$.

Since we are interested in determining $\mathrm{rank}|A|$, relation (18) allows us to reduce this question to the evaluation of the rank of the diagonal representative ϱ_{diag}. For diagonal matrices, the Bloch vector is $\boldsymbol{\xi}^{\mathrm{diag}} = (0, 0, 0, \xi_3, 0, 0, 0, \xi_8)$. Taking into account the values of the structure constants from (13), the expression for $|A^{\mathrm{diag}}|$ reads

$$A^{\mathrm{diag}} = \frac{1}{3} \, \mathrm{diag}\Big(4\xi_3^2, 4\xi_3^2, 0, (\xi_3 + \sqrt{3}\xi_8)^2, (\xi_3 + \sqrt{3}\xi_8)^2, (\xi_3 - \sqrt{3}\xi_8)^2, (\xi_3 - \sqrt{3}\xi_8)^2, 0 \Big). \tag{19}$$

From (19) we conclude that there are orbits of three different dimensions:

- orbits of maximal dimension $\dim(\mathcal{O}_\varrho) = 6$,
- orbits of dimension $\dim(\mathcal{O}_\varrho) = 4$,
- the zero-dimensional orbit, the single point $\boldsymbol{\xi} = 0$.

The above algebraic description of the orbits $\mathcal{O}_\varrho$ corresponds to their classification based on the analysis of the group of transformations G_ϱ, the isotropy group (or stability group) that stabilizes the point $\varrho \in \mathcal{O}_\varrho$. Orbits of different dimensions have different stability groups; for points lying on an orbit of maximal dimension, the stability group is the Cartan subgroup $\mathrm{U}(1) \otimes \mathrm{U}(1) \otimes \mathrm{U}(1)$, while the stability group of points with diagonal representative λ_8 is $\mathrm{U}(2) \otimes \mathrm{U}(1)$. The dimensions of the listed orbits agree with the general formula

$$\dim \mathcal{O}_\varrho = \dim G - \dim G_\varrho \,. \tag{20}$$

Since the isotropy groups of any two points on an orbit coincide up to conjugation, the orbits can be partitioned into sets with equivalent isotropy groups.[3] These sets are known as *"strata."*

In conclusion, we mention the relations between the triangle ABC depicted in Fig. 1 and the corresponding strata. The domain inside the triangle ABC corresponds to the principal stratum with stability group $U(1) \times U(1) \times U(1)$. The discriminant is positive, $|\mathrm{Disc}| > 0$, the density matrix has three distinct real eigenvalues, and the representative matrix reads $\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\,(\xi_3\lambda_3 + \xi_8\lambda_8))$ with ξ_3 and ξ_8 subject to the following constraints:

$$0 < 1 - \xi_3^2 - \xi_8^2 < 1,$$
$$0 < (2\xi_8 - 1)(1 - \sqrt{3}\xi_3 + \xi_8)(1 + \sqrt{3}\xi_3 + \xi_8) < 1.$$

The coefficient S_3 vanishes at the line BC. The boundary line BC except for the vertices B and C also belongs to the principal stratum, while the points B and C belong to a stratum of lower dimension. On the sides AB and AC the discriminant is zero, $|\mathrm{Disc}| = 0$, hence the density matrix has three real eigenvalues and two of them are equal. At the point B, two eigenvalues of ϱ vanish. The lines $AB \setminus \{A\}$ and $AC \setminus \{A\}$ represent the degenerate 4-dimensional orbits whose stability group is $U(2) \otimes U(1)$. Finally, the point A is the zero-dimensional stratum corresponding to the maximally mixed state $\varrho = \frac{1}{3}\mathbb{I}_3$. The details of the orbit types are collected in the table below.

$\dim \mathcal{O}$	Stratum	Stability group	Representative matrix	Constraints
6	The interior of the triangle ABC	$U(1) \otimes U(1) \otimes U(1)$	$\frac{1}{3}(\mathbb{I}_3+\sqrt{3}(\xi_3\lambda_3 + \xi_8\lambda_8))$	$\mathrm{Disc}>0$, $S_2>0$, $S_3>0$
	Boundary: $BC/\{B,C\}$	$U(1) \otimes U(1) \otimes U(1)$	$\frac{1}{3}(\mathbb{I}_3+\sqrt{3}(\xi_3\lambda_3 + \frac{1}{2}\lambda_8))$	$\mathrm{Disc}>0$, $S_2>0$, $S_3=0$
4	Boundary: $AB/\{A\}$ $AC/\{A\}$	$U(2) \otimes U(1)$	$\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\,\xi_8\lambda_8)$ $\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\xi_8(\sqrt{3}\lambda_3 + \lambda_8))$ $\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\xi_8(-\sqrt{3}\lambda_3 + \lambda_8))$	$\mathrm{Disc}=0$, $S_2\geq 0$, $S_3\geq 0$
0	Point: $\{A\}$	$U(3)$	$\frac{1}{3}\mathbb{I}_3$	$\mathrm{Disc} = S_2 = S_3 = 0$

Table 1. The strata decomposition for the orbit space of the qutrit.

[3] The isotropy group of a point ϱ depends only on the algebraic multiplicity of the eigenvalues of the matrix ϱ.

4.2. The orbit space of a four-level quantum system. The density matrix ϱ of a 4-level quantum system in the Bloch form reads

$$\rho = \frac{1}{4}\left(\mathbb{I}_4 + \sqrt{6}\,\vec{\xi}\cdot\vec{\lambda}\right)\,, \tag{21}$$

where the traceless part of ϱ is given by the inner product of the 15-dimensional Bloch vector $\vec{\xi} = \{\xi_1, \ldots, \xi_{15}\} \in \mathbb{R}^{15}$ with the λ-vector whose components are the elements of the Hermitian basis of the Lie algebra $\mathfrak{su}(4)$:

$$\lambda_i\lambda_j = \frac{1}{2}\,\delta_{ij}\mathbb{I}_4 + (d_{ijk} + i\,f_{ijk})\lambda_k, \quad i, j, k = 1, \ldots, 15.$$

The corresponding integrity basis for the polynomial ring $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(4)}$ consists of three U(4)-invariant polynomials, the Casimir scalars $\mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4$:

$$\mathfrak{C}_2 = \vec{\xi}\cdot\vec{\xi}, \quad \mathfrak{C}_3 = \sqrt{\frac{3}{2}}\,d_{ijk}\xi_i\xi_j\xi_k\,, \quad \mathfrak{C}_4 = \frac{3}{2}\,d_{ijk}d_{lmk}\xi_i\xi_j\xi_l\xi_m. \tag{22}$$

The semi-positivity of (21) can be stated as the nonnegativity of the coefficients S_2, S_3, and S_4 of the characteristic polynomial (8):[4]

$$\begin{aligned} S_2 &= \frac{3}{8}(1-\mathfrak{C}_2) \geq 0, \\ S_3 &= \frac{1}{16}(1 - 3\mathfrak{C}_2 + 2\mathfrak{C}_3) \geq 0, \\ S_4 &= \det\rho = \frac{1}{256}((1-3\mathfrak{C}_2)^2 + 8\mathfrak{C}_3 - 12\mathfrak{C}_4) \geq 0. \end{aligned} \tag{23}$$

Now we are in a position to compute the Grad matrix in terms of the SU(4) Casimir scalars:

$$\mathrm{Grad} = \begin{pmatrix} 4\mathfrak{C}_2 & 6\mathfrak{C}_3 & 8\mathfrak{C}_4 \\ 6\mathfrak{C}_3 & 9\mathfrak{C}_4 & 12\mathfrak{C}_2\mathfrak{C}_3 \\ 8\mathfrak{C}_4 & 12\mathfrak{C}_2\mathfrak{C}_3 & 4(\mathfrak{C}_3^2 + 3\mathfrak{C}_2\mathfrak{C}_4) \end{pmatrix}. \tag{24}$$

Passing to the equivalent matrix $Q\,\mathrm{Grad}\,Q^T$ with $Q = \mathrm{diag}(2,3,2)$, we arrive at the following form for the Procesi–Schwarz inequalities:

$$\mathfrak{C}_2 + \mathfrak{C}_3^2 + 3\mathfrak{C}_2\mathfrak{C}_4 + \mathfrak{C}_4 \geq 0\,, \tag{25}$$

$$\mathfrak{C}_3^2\left(-4\mathfrak{C}_2^2 + \mathfrak{C}_2 + \mathfrak{C}_4 - 1\right) + \mathfrak{C}_4\left(3\mathfrak{C}_2^2 + 3\mathfrak{C}_2\mathfrak{C}_4 + \mathfrak{C}_2 - 4\mathfrak{C}_4\right) \geq 0\,, \tag{26}$$

$$-4\mathfrak{C}_2^3\mathfrak{C}_3^2 + 3\mathfrak{C}_2^2\mathfrak{C}_4^2 + 6\mathfrak{C}_2\mathfrak{C}_3^2\mathfrak{C}_4 - \mathfrak{C}_3^4 - 4\mathfrak{C}_4^3 \geq 0\,. \tag{27}$$

The domain describing the semi-positivity (23)–(25) of ρ and its residual part obtained by imposing the condition of the semi-positivity of the Grad matrix (25)–(27) are depicted in Fig. 2.

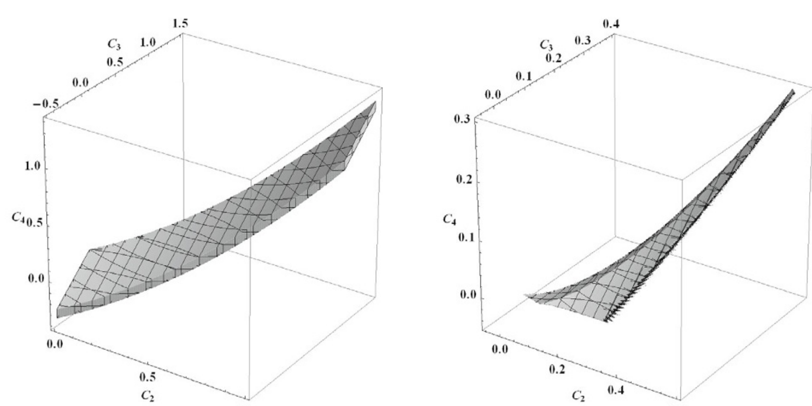

Fig. 2. On the left: $\rho \geq 0$. On the right: $\rho \geq 0 \cap \mathrm{Grad} \geq 0$.

[4]For details, we refer to [9].

The work was supported in part by the Ministry of Education and Science of the Russian Federation (grant 3802.2012.2) and the Russian Foundation for Basic Research (grant 13-01-0068). A. Kh. acknowledges support from the University of Georgia, under the grant 07-01-2013.

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