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CONSTRAINTS ON $SU(2) \otimes SU(2)$ INVARIANT POLYNOMIALS FOR A PAIR OF ENTANGLED QUBITS

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We discuss the entanglement properties of two qubits in terms of polynomial invariants of the adjoint action of $SU(2) \otimes SU(2)$ group on the space of density matrices $\mathfrak{P}_+$. Since elements of $\mathfrak{P}_+$ are Hermitian, non-negative fourth-order matrices with unit trace, the space of density matrices represents a semi-algebraic subset, $\mathfrak{P}_+ \in \mathbb{R}^{15}$. We define $\mathfrak{P}_+$ explicitly with the aid of polynomial inequalities in the Casimir operators of the enveloping algebra of $SU(4)$ group. Using this result the optimal integrity basis for polynomial $SU(2) \otimes SU(2)$ invariants is proposed and the well-known Peres–Horodecki separability criterion for 2-qubit density matrices is given in the form of polynomial inequalities in three $SU(4)$ Casimir invariants and two $SU(2) \otimes SU(2)$ scalars; namely, determinants of the so-called *correlation* and the *Schlienz–Mahler entanglement* matrices.

1. INTRODUCTION

Attempting to understand the nature of quantum information we strongly rely on a knowledge from the classical background [1, 2]. A fundamental unit of quantum information, "qubit", was introduced by analogy with the classical binary alternatives as the information associated with a 2-level quantum-mechanical system[4)]. Follow further this correspondence, the quantum relative of the classical n-bit string is composite object constructed as quantum superposition of n-qubit states. In the same way, for information processing on quantum level, instead of the classical logical gates the manipulations based on the unitary transformations of n-qubit density matrix ρ

$$\rho \quad \rightarrow \quad \rho' = U\rho U^{\dagger}, \quad U \in U(2^n), \qquad (1)$$

are used. However, exactly at this place the quantum–classical analogies cease to work. While *classical gates act transitively on all n-bit strings*, i.e., an arbitrary n-bit string can be transformed to a fixed n-bit string applying logical operations on individual bits, quantum local transformations act *nontransitively* on a multiqubit states [3][5)]. The qubits states that are not related by local transformations are "different" as far as their nonlocal properties are concerned. This means that regardless of the actual physical nature of gates and qubits nonlocal properties of composed system are encoded in equivalence relations between states provided by local transformations. In mathematical terms local unitary transformations acting on the state space partition it into equivalent classes — orbits. Each orbit is a representative of state with definite nonlocal characteristics and the set of all orbits of n qubits forms the *"entanglement space"*, $\mathcal{E}_n$ [3, 4]. Therefore, quantification and classification of all possible nonlocalities in multiqubit system reduces to the analysis of $\mathcal{E}_n$. For the case of 2-qubits in pure state the entanglement space represents the closed interval $[0, 1]$, parameterized by the concurrence [5]. However, for mixed states, especially if the number of qubits is more than two, the constructive description of geometry and topology of $\mathcal{E}_n$ becomes very challenging. The canonical way to describe the entanglement space can be adopted from the classical theory of invariants [6]; to separate orbits, giving coordinates for points in $\mathcal{E}_n$, the polynomial functions of the density matrix elements which are invariant under the local operations

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4)Qubits in question could be of quite different kinds, e.g., correspond to spin-1/2 degrees of freedom of an atom or a single photon's helicity.

5)The *local* operations, acting independently on each of n qubits, are defined as unitary transformations of the form $SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2)$. The complementary ones, $SU(2^n)/SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2)$ represent the *nonlocal* transformations.

can be used[6]. A series of important results in this direction has been obtained since Linden and Popescu addressed this issue to characterize the entanglement by polynomial invariants [8]. Particularly, for a mixed state of 2-qubits a complete set of fundamental local invariants that are homogeneous polynomials, from which all others can be constructed as the sums of products, has been determined [9]. Furthermore, the algebraic structure of the corresponding polynomial ring, $\mathbb{C}[\mathbb{R}^{15}]^{SU(2)\otimes SU(2)}$, that is necessarily Cohen–Macaulay type, has been recently identified [10].

All above studies were based on the assumption that the action of the local group is a *linear representation on a vector space*. However, according to the definition, the density matrix of n-level system $\rho \in \mathfrak{P}_+(\mathbb{R}^{n^2-1})$, where $\mathfrak{P}_+(\mathbb{R}^{n^2-1})$ is a space of $n \times n$ Hermitian, semi-definite matrices with trace normalized to unity, $\mathrm{Tr}(\rho) = 1$. The requirement of positive semi-definiteness means that space $\mathfrak{P}_+(\mathbb{R}^{15})$ in question is a *nonlinear subset* of $\mathbb{R}^{15}$. Therefore semi-definiteness implies that the local group $SU(2) \otimes SU(2)$ acts not on elements of vector space but on a certain semi-algebraic variety, given by the set of polynomial inequalities in elements of the density matrix. Consequently, values that local invariants can have are not arbitrary ones but are constrained. This in turn constraints characteristics of the entanglement space $\mathcal{E}_n$ and requires therefore the detailed analysis.

Below to gain some insight into this important issue we discuss the question: *How does semi-definiteness requirement affect the local invariants of bipartite quantum system?*[7]

In the present article this problem is analyzed for the simplest bipartite system, pair of qubits. Below we define $\mathfrak{P}_+(\mathbb{R}^{15})$ explicitly as solution of the system of linear and second-order polynomial inequalities in the Casimir invariants of the enveloping algebra of $SU(4)$ group. Having in mind this result the special integrity basis for local $SU(2) \otimes SU(2)$ polynomial invariants is constructed, with minimal number of elements constrained by requirement of positive semi-definiteness.

Our plan is as follows. We start with preliminaries, introducing the basic notions and present one vivid example. Sections 3 and 4 contain main results; the special, optimal basis for local polynomial scalars is described and inequalities in global $SU(4)$ scalars that guarantee the semi-definiteness of density matrices are presented. In Section 4 the well-known separability criterion for two qubit density matrices, the Peres–Horodecki condition, is reformulated in the form of polynomial inequalities in local invariants.

2. PRELIMINARIES

In this section the basic conventions and terminology are presented for the reader's convenience.

2.1. Mixed States for One and Two Qubits

Irrelevant to realization of a qubit its mixed state is described by analog of the density matrix of a nonrelativistic spin-1/2

$$\rho = \frac{1}{2}\,(1 + \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}). \tag{2}$$

Here, $\boldsymbol{\sigma}$ are the standard Pauli matrices and $\boldsymbol{\alpha}$ is defined as mathematical expectation:

$$\boldsymbol{\alpha} = \langle\boldsymbol{\sigma}\rangle = \mathrm{Tr}(\rho\boldsymbol{\sigma}).$$

If $\rho^2 = \rho$, a qubit is in "pure state", otherwise in the "mixed" one. For pure states $\boldsymbol{\alpha}$ parameterizes points on the so-called Bloch 2-sphere, $\boldsymbol{\alpha}^2 = 1$, while for mixed states the positive semi-definiteness of the density matrix is provided inside the Bloch ball $\boldsymbol{\alpha}^2 < 1$.

The generic form for an arbitrary mixed 2-qubit state is given by decomposition [14]:

$$\rho = \frac{1}{4}\,[\mathbb{I}_4 + \mathbf{a}\cdot\boldsymbol{\sigma}\otimes\mathbb{I}_2 + \mathbb{I}_2\otimes\boldsymbol{\sigma}\cdot\mathbf{b} + c_{ij}\boldsymbol{\sigma}_i\otimes\boldsymbol{\sigma}_j]\,. \tag{3}$$

The state is characterized by 15 expectation values: $\boldsymbol{a}$, $\mathbf{b}$ and c_{ij}, $i, j = 1, 2, 3$. The parameters $\boldsymbol{a}$, $\mathbf{b}$ are related with density matrices ρ_A and ρ_B of individual qubit's (A, B), extracted from ρ by taking the partial traces:

$$\rho_A = \mathrm{Tr}_B\rho, \quad \rho_B = \mathrm{Tr}_A\rho. \tag{4}$$

The coefficients c_{ij} are entries of the so-called *"correlation matrix"*, $C := ||c_{ij}||$[8].

Similarly to the one-qubit case, using the expansion for traceless part of ρ over the basis $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_{15}\}$ of the $\mathfrak{su}(4)$ algebra, a density matrix for 2-qubits can be characterized by 15-dimensional Bloch vector $\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_{15}\} \in \mathbb{R}^{15}$ [15]:

$$\rho = \frac{1}{4}\left(\mathbb{I}_4 + \sqrt{6}\boldsymbol{\xi}\cdot\boldsymbol{\lambda}\right). \tag{5}$$

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[6]Note that usage of polynomial functions is a reasonable restriction since according to the Schwarz theorem [7] any C^∞ class function, that is invariant under finite-dimensional linear orthogonal representation of a compact group, is C^∞ function of invariant polynomials.

[7]Also this question has been studied for many years (see, e.g., [11–13] and references therein) the elaboration of efficient practical computational methods is still under question.

[8]Under the local transformations, acting on each qubit independently, parameters $\mathbf{a}$ and $\mathbf{b}$ transform as 3-vectors, while c_{ij} as dyadic.

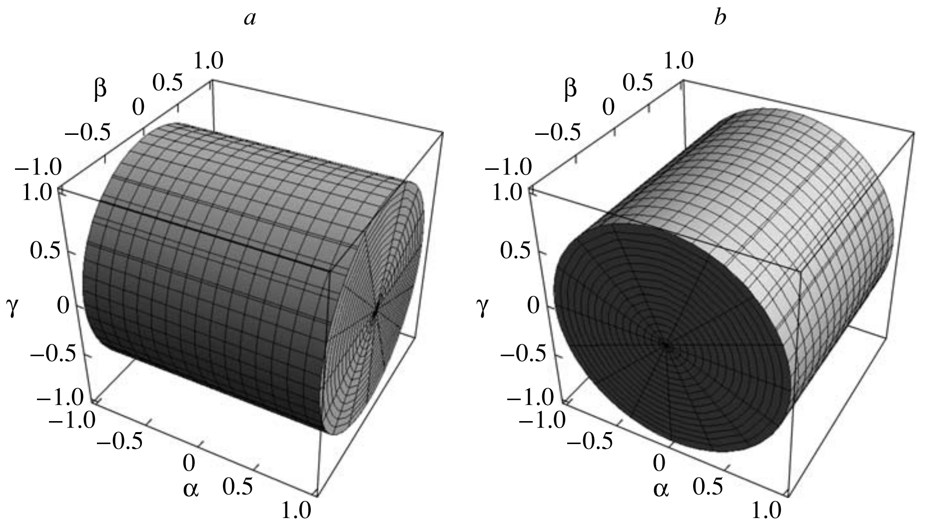

Fig. 1. The positivity domain for ρ_0 (a) and $\rho_0^{T_B}$ (b).

2.2. The Entangled Mixed States

The useful quantity for measure of correlation in composed bipartite system is the so-called *Schlienz–Mahler* matrix [16]:

$$M = \rho - \rho_A \otimes \rho_B. \tag{6}$$

The invariants constructed with the aid of (6) are aimed to describe the correlations between subsystems in the the so-called *"entangled states"*. The mixed entangled states are states complementary to those representable in the following separable form [3]:

$$\rho_{\text{sep}} = \sum \omega_k \frac{1}{2}\left(1 + \boldsymbol{\alpha}_k \cdot \boldsymbol{\sigma}_k\right) \otimes \frac{1}{2}\left(1 + \boldsymbol{\beta}_k \cdot \boldsymbol{\sigma}_k\right) \tag{7}$$

with

$$\sum \omega_k = 1, \quad \omega_k > 0.$$

The well-known test for detecting entanglement states for 2-qubits is based on the Peres [17] and Horedecki [18] observation.

2.3. The Peres–Horodecki Separability Criterion

According to the Peres–Horodecki a given state ρ is separable if it's partially transpose is positive and only then. The partial transpose ρ^{T_B} of a 2-qubit's density matrix is defined as

$$\rho^{T_B} = I \otimes T\rho, \tag{8}$$

where T is transposition operation. Under the transposition the Pauli matrices change as $T(\sigma_1, \sigma_2, \sigma_3) \to$ $\to (\sigma_1, -\sigma_2, \sigma_3)$.

2.4. An Example: $\alpha\beta\gamma$ States

Here we present an illustrative example of 3-parameter family of density matrices showing how the non-negativity of density matrices constraints the moduli space.

Consider 3-parameter family of density matrices of the following form:

$$\rho_0 = \frac{1}{4}\begin{pmatrix} 1+\alpha & 0 & 0 & 0 \\ 0 & 1-\beta & i\gamma & 0 \\ 0 & -i\gamma & 1+\beta & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix}, \tag{9}$$

$$\rho_0^{T_B} = \frac{1}{4}\begin{pmatrix} 1+\alpha & 0 & 0 & i\gamma \\ 0 & 1-\beta & 0 & 0 \\ 0 & 0 & 1+\beta & 0 \\ -i\gamma & 0 & 0 & 1-\alpha \end{pmatrix}.$$

Note that the coefficients $\mathbf{a}$, $\mathbf{b}$ and entries of the correlations matrix C for ρ_0, when represented in the Fano form (3), read:

$$a_3 = (\alpha - \beta)/2, \quad b_3 = (\alpha + \beta)/2, \tag{10}$$
$$c_{12} = -c_{21} = \gamma/2.$$

Straightforward calculations show that the introduced density matrices ρ_0 and $\rho_0^{T_B}$ are non-negative when the real parameters α, β, and γ satisfy inequalities

$$\alpha^2 \le 1, \quad \beta^2 + \gamma^2 \le 1 \quad (\rho_0 \ge 0), \tag{11}$$

$$\beta^2 \le 1, \quad \alpha^2 + \gamma^2 \le 1 \quad (\rho_0^{T_B} \ge 0). \tag{12}$$

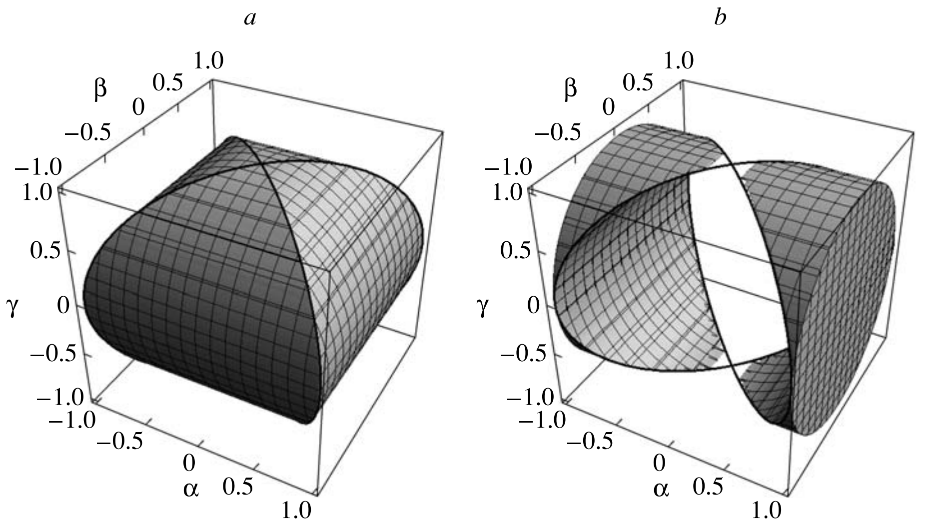

Fig. 2. Moduli domains for separable (a) and entangled states (b).

The moduli spaces of matrices ρ_0 and $\rho_0^{T_B}$ are cylinders depicted on Fig. 1. Their intersection shown on the Fig. $2a$, characterizes separable states, while the cylinder with "drill hole" on Fig. $2b$ represents the entangled ones.

Having in mind the last example we pass to the description of the generic constraints on the parameters space of arbitrary 2-qubit density matrices.

3. CONSTRUCTING LOCAL INVARIANTS ON $\mathfrak{P}_+$

In this section we aim to describe the fundamental set of polynomial functions of elements of the density matrix that are invariant under the local group $SU(2) \otimes SU(2)$ viewed as subgroup of the global $SU(4)$.

At first the positive semi-definiteness requirement $\rho \geq 0$ is relaxed and the polynomial invariants of the linear action of $SU(2) \otimes SU(2)$ on $\mathbb{R}^{16}$ is constructed. Finally, based on the explicit definition of the semi-algebraic subset $\mathfrak{P}_+ \in \mathbb{R}^{16}$ given in terms of the Casimir invariants of the enveloping algebra of $SU(4)$ group, the "optimal" basis for $\mathbb{C}[\mathfrak{P}_+]^{SU(2)\otimes SU(2)}$ is presented.

3.1. The Integrity Basis for Ring $\mathbb{C}[\mathbb{R}^{16}]^{SU(2)\otimes SU(2)}$ *of Polynomial Invariants*

As mathematical issue this task was risen in nuclear physics already in seventeenth in the context of the so-called *"missing label"* problem (see [19, 20] and references therein). The recent studies of algebraic structure of this polynomial ring in relation with 2-qubit system has been done by several authors (e.g., [9, 10] and references therein). Particularly, based on the known construction of invariants[9)] it was shown that the ring of invariants admits the Hironaka decomposition. More precisely it was proved that the ring $\mathbb{C}[\mathbb{R}^{16}]^{SU(2)\otimes SU(2)}$ of polynomial invariants admits the following decomposition

$$\mathbb{C}[\mathfrak{P}_+]^{SU(2)\otimes SU(2)} = \qquad (13)$$
$$= \bigoplus_{k=0}^{15} J_k \mathbb{C}[K_1, K_2, \ldots, K_{10}],$$

where $K_r, r = 1, 2, \ldots, 10$ are primary algebraically independent polynomials of degrees 1, 2, 2, 2, 3, 3, 4, 4, 4, 6 and $J_k, k = 0, 1, 2, \ldots, 15, J_0 = 1$, are secondary linearly independent invariants of degrees 4, 5, 6, 6, 6, 7, 7, 8, 8, 9, 9, 9, 10, 11, 15, respectively.

To write down these basic polynomials in components of the density matrix elements it is convenient at first to introduce the following homogeneous polynomial scalars [20][10)]:

3 invariants of second order

$$C^{(002)} = c_{ij}c_{ij}, \quad C^{(200)} = a_i a_i, \qquad (14)$$
$$C^{(020)} = b_i b_i,$$

2 invariants of third order

$$C^{(003)} = \frac{1}{3!}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}c_{i\alpha}c_{j\beta}c_{k\gamma}, \qquad (15)$$
$$C^{(111)} = a_i c_{ij} b_j,$$

[9)] The systematic procedure for the invariant construction is based on the coupling of irreducible tensors using the Clebsch–Gordan or Wigner coefficients.

[10)] The summation from one to three over the repeated indexes in all expressions below is assumed.

4 invariants of fourth order

$$C^{(004)} = c_{i\alpha}c_{i\beta}c_{j\alpha}c_{j\beta}, \quad C^{(202)} = a_i a_j c_{i\alpha}c_{j\alpha}, \quad (16)$$

$$C^{(022)} = b_\alpha b_\beta c_{i\alpha}c_{i\beta}, \quad (17)$$

$$C^{(112)} = \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}a_i b_\alpha c_{j\beta}c_{k\gamma},$$

1 invariant of fifth order

$$C^{(113)} = a_i c_{i\alpha}c_{\beta\alpha}c_{\beta j}b_j, \quad (18)$$

4 invariants of six order

$$C^{(123)} = \epsilon_{ijk}b_i c_{\alpha j}a_\alpha c_{\beta k}c_{\beta l}b_l, \quad (19)$$

$$C^{(204)} = a_i c_{i\alpha}c_{j\alpha}c_{j\beta}c_{k\beta}a_k,$$

$$C^{(024)} = b_i c_{\alpha i}c_{\alpha j}c_{\beta j}c_{\beta,k}b_k, \quad (20)$$

$$C^{(213)} = \epsilon_{\alpha\beta\gamma}a_\alpha c_{\beta i}b_i c_{\gamma j}c_{\delta j}a_\delta,$$

2 invariants of seventh order

$$C^{(214)} = \epsilon_{ijk}b_i c_{\alpha j}a_\alpha c_{\beta k}c_{\beta l}c_{\gamma l}a_l, \quad (21)$$

$$C^{(124)} = \epsilon_{\alpha\beta\gamma}a_\alpha c_{\beta j}b_j c_{\gamma k}c_{\delta k}c_{\delta l}b_l,$$

2 invariants of eighth order

$$C^{(125)} = \epsilon_{ijk}b_i c_{\alpha j}c_{\alpha l}b_l c_{\beta k}c_{\beta m}c_{\gamma m}a_\gamma, \quad (22)$$

$$C^{(215)} = \epsilon_{\alpha\beta\gamma}a_\alpha c_{\beta i}c_{\delta i}a_\delta c_{\gamma k}c_{\varrho k}c_{\varrho l}b_l,$$

2 invariants of ninth order

$$C^{(306)} = \epsilon_{\alpha\beta\gamma}a_\alpha c_{\beta i}c_{\delta i}a_\delta c_{\gamma j}c_{\varrho j}c_{\varrho k}c_{\sigma k}a_\sigma, \quad (23)$$

$$C^{(036)} = \epsilon_{ijk}b_i c_{\alpha j}c_{\alpha l}b_l c_{\beta k}c_{\beta m}c_{\gamma m}c_{\gamma s}b_s.$$

These invariants were used in [10] to construct the integrity basis for $\mathbb{C}[\mathbb{R}^{16}]^{SU(2)\otimes SU(2)}$. However, having in mind that we are interested in description of invariant functions only on the subset $\mathfrak{P}_+ \subset \mathbb{R}^{16}$, below the new integrity basis is proposed. It is optimal in a sense that only a minimal number of basic invariants are subject to the constraints arising due to the projection $\mathbb{R}^{16} \to \mathfrak{P}_+$.

3.2. Semi-Definiteness of Density Matrices in Terms of $SU(4)$ Casimir Operators

The semi-definiteness of the density matrix for 2-qubit system being the property which is invariant under the group $SU(4)$ can be formulated directly in terms of the invariants of the corresponding enveloping algebra.

Invariant properties of n-level system are encoded in the characteristic equation for the density matrix

$$||x\mathbb{I}_4 - \rho|| = x^n - S_1 x^{n-1} + S_2 x^{n-2} - \cdots + (-1)^n S_n = 0. \quad (24)$$

Since (24) is characteristic polynomial of a Hermitian matrix all its zeroes are real numbers $x_1, x_2, \ldots, x_n$. The positive semi-definiteness of the density matrix implies the non-negativity of all roots,

$$x_k \geq 0, \quad \text{for all } k. \quad (25)$$

However, because for a generic matrix precise determination of the actual eigenvalues is not a simple task, it is instructive to formulate the non-negativity in terms of coefficients of the characteristic equation (24) straightforwardly. Fortunately, this is possible to do for any Hermitian matrix in a very simply way. Indeed (see the proof in [12, 13]), if all roots of (24) are real than for their non-negativity it is necessary and sufficient that

$$S_k \geq 0, \quad \text{for all } k. \quad (26)$$

Coefficients S_k are expressible in terms of the traces of the powers of the density matrices $t_k = \mathrm{Tr}(\rho^k)$, as determinants

$$k!S_k = \det \begin{bmatrix} t_1 & 1 & 0 & \cdots & \cdots \\ t_2 & t_1 & 2 & \cdots & \cdots \\ t_3 & t_2 & t_1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{k-1} & t_{k-2} & \cdots & \cdots & k-1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & t_1 \end{bmatrix} \quad (27)$$

and finally in terms of the Casimir operators of $SU(n)$.

For a system of 2-qubits $n = 4$ and the group $SU(4)$ has three homogeneous invariants, the Casimir operators of order 2, 3, and 4. Using the generalized Bloch vector representation (5) the Casimir operators of $SU(4)$ read

$$\mathfrak{C}_2 = \boldsymbol{\xi} \cdot \boldsymbol{\xi}, \quad (28)$$

$$\mathfrak{C}_3 = \sqrt{\frac{3}{2}} d_{ijk}\xi_i\xi_j\xi_k = \boldsymbol{\xi} \vee \boldsymbol{\xi} \cdot \boldsymbol{\xi}, \quad (29)$$

$$\mathfrak{C}_4 = \frac{3}{2} d_{ijk}d_{lmk}\xi_i\xi_j\xi_l\xi_m = \boldsymbol{\xi} \vee \boldsymbol{\xi} \cdot \boldsymbol{\xi} \vee \boldsymbol{\xi}. \quad (30)$$

Rewriting traces $t_k = \mathrm{Tr}(\rho^k)$ with the help of $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$ and compare them with (27) one can derive the expressions for coefficients S_k directly in terms of the Casimirs. The results of calculations for $n = 4$ are

$$S_2 = \frac{3}{8}(1 - \mathfrak{C}_2), \quad (31)$$

$$S_3 = \frac{1}{16}\left[1 - 3\mathfrak{C}_2 + 2\mathfrak{C}_3\right], \quad (32)$$

$$S_4 = \frac{1}{256}\left[(1 - 3\mathfrak{C}_2)^2 + 8\mathfrak{C}_3 - 12\mathfrak{C}_4\right]. \quad (33)$$

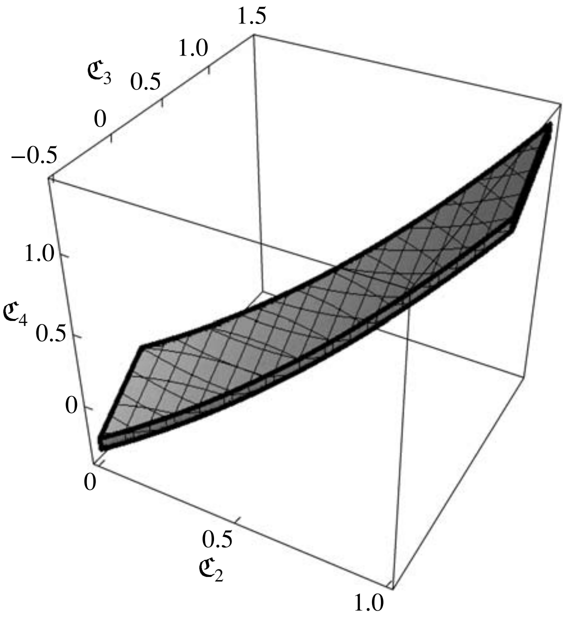

Fig. 3. The allowed region for the Casimirs $\mathfrak{C}_2$, $\mathfrak{C}_3$, and $\mathfrak{C}_4$.

Now we are in position to write down the set of inequalities in the Casimir operators that guarantee the semi-definiteness of density matrices. For this purpose let us finally note that since the density matrix has a unit trace $\mathrm{Tr}(\rho) = 1$, the maximum of S_k is attainable when all eigenvalues are equal $x_1 = x_2 = x_3 = x_4 = 1/4$. Therefore the least upper bounds for coefficients are

$$\sup\{S_2\} = \frac{3}{8}, \quad \sup\{S_3\} = \frac{1}{16}, \tag{34}$$
$$\sup\{S_4\} = \frac{1}{256}.$$

Gathering all above results together the positive semi-definite density matrices with unit trace are finally formulated as inequalities in three $SU(4)$ Casimirs, $\mathfrak{C}_2$, $\mathfrak{C}_3$, and $\mathfrak{C}_4$:

$$0 \le \mathfrak{C}_2 \le 1, \tag{35}$$
$$0 \le 3\mathfrak{C}_2 - 2\mathfrak{C}_3 \le 1, \tag{36}$$
$$0 \le (1 - 3\mathfrak{C}_2)^2 + 8\mathfrak{C}_3 - 12\mathfrak{C}_4 \le 1. \tag{37}$$

From (35), (36) it follows that the allowed region on $\mathfrak{C}_2\mathfrak{C}_3$ plane is a strip bounded by the straight lines

$$\mathfrak{C}_3 = \frac{3}{2}\mathfrak{C}_2, \quad \mathfrak{C}_3 = -\frac{1}{2} + \frac{3}{2}\mathfrak{C}_2, \tag{38}$$
$$\mathfrak{C}_2 = 0, \quad \mathfrak{C}_2 = 1.$$

Taking into account (36) we conclude that all Casimirs are bounded

$$0 \le \mathfrak{C}_2 \le 1, \quad -1/2 \le \mathfrak{C}_3 \le 3/2, \tag{39}$$
$$-1/3 \le \mathfrak{C}_4 \le 4/3.$$

The details of the allowed 3-dimensional region in the $\mathfrak{C}_2\mathfrak{C}_3\mathfrak{C}_4$ space can be visualized from Fig. 3. Its typical 2-dimensional slice obtained by fixing the fourth Casimir $\mathfrak{C}_4$, say $\mathfrak{C}_4 = 0$, represents the region bounded by two parabolas

$$\mathfrak{C}_3 = -\frac{1}{8}(1 - 3\mathfrak{C}_2)^2, \quad \mathfrak{C}_3 = -\frac{9}{8}\mathfrak{C}_2^2 + \frac{3}{4}\mathfrak{C}_2.$$

3.3. The Integrity Basis for Ring $\mathbb{C}[\mathfrak{P}_+]^{SU(2)\otimes SU(2)}$ of Polynomial Invariants

The global $SU(4)$ Casimir operators admit the expansion over the integrity basis of local $SU(2) \otimes \otimes SU(2)$ invariants introduced in the previous section[11)]

$$\mathfrak{C}_2 = \frac{1}{3}(C^{(200)} + C^{(020)} + C^{(002)}), \tag{40}$$
$$\mathfrak{C}_3 = C^{(111)} - C^{(003)}, \tag{41}$$
$$\mathfrak{C}_4 = \frac{1}{6}[2(C^{(200)}C^{(020)} + C^{(202)} + \tag{42}$$
$$+ C^{(022)} - C^{(112)}) + (C^{(002)})^2 - C^{(004)}].$$

The expansions (40)–(42) involve nine local invariants from the King–Jararvis–Welsh (KJW) integrity basis [10]. However, because the constraints are imposed only on three Casimirs, $\mathfrak{C}_2$, $\mathfrak{C}_3$, and $\mathfrak{C}_4$, it seems practical to choose instead of the KWJ basis another one, whose elements contain among others these Casimir scalars instead of invariants $C^{(002)}$, $C^{(111)}$, and $C^{(112)}$.

Follow this idea we define the set of local invariants, which includes 10 *primary invariants*:

$$\deg = 1, \quad K_1 = 1, \tag{43}$$
$$\deg = 2, \quad K_2 = \mathfrak{C}_2, \quad K_3 = C^{(200)}, \quad K_4 = C^{(020)},$$
$$\deg = 3, \quad K_5 = C^{(003)}, \quad K_6 = \mathfrak{C}_3,$$
$$\deg = 4, \quad K_7 = C^{(004)}, \quad K_8 = C^{(202)},$$
$$K_9 = C^{(022)},$$
$$\deg = 6, \quad K_{10} = C^{(204)} + C^{(024)},$$

and 15 *secondary invariants*:

$$\deg = 4, \quad J_1 = \mathfrak{C}_4, \tag{44}$$
$$\deg = 5, \quad J_2 = C^{(113)},$$
$$\deg = 6, \quad J_3 = C^{(204)} - C^{(024)},$$
$$J_8 = C^{(124)}, \quad J_9 = C^{(213)},$$
$$\deg = 7, \quad J_{10} = C^{(214)}, \quad J_{11} = C^{(113)},$$

11) This can be easily done with the aid of the Gröbner bases technique [21].

$$\deg = 8, \quad J_{12} = C^{(215)}, \quad J_{13} = C^{(125)},$$
$$\deg = 9, \quad J_4 = J_1 J_2, \quad J_{14} = C^{(306)},$$
$$J_{15} = C^{(036)},$$
$$\deg = 10, \quad J_5 = J_1 J_3, \tag{45}$$
$$\deg = 11, \quad J_6 = J_2 J_3, \tag{46}$$
$$\deg = 15, \quad J_7 = J_1 J_2 J_3. \tag{47}$$

So, we finalize with the statement: for local polynomial invariants of 2-qubits in an arbitrary mixed state there is an optimal integrity basis which has the property that only a minimal number of primary invariants of degree 2, 3 and one lowest degree 4 secondary invariant that appear in the Hironaka decomposition (13) are subject to the constraints (35)–(37), arising from the requirement of positive semi-definiteness of 2-qubit density matrices.

4. THE SEPARABILITY CRITERION IN TERMS OF $SU(2)\otimes SU(2)$ SCALARS

Now we can reformulate the Peres–Horodecki separability criterion for pair of qubits as a set of inequalities in the local scalars. Consider the partially transposed matrix ρ^{T_B} and following Section 3.2 express its positive definiteness in terms of the non-negativity of the coefficients $S_k^{T_B}$ of the corresponding characteristic equation. As calculations show, the second coefficient $S_2^{T_B}$ of characteristic equation of transposed matrix ρ^{T_B} coincides with the coefficient S_2, while the third and fourth coefficients $S_3^{T_B}$ and $S_4^{T_B}$ can be written with the aid of determinants of the correlation matrix C and Schlienz–Mahler matrix M:

$$S_3^{T_B} = S_3 + \frac{1}{4}\det ||C||, \tag{48}$$

$$S_4^{T_B} = S_4 + \frac{1}{16}\det ||M||. \tag{49}$$

Using relations (48), (49) the requirement of the positive definiteness of ρ^{T_B} leads to the inequalities:

$$0 \le 3\mathfrak{C}_2 - 2\mathfrak{C}_3 - 4\det ||C|| \le 1, \tag{50}$$

$$0 \le (1 - 3\mathfrak{C}_2)^2 + 8\mathfrak{C}_3 - 12\mathfrak{C}_4 + \tag{51}$$
$$+ 16\det ||M|| \le 1.$$

Therefore we conclude that an arbitrary mixed 2-qubit state is separable if two sets of Eqs. (35)–(37) and (50), (51) are satisfied simultaneously.

5. CONCLUSION

In the present paper the construction of the integrity polynomial basis for local invariants of two qubits has been discussed. Our aim was to find a special basis, such that only a minimal number of its elements are constrained due to the positive semi-definiteness of density matrices. The basis was found and the constraints were presented in the form of inequalities in its primary and one secondary elements. We conclude with the final remark about the significance of those constraints; the bounds on values of local invariants can exert strong influence on characteristics of the entangled states.

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