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ON THE HAMILTONIAN FORMULATION OF GAUGE THEORIES IN TERMS OF PHYSICAL VARIABLES

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We examine the Hamiltonian aspects of gauge theories related to the problem of description of gauge-invariant observables. The method of reformulation of non-Abelian gauge theory entirely in terms of gauge-invariant unconstrained canonical variables is exemplified in detail for the $SU(2)$ Yang–Mills theory. We present the equivalent unconstrained Hamiltonian system in the form of a nonlocal field theory of six fields which can be identified with particles carrying "nonrelativistic spin 2 and spin 0." Based on this form of unconstrained theory, an effective, low-energy, nonlinear, sigma-model-type Lagrangian is derived. The long-wavelength approximation for the obtained nonlocal unconstrained theory is studied. In particular, for zero-order approximation the relation between $SU(2)$ gauge theory and the well-known integrable many-body Euler–Calogero–Moser system is discussed.

1. Introduction

The notion of the gauge invariance is the basis of modern theories of fundamental electroweak and strong interactions. Despite the fact that the roots of gauge invariance go back to the classical electrodynamics developed in the XIXth century, the complete understanding of its importance belongs to the XXth century. Inspired by the ideas of Riemannian geometry, Herman Weyl was the first who used invariance under an arbitrary scale change of metric tensor as a constructive principle to provide the unification of gravitation and electromagnetism [123]. Although this first form of gauge theory was not successful, later works of Fock [43], London [80, 81], and Weyl [124] gave a form of the gauge description of electromagnetism which is today referred to as $U(1)$*-gauge theory.*

A generalization of the $U(1)$-gauge theory was suggested in 1938 by Klein [70]. He made the first attempt to apply the non-Abelian gauge symmetry $SU(2)\times U(1)$ to electromagnetic and weak interactions, but it was too early for the physical community to accept this idea. Only after the work of Yang and Mills [128] did the concept of isotopic local gauge invariance become the subject of intensive study. Great progress was achieved in the next two decades; the non-Abelian electroweak theory and non-Abelian theory of strong interaction of quarks and gluons (quantum chromodynamics) were created. There are several reviews on the very interesting history of the appearance, transmutation, and understanding of the gauge invariance in physics [63, 97, 107, 120, 127].

As was emphasized in [63], it took almost a century to formulate the nonuniqueness of the gauge potential that exists despite the uniqueness of the electromagnetic fields. Passing from the geometric setting of gauge theory to the dynamical realization in the form of a variational problem for gauge potential encounters problems unknown to the classical mechanics of the XIXth century. Treating Maxwell's equations as Euler–Lagrange equations for a certain variational problem leads to a Lagrangian function which is degenerate (singular), i.e., its Hessian is zero. The conventional classical treatment including the Hamiltonian formulation of such a dynamical system is impossible. This property is common for all theories possessing local invariance, i.e., invariance under the transformations whose parameters are arbitrary functions of space-time points. In the middle of the XXth century, Dirac and Bergman elaborated a new Hamiltonian formalism, which is now referred to as the generalized Hamiltonian formalism applicable to theories with a degenerate Lagrangian function. Theories whose singularity is caused by the local gauge invariance strongly differ from the other ones; for such theories, the principle of deterministic evolution

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is violated. Gauge systems have so-called pure gauge degrees of freedom, whose dynamics is completely unpredictable. Only the variables with uniquely predictable dynamics have a physical meaning, and the complete set of these variables represents the physical variables in gauge theory. For the case of Abelian electrodynamics, the conventional choice of physical variables consists of their identification with the transverse components of the gauge potential. In the case of non-Abelian gauge theories, this problem is much more complicated. Dealing with the problem of the identification and elimination of the pure gauge degrees of freedom, two approaches, perturbative and nonperturbative, are used. The conventional perturbative gauge-fixing method [42] works successfully for the description of high-energy phenomena; it ascribes the transverse components of the non-Abelian gauge field as physical variables, but fails in applications in the infrared region. The different nonperturbative reductions of gauge theories were elaborated during the last few decades [8,10,22–24,31,51,55,57,60,66,67,69,77,94,96,103,108,113,122]. The object of these investigations is the search for a representation of the gauge-invariant variables which is suitable for a description of the infrared limit of Yang–Mills theory but unfortunately up to now has been rather complicated for practical calculations. In this paper, we state one such attempt to represent the gauge theory in terms of the canonical physical variables only, following the Dirac generalized Hamiltonian formalism [38, 58, 117] and using the method of Hamiltonian reduction (see [47–50, 111] and the references therein). In the next section, we briefly recall this general reduction-scheme formalism in order to set the formalism. Section 3 presents the application of this formalism to the $SU(2)$ Yang–Mills field theory. A canonical transformation to the set of adapted coordinates is performed in terms of which the Abelianization of the Gauss law constraints reduces to an algebraic operation and the pure gauge degrees of freedom drop out from the Hamiltonian after projection onto the constraint shell. For the remaining gauge-invariant fields, two representations are introduced, where the three fields which transform as scalars under spatial rotations are separated from the three rotational fields. In Sec. 4, relations between the Yang–Mills theory and the well-known integrable many-body Calogero–Moser system are discussed. In Sec. 5, we present the generalization of our formalism to the case of classical action for Yang–Mills fields including the boundary terms, the so-called topological Pontryagin invariant. Section 6 is devoted to the discussion of effective theories corresponding to non-Abelian gauge theories in the low-energy region. An effective, low-energy, nonlinear sigma-model-type Lagrangian is derived, which out of the six physical fields involves only one of the three scalar fields and two rotational fields summarized in a unit vector. Finally, in Sec. 7, we make several remarks on quantization of gauge theories and give our conclusions.

In the Appendix, we list several notations and formulas for nonrelativistic spin 0, 1, and 2 used in the text.

2. Gauge Invariance and Constrained Dynamics

Below, we briefly recall the general reduction formalism for obtaining the unconstrained Hamiltonian system from the initial gauge-invariant formalism in the framework of Dirac constraint theory in order to set the formalism. The Dirac and Faddeev gauge-fixing methods as well as the Hamiltonian reduction method are also described. The Hamiltonian reduction method is exemplified by considering the Yang–Mills system in $(0 + 1)$-dimensions.

2.1. Reduction of constrained systems with first-class constraints. The procedure of reduction of the phase space of a singular system is a generalization of the reduction method of a system of differential equations possessing a Lie group symmetry. The well-known results for this type of reduction in the number of the degrees of freedom are embodied in the famous Liouville theorem on first integrals in involution. Interest in these has revived in connection with the study of Hamiltonian systems with a local (gauge) symmetry. Since the work of Bergmann and Dirac at the beginning of the 1950s, it has become clear that the role of integrals of motion in a Hamiltonian system with gauge symmetry is played by the first-class constraints. Although the reduction in the number of degrees of freedom due to first-class constraints has many common features with the classical case, there are very important differences.[1] In order to explain these peculiarities of the reduction procedure and to make the notes self-contained, we first summarize some definitions and recall the main facts of the Dirac theory of generalized Hamiltonian dynamics into the appropriate context. We discuss these ideas for a mechanical system, i.e., a system with a finite number of degrees of freedom, to separate these aspects from the difficulties connected with infinitely many degrees of freedom in field theory.

2.1.1. The definition of reduced phase space. Let us consider a system with the $2n$-dimensional Euclidean phase space Γ spanned by the canonical coordinates q_i and their conjugate momenta p_i and endowed with the canonical simplectic structure $\{q_i, p^j\} = \delta_i^j$. Suppose that the dynamics is constrained to a certain $(2n-m)$-dimensional submanifold Γ_c determined by m functionally independent constraints

$$\varphi_\alpha(p,q) = 0, \tag{2.1.1}$$

which are assumed to be of first class,

$$\{\varphi_\alpha(p,q), \varphi_\beta(p,q)\} = f_{\alpha\beta\gamma}(p,q)\varphi_\gamma(p,q), \tag{2.1.2}$$

and complete in the sense that

$$\{\varphi_\alpha(p,q), H_C(p,q)\} = g_{\alpha\gamma}\varphi_\gamma(p,q), \tag{2.1.3}$$

where $H_C(p,q)$ is the canonical Hamiltonian. Due to the presence of these constraints, the Hamiltonian system admits a generalized dynamics described by the extended Poincare–Cartan form

$$\Theta := \sum_{i=1}^{n} p_i dq_i - H_E(p,q)dt \tag{2.1.4}$$

with the extended Hamiltonian $H_E(p,q)$ that differs from the canonical Hamiltonian $H_C(p,q)$ by a linear combination of constraints with arbitrary multipliers $u_\alpha(t)$:

$$H_E(p,q) := H_C(p,q) + u_\alpha(t)\varphi_\alpha(p,q). \tag{2.1.5}$$

The completeness condition (2.1.3) with H_C replaced by H_E implies that for first-class constraints, the functions $u_\alpha(t)$ cannot be fixed in internal terms of the theory. This implies that the system possesses a local symmetry and that the coordinates split up into two sets: one set whose dynamics is governed in an arbitrary way and another set with a uniquely determined behavior. Recalling the Dirac definition [33] of a *physical variable* as a dynamical variable F with the property

$$\{F(p,q), \varphi_\alpha(p,q)\} = d_{\alpha\gamma}(p,q)\varphi_\gamma(p,q), \tag{2.1.6}$$

one can conclude that the first set of coordinates does not affect the physical quantities which are defined on some subspace of the constraint surface Γ_c. Indeed, if we consider (2.1.6) as a set of m first-order linear differential equations for F, then, due to the integrability condition (2.1.2), this function can be completely determined by its values in the $2(n-m)$-dimensional submanifold of its initial conditions [40]. This subspace of the constraint shell represents the *reduced phase space* Γ^*. This definition of reduced phase space is implicit. The main problem is to find the set of $2(n-m)$ "physical coordinates" Q_i^* and P_i^* that span this reduced phase space and pick out the other additional m pairs which have no physical significance and represent the pure gauge degrees of freedom. Several approaches to its solution are known. Below we briefly discuss the corresponding methods of practical construction of the physical and the gauge degrees of freedom with and without gauge fixing.

[1] Presumably, Shanmugadhasan [111] was the first who employed the classical Lie–Cartan reduction method (see, e.g., [1, 21, 85, 100, 102]) in the framework of generalized Hamiltonian dynamics to the reduction in the number of degrees of freedom instead of the conventional gauge-fixing method.

2.1.2. Reduced phase space with the Dirac gauge-fixing method. General principles for imposing gauge fixing constraints onto the canonical variables in the Hamiltonian approach were proposed by Dirac in connection with the canonical formulation of gravity [35]. According to the Dirac gauge-fixing prescription, one starts with the introduction of as many new "gauge" constraints

$$\chi_\alpha(p,q) = 0 \tag{2.1.7}$$

as there are first-class constraints (2.1.1), with the requirement

$$\det \|\{\chi_\alpha(p,q), \varphi_\beta(p,q)\}\| \neq 0. \tag{2.1.8}$$

This allows one to find the unknown Lagrange multipliers $u_\alpha(t)$ from the requirement of conservation of the gauge conditions (2.1.7) in time[2]

$$\dot{\chi}_\alpha = \{\chi_\alpha, H_C\} + \sum_\beta \{\chi_\alpha, \varphi_\beta\} u_\beta = 0 \tag{2.1.9}$$

and thus to determine uniquely the dynamics of the system. A striking result of Dirac is the observation that such a kind of fixation of Lagrange multipliers $u(t)$ is equivalent to the following way of proceeding. One can drop both constraints (2.1.1) and the gauge-fixing conditions (2.1.7) and at the same time achieve the reduction to the unconstrained theory by using the Dirac brackets

$$\{F, G\}_D := \{F, G\} - \{F, \xi_s\} C^{-1}_{ss'} \{\xi_{s'}, G\} \tag{2.1.10}$$

instead of the Poisson brackets, where ξ denotes the set of all constraints (2.1.1) and (2.1.7) and C^{-1} is the inverse of the Poisson matrix $C_{\alpha\beta} := \{\xi_\alpha, \xi_\beta\}$. In this method, all coordinates of the phase space are treated on an equal footing and all information on both initial and gauge constraints is absorbed into the Dirac brackets, which describe an effective reduction in the number of degrees of freedom from n to $n - m$:

$$\sum_{i=1}^{n} \{q_i, p_i,\}_{P.B.} = n, \quad \sum_{i=1}^{n} \{q_i, p_i,\}_{D.B.} = n - m.$$

The inclusion of gauge constraints in addition to the initial constraints allows one to take the constraint nature of the canonical variables into account by changing the initial canonical symplectic structure to a new one defined by the Dirac brackets. The new canonical structure, being dependent on the choice of gauge-fixing conditions, is very complicated in general, and it is not clear how to deal with it, in particular, when we are quantizing the theory. However, there is a special case where the Dirac bracket coincides with the canonical one and looks like the Poisson bracket for an unconstrained system defined on Γ^*:

$$\{F, G\}_D \Big|_{\varphi=0,\ \chi=0} = \sum_{i=1}^{n-m} \left\{ \frac{\partial \overline{F}}{\partial Q_i^*} \frac{\partial \overline{G}}{P_i^*} - \frac{\partial \overline{F}}{\partial P_i^*} \frac{\partial \overline{G}}{Q_i^*} \right\}. \tag{2.1.11}$$

This representation of the Dirac bracket means that in terms of the conjugate coordinates Q_i^* and P_i^* $(i = 1, \ldots, n - m)$, the reduced phase space is parametrized such that the constraints vanish identically and any function $F(p,q)$ given on the reduced phase space becomes

$$F(p,q)\Big|_{\varphi=0,\chi=0} = \overline{F}(P^*, Q^*)$$

(see [117]). Thus, in the Dirac gauge-fixing method, the problem of definition of "true dynamical degrees of freedom" reduces to the problem of a "lucky" choice of the gauge condition.

[2]Everywhere below, the dot over a letter denotes the derivative with respect to the time variable.

2.1.3. Reduced phase space with the Faddeev gauge-fixing method. An alternative to the Dirac gauge-fixing procedure was proposed in the well-known paper of Faddeev [40] devoted to the method of path-integral quantization of a constrained system. In contrast to the Dirac method, the main idea of the Faddeev method is to introduce an explicit parametrization of the reduced phase space. As in the Dirac method, one introduces gauge-fixing constraints $\chi_\alpha(p,q) = 0$, but now with the additional "Abelian" property

$$\{\chi_\alpha(p,q), \chi_\beta(p,q)\} = 0, \tag{2.1.12}$$

and requirement (2.1.8) is fulfilled. Now, in accordance with the Abelian character of gauge conditions (2.1.12), there exists a canonical transformation to new coordinates

$$q_i \mapsto Q_i := Q_i\,(q,p)\,, \quad p_i \mapsto P_i := P_i\,(q,p)$$

such that m of the new P's coincide with the constraints χ_α:

$$P_\alpha = \chi_\alpha\,(q,p)\,. \tag{2.1.13}$$

Condition (2.1.8) allows one to resolve constraints (2.1.1) for the coordinates Q_α in terms of $(n-m)$ canonical pairs (Q_i^*, P_i^*), which span the $2(n-m)$-dimensional surface Σ determined by the equations

$$P_\alpha = 0, \quad Q_\alpha = Q_\alpha\,(Q^*, P^*)\,.$$

After this construction has been carried out, the problem is to prove that the surface Σ coincides with the true reduced phase space Γ^*, which is independent of the choice of the gauge-fixing conditions. In other words, it is necessary to find a criterion for gauge conditions to be admissible. A radical method to solve this problem is not to use any gauge conditions at all. In Sec. 2.1.4, we give a brief description of such an alternative gaugeless scheme inspired by the classical Hamiltonian reduction approach to construct the reduced phase space of systems possessing rigid symmetries.

2.1.4. The Hamiltonian reduction method for constrained systems. If the theory contains only Abelian constraints, one can find a parametrization of the reduced phase space as follows. According to a well-known theorem (see, e.g., [125]), it is always possible to find a canonical transformation to a new set of canonical coordinates

$$q_i \mapsto Q_i := Q_i\,(q,p)\,, \quad p_i \mapsto P_i := P_i\,(q,p) \tag{2.1.14}$$

such that m of the new momenta, say, $(\overline{P}_1, \ldots, \overline{P}_m)$, become equal to the Abelian constraints φ_α:

$$\overline{P}_\alpha = \varphi_\alpha(q,p). \tag{2.1.15}$$

In terms of the new coordinates $(\overline{Q}, \overline{P})$ and (Q^*, P^*), the canonical equations have the form

$$\begin{aligned} \dot{Q}^* &= \{Q^*, H_{\text{phys}}\}, \qquad & \dot{\overline{Q}} &= u(t), \\ \dot{P}^* &= \{P^*, H_{\text{phys}}\}, \qquad & \dot{\overline{P}} &= 0, \end{aligned} \tag{2.1.16}$$

with the physical Hamiltonian

$$H_{\text{phys}}(P^*, Q^*) \equiv H_C(P,Q)\Big|_{\overline{P}_\alpha = 0}, \tag{2.1.17}$$

where H_{phys} depends only on $(n-m)$ pairs of new gauge-invariant canonical coordinates (Q^*, P^*) and the form of the canonical system (2.1.16) expresses the explicit separation of the phase space into physical and unphysical sectors:

$$2n \left\{ \begin{pmatrix} q_1 \\ p_1 \\ \vdots \\ q_n \\ p_n \end{pmatrix} \right. \mapsto \begin{array}{ll} 2(n-m) \left\{ \begin{pmatrix} Q^* \\ P^* \end{pmatrix} \right. & \text{physical variables.} \\ \\ 2m \left\{ \begin{pmatrix} \overline{Q} \\ \overline{P} \end{pmatrix} \right. & \text{unphysical variables.} \end{array} \tag{2.1.18}$$

Arbitrary functions $u(t)$ enter into part of the system of equations containing only the ignorable coordinates $\overline{Q}_\alpha$ and momenta $\overline{P}_\alpha$. A straightforward generalization of this method to the non-Abelian case is impossible since the identification of momenta with constraints is forbidden due to the non-Abelian character of the constraints. However, there exists the possibility of replacement of the constraints φ_α by an equivalent set of new constraints Φ_α,

$$\Phi_\alpha = D_{\alpha\beta}\varphi_\beta, \qquad \det \|D\| \Big|_{\varphi=0} \neq 0, \tag{2.1.19}$$

describing the same surface Γ_c but forming an Abelian algebra. There are different proofs of this statement based on the resolution of constraints [58,117], exploiting gauge-fixing conditions [9], or using the direct method of constructing the Abelianization matrix as the solution of a certain system of linear first-order differential equations [48].[3] Therefore, for non-Abelian systems, the construction of the Abelianization matrix and the implementation of the above-mentioned transformation (2.1.14) to the new set of Abelian constraint functions Φ_α completes the reduction of the phase space without using gauge-fixing functions, solely in internal terms of the theory.

Before applying the Hamiltonian reduction method to the construction of the reduced phase space of Yang–Mills fields in $(3+1)$-dimensional space, it seems worth setting forth our approach to the same problem in $(0+1)$-dimensional space.

2.2. Example: $SU(2)$ Yang–Mills fields in $(0+1)$ dimensions. In order to explain our main idea on how to construct the physical variables, we start with the non-Abelian Christ–Lee model [23,106]. The Lagrangian of this model is

$$L := \frac{1}{2}\,(D_t x)_i (D_t x)_i - \frac{1}{2}\,V(x^2), \tag{2.2.1}$$

where x_i and y_i are the components of three-dimensional vectors and the covariant derivative D_t is defined by the formula

$$(D_t x)_i := \dot{x}_i + g\epsilon_{ijk} y_j x_k. \tag{2.2.2}$$

This model represents the Dirac–Yang–Mills theory in $(0+1)$-dimensional space-time and inherits its local gauge invariance in the form of $SO(3)$ gauge invariance.

Performing the Legendre transformations

$$p_y^i = \frac{\partial L}{\partial \dot{y}_i}, \quad p^i = \frac{\partial L}{\partial \dot{x}_i} = \dot{x}_i + g\epsilon^{ijk} y_j x_k,$$

we obtain the canonical Hamiltonian

$$H_C = \frac{1}{2} p_i p_i - \epsilon_{ijk} x_j p_k y_i + V(x^2), \tag{2.2.3}$$

and identify three primary constraints $p_y^i = 0$ as well as three secondary constraints

$$\Phi_i = \epsilon_{ijk} x_j p_k = 0 \tag{2.2.4}$$

[3] In all cases, the proofs use the large freedom in the canonical description of the constrained systems. Apart from the ordinary canonical transformations, there exist generalized canonical transformations [12], i.e., those which preserve the form of all constraints of the theory as well as the canonical form of the equations of motion. The Abelianization transformation (2.1.19) is of course noncanonical, but it belongs to this class of generalized canonical transformations.

forming the $SO(3)$ algebra

$$\{\Phi_i, \Phi_j\} = \epsilon_{ijk}\Phi_j. \tag{2.2.5}$$

One easily verifies that the secondary constraints are functionally dependent, $x_i\Phi_i = 0$. Now we perform the Abelianization procedure and choose

$$\Phi_1^{(0)} := x_2p_3 - x_3p_2, \quad \Phi_2^{(0)} := x_3p_1 - x_1p_3$$

as the two independent constraints forming the algebra

$$\{\Phi_1^{(0)}, \Phi_2^{(0)}\} = -\frac{x_1}{x_3}\,\Phi_1^{(0)} - \frac{x_2}{x_3}\,\Phi_2^{(0)}. \tag{2.2.6}$$

The general iterative scheme of the construction of the Abelianization matrix [48] consists of two steps for this simple case. First, let us exclude $\Phi_1^{(0)}$ from the right-hand side of Eq. (2.2.6). This can be achieved by performing the transformation

$$\Phi_1^{(1)} := \Phi_1^{(0)}, \quad \Phi_2^{(1)} := \Phi_2^{(0)} + C\Phi_1^{(0)},$$

where the function C satisfies the partial differential equation

$$\{\Phi_1^{(0)}, C\} = -\frac{x_2}{x_3}C + \frac{x_1}{x_3}.$$

Writing down a particular solution of this equation,

$$C(x) = \frac{x_1x_2}{x_2^2 + x_3^2},$$

we obtain the algebra for new constraints

$$\{\Phi_1^{(1)}, \Phi_2^{(1)}\} = -\frac{x_2}{x_3}\Phi_2^{(1)}.$$

Now let us perform the second transformation

$$\Phi_1^{(2)} := \Phi_1^{(1)}, \quad \Phi_2^{(2)} := B\Phi_2^{(1)},$$

where the function B satisfies the equation

$$\{\Phi_1^{(2)}, B\} = \frac{x_2}{x_3}B.$$

A particular solution of this equation is $B(x) = \frac{1}{x_3}$. As a result of the above two transformations, the Abelian constraints equivalent to the initial non-Abelian ones have the form

$$\begin{aligned}\Phi_1^{(2)} &= x_2p_3 - x_3p_2,\\ \Phi_2^{(2)} &= \frac{1}{x_3}\left[(x_3p_1 - x_1p_3) + \frac{x_1x_2}{x_2^2 + x_3^2}(x_2p_3 - x_3p_2)\right].\end{aligned} \tag{2.2.7}$$

Now we are ready to perform a canonical transformation to new variables so that the two new momenta will coincide with the Abelian constraints (2.2.7)[4]

$$p_\theta := \frac{(\vec{x}\cdot\vec{p})x_1 - \vec{x}^2p_1}{\sqrt{{x_2}^2 + {x_3}^2}}, \quad p_\phi := x_2p_3 - x_3p_2. \tag{2.2.8}$$

[4]Here we introduce the compact notation for the three-dimensional vectors $\vec{x}$ and $\vec{p}$ and multiply the constraint $\Phi_2^{(2)}$ by the factor $\sqrt{{x_2}^2 + {x_3}^2}$ to deal with constraints of the same dimension. This multiplication conserves the Abelian character of the constraints since $\{\Phi_1^{(2)}, \sqrt{{x_2}^2 + {x_3}^2}\} = 0$.

It is easy to verify that the contact transformation from the Cartesian coordinates to the spherical coordinates

$$
\begin{aligned}
x_1 &= r\cos\theta, \\
x_2 &= r\sin\phi\sin\theta, \\
x_3 &= r\cos\phi\sin\theta,
\end{aligned}
\qquad
\begin{aligned}
r &= \sqrt{{x_1}^2+{x_2}^2+{x_3}^2}, \\
\theta &= \arccos\frac{x_1}{\sqrt{{x_1}^2+{x_2}^2+{x_3}^2}}, \\
\phi &= \arctan\left(\frac{x_2}{x_3}\right),
\end{aligned}
\tag{2.2.9}
$$

is just the required transformation. Indeed, using the corresponding generating function

$$
F[\vec{x};\ p_r, p_\theta, p_\phi] = p_r\sqrt{{x_1}^2+{x_2}^2+{x_3}^2} + p_\theta\arccos\frac{x_1}{\sqrt{{x_1}^2+{x_2}^2+{x_3}^2}} + p_\phi\arctan\left(\frac{x_2}{x_3}\right),
$$

we obtain

$$
\begin{aligned}
p_1 &= \frac{\partial F}{\partial x_1} = p_r\cos\theta - p_\theta\frac{\sin\theta}{r}, \\
p_2 &= \frac{\partial F}{\partial x_2} = p_r\sin\theta\sin\phi + p_\theta\frac{\sin\phi\cos\theta}{r} + p_\phi\frac{\cos\phi}{r\sin\theta}, \\
p_3 &= \frac{\partial F}{\partial x_3} = p_r\sin\theta\cos\phi + p_\theta\frac{\cos\phi\cos\theta}{r} - p_\phi\frac{\sin\phi}{r\sin\theta},
\end{aligned}
$$

and convince ourselves that in terms of these new variables, two independent constraints $p_\theta = 0$ and $p_\phi = 0$ are in accordance with (2.2.8). It is worth noting here that, starting with the set of reducible constraints (2.2.4) and performing the above transformation (2.2.9), one obtains the representation

$$
\begin{aligned}
\Phi_1 &= -p_\phi, \\
\Phi_2 &= -p_\theta\cos\phi + p_\phi\sin\phi\cot\theta, \\
\Phi_3 &= p_\theta\sin\phi + p_\phi\cos\phi\cot\theta,
\end{aligned}
$$

adapted to the Abelianization. The corresponding Abelianization matrix for the reducible set of constraints is

$$
D := \frac{1}{d}\begin{pmatrix}
-d_2\sin\phi - d_3\cos\phi, & d_1\sin\phi, & d_1\cos\phi, \\
(d_2\cos\phi - d_3\sin\phi)\cot\theta, & -d_3 - d_1\cos\phi\cot\theta, & d_2 + d_1\sin\phi\cot\theta, \\
\cot\theta, & \sin\phi, & \cos\phi,
\end{pmatrix},
$$

with arbitrary $\vec{d}$ and $d := d_1\cot\theta + d_2\sin\phi + d_3\cos\phi$. This example demonstrates two important features of the Abelianization procedure:

(i) it is not necessary to work with an irreducible set of constraints since the Abelianization procedure automatically leads to an irreducible set of constraints;

(ii) in certain special coordinates, the problem of the solution of differential equations reduces to the solution of a simple algebraic problem.

In terms of the new canonical variables, the canonical Hamiltonian (2.2.3) becomes

$$
H_C = \frac{1}{2}p_r^2 + \frac{1}{2r^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - p_\phi y_\phi - p_\theta y_\theta + V(r) \tag{2.2.10}
$$

with the physical momentum

$$
p_r = \frac{(\vec{x}\cdot\vec{p})}{\sqrt{{x_1}^2+{x_2}^2+{x_3}^2}}
$$

and

$$
\begin{aligned}
y_\phi &:= y_1 + y_2\sin\phi + y_3\cos\phi\cot\theta, \\
y_\theta &:= y_2\cos\phi - y_3\sin\phi.
\end{aligned}
$$

As a result, all unphysical variables are separated from the physical variables r and p_r and their dynamics is governed by the physical Hamiltonian obtained from the canonical Hamiltonian by setting p_ϕ and p_θ in (2.2.10) equal to zero:

$$H_{\text{phys}} = \frac{1}{2}p_r^2 + V(r). \tag{2.2.11}$$

This simple example of a model with non-Abelian gauge symmetry shows that the reduced system has a nontrivial topological structure of a phase space [106] (the domain of the configuration variable r is $[0, \infty]$) and this fact leads to important consequences after the quantization of the system.

3. Unconstrained $SU(2)$ Yang–Mills Theory

In this section, we give a Hamiltonian formulation of classical $SU(2)$ Yang–Mills field theory entirely in terms of gauge-invariant variables and separate these variables into scalars under ordinary space rotations and into "rotational" degrees of freedom. It will be shown that this naturally leads to their identification as fields with "nonrelativistic spin 2 and spin 0." Furthermore, the separation into scalar and rotational degrees of freedom will turn out to be very well suited for the study of the infrared limit of unconstrained Yang–Mills theory.

3.1. Elimination of gauge degrees of freedom. The conventional Yang–Mills action for $SU(2)$ gauge fields $A_\mu^a(x)$

$$\mathcal{S}[A] := -\frac{1}{4}\int d^4x\ F^a_{\mu\nu}F^{a\mu\nu}, \qquad F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu \tag{3.1.1}$$

is degenerate. From the definition

$$P_a := \frac{\partial L}{\partial(\partial_0 A_0^a)}, \qquad E_{ai} := \frac{\partial L}{\partial(\partial_0 A_{ai})}$$

of the canonical momenta it follows that the phase space spanned by the variables (A_0^a, P^a) and (A_{ai}, E_{ai}) is restricted by three primary constraints $P^a(x) = 0$. In this case, according to the Dirac procedure, the evolution of the system is governed by the total Hamiltonian containing three arbitrary functions $\lambda_a(x)$:

$$H_T := \int d^3x \left[\frac{1}{2}\left(E^2_{ai} + B^2_{ai}(A)\right) - A_0^a\left(\partial_i E_{ai} + g\epsilon_{abc}A_{bi}E_{ci}\right) + \lambda_a(x)P^a(x)\right], \tag{3.1.2}$$

where $B_{ai}(A) := \epsilon_{ijk}\left(\partial_j A_{ak} + \frac{1}{2}g\epsilon_{abc}A_{bj}A_{ck}\right)$ is the non-Abelian magnetic field. From the conservation of the primary constraints $P^a = 0$ in time, one obtains the non-Abelian Gauss-law constraints

$$\Phi_a := \partial_i E_{ai} + g\epsilon_{abc}A_{ci}E_{bi} = 0. \tag{3.1.3}$$

Although the total Hamiltonian (3.1.2) depends on arbitrary functions $\lambda_a(x)$, it is possible to extract the dynamical variables which have uniquely predictable dynamics. Furthermore, they can be chosen to be free of any constraints. Such an extracted system with predictable dynamics without constraints is called unconstrained.

The non-Abelian character of the secondary constraints

$$\{\Phi_a(x), \Phi_b(y)\} = g\epsilon_{abc}\Phi_c(x)\delta(x-y) \tag{3.1.4}$$

is the main obstacle for the corresponding projection to the unconstrained phase space. One can try to proceed here in the same way as was explained for the finite-dimensional mechanical system. For Abelian constraints $\Psi_\alpha(\{\Psi_\alpha, \Psi_\beta\} = 0)$, the projection to the reduced phase space can be simply achieved in the following two steps. One performs a canonical transformation to new variables such that part of the new momenta P_α coincide with the constraints Ψ_α. After the projection onto the constraint shell, i.e., setting in all expressions $P_\alpha = 0$, the coordinates canonically conjugate to the P_α drop out from the physical quantities. The remaining canonical pairs are then gauge invariant and form the basis for the unconstrained system. For the case of non-Abelian constraints (3.1.4), in previous section the mechanism of conversion constraints into the Abelian form was described. However, in the case of field theory, this approach becomes very complicated because it is necessary in general to solve certain functional equations. Below, we show how it is possible to avoid this complicated step by a "clever" choice of the canonical variable. We show that one can find canonical coordinates such that the Abelianization procedure of constraints becomes a simple algebraic procedure.

3.1.1. Canonical transformation and Abelianization of the Gauss law. The problem of Abelianization is considerably simplified when studied in terms of coordinates adapted to the action of the gauge group. The knowledge of the $SU(2)$ gauge transformations

$$A_\mu \to A'_\mu = U^{-1}(x)\left(A_\mu - \frac{1}{g}\partial_\mu\right)U(x), \tag{3.1.5}$$

which leaves Yang–Mills action (3.1.1) invariant, directly prompts us with the choice of adapted coordinates by using the following point transformation to the new set of Lagrangian coordinates q_j $(j = 1, 2, 3)$ and six elements $S_{ik} = S_{ki}$ $(i, k = 1, 2, 3)$ of the positive-definite symmetric (3×3) matrix S

$$A_{ai}\left(q, S\right) := O_{ak}\left(q\right) S_{ki} - \frac{1}{2g}\epsilon_{abc}\left(O\left(q\right)\partial_i O^T\left(q\right)\right)_{bc}, \tag{3.1.6}$$

where $O(q)$ is an orthogonal (3×3)-matrix parameterized by q_i.[5] In what follows, we show that in terms of these variables, the non-Abelian Gauss law constraints (3.1.3) depend only on the q_i's and their conjugated momenta p_i's and after Abelianization become $p_i = 0$. The unconstrained variables S_{ik} and their conjugate P_{ik} are gauge invariant, i.e., commute with the Gauss law, and represent the basic variables for all observable quantities.[6] Transformation (3.1.6) induces a point canonical transformation linear in the new canonical momenta P_{ik} and p_i. Using the corresponding generating functional depending on the old momenta and the new coordinates,

$$F_3\left[E; q, S\right] := \int d^3z\; E_{ai}(z) A_{ai}\left(q(z), S(z)\right), \tag{3.1.7}$$

one can obtain the transformation to new canonical momenta p_j and P_{ik}

$$p_j(x) := \frac{\delta F_3}{\delta q_j(x)} = -\frac{1}{g}\Omega_{jr}\left(D_i(S) O^T E\right)_{ri}, \tag{3.1.8}$$

$$P_{ik}(x) := \frac{\delta F_3}{\delta S_{ik}(x)} = \frac{1}{2}\left(E^T O + O^T E\right)_{ik}, \tag{3.1.9}$$

where

$$\Omega_{ji}(q) := -\frac{i}{2}\operatorname{Tr}\left(O^T\left(q\right)\frac{\partial O\left(q\right)}{\partial q_j} J_i\right), \tag{3.1.10}$$

$(J_i)_{mn} := i\epsilon_{min}$ are the (3×3)-matrix generators of $SO(3)$, and the corresponding covariant derivative $D_i(S)$ in the adjoint representation

$$\left(D_i(S)\right)_{mn} := \delta_{mn}\,\partial_i - ig\left(J^k\right)_{mn} S_{ki}. \tag{3.1.11}$$

[5] In the strong coupling limit, representation (3.1.6) reduces to the so-called polar representation for arbitrary quadratic matrices for which the decomposition can be proven to be well defined and unique (see, e.g., [84]). In the general case, we have the additional second term which takes into account the inhomogeneity of the gauge transformation and (3.1.6) has to be regarded as a set of partial differential equations for q_i's. The uniqueness and regularity of the suggested transformation (3.1.6) depends on the imposed boundary conditions. In the present work, the uniqueness and regularity of the change of coordinates is assumed as a reasonable conjecture without a search for the appropriate boundary conditions.

[6] The freedom to use other canonical variables in the unconstrained phase space corresponds to another fixation of the six variables S in representation (3.1.6). This observation clarifies the connection with the conventional gauge-fixing method (see [24]).

A straightforward calculation based on linear relations (3.1.8) and (3.1.9) between the old and new momenta leads to the following expression for the field strengths E_{ai} in terms of the new canonical variables:

$$E_{ai} = O_{ak}\,(q)\left[P_{\,ki} + \epsilon_{kis}{}^{*}D^{-1}_{sl}(S)\left[\left(\Omega^{-1}p\right)_l - \mathcal{S}_l\right]\right], \tag{3.1.12}$$

where ${}^{*}D^{-1}$ is the inverse of the matrix operator

$${}^{*}D_{ik}(S) := -i\,(J^m D_m(S))_{ik} \tag{3.1.13}$$

and

$$\mathcal{S}_k(x) := \epsilon_{klm}\,(PS)_{lm} - \frac{1}{g}\partial_l P_{kl}. \tag{3.1.14}$$

Using representations (3.1.6) and (3.1.12), one can easily convince oneself that the variables S and P make no contribution to the Gauss law constraints (3.1.3):

$$\Phi_a = O_{as}(q)\Omega^{-1}_{sj}(q)p_j = 0. \tag{3.1.15}$$

Here and in (3.1.12), we assume that the matrix Ω is invertible. The equivalent set of Abelian constraints is

$$p_a = 0. \tag{3.1.16}$$

They are Abelian due to the canonical structure of the new variables.

3.1.2. Projection to the reduced phase space. After having rewritten the model in terms of the new canonical coordinates and after the Abelianization of the Gauss law constraints, the construction of the unconstrained Hamiltonian system is straightforward. In all expressions, we can simply set $p_a = 0$. In particular, the Hamiltonian in terms of the unconstrained canonical variables S and P can be represented by the sum of three terms:

$$H[S,P] = \frac{1}{2}\int d^3x\left[\mathrm{Tr}(P)^2 + \mathrm{Tr}(B^2(S)) + \frac{1}{2}\vec{E}^2(S,P)\right]. \tag{3.1.17}$$

The first term is the conventional quadratic "kinetic" part and the second term the "magnetic potential" term, which is the trace of the square of the non-Abelian magnetic field

$$B_{sk} := \epsilon_{klm}\left(\partial_l S_{sm} + \frac{g}{2}\epsilon_{sbc}S_{bl}S_{cm}\right). \tag{3.1.18}$$

It is interesting that after the elimination of the pure gauge degrees of freedom, the magnetic field strength tensor is the commutator of the covariant derivatives (3.1.11): $F_{ij} = [D_i(S), D_j(S)]$.

The third, nonlocal term in Hamiltonian (3.1.17) is the square of the antisymmetric part of the electric field (3.1.12), $E_s := (1/2)\epsilon_{sij}E_{ij}$, after projection onto the constraint surface. It is given as the solution of the system of differential equations[7]

$${}^{*}D_{ls}(S)E_s = g\mathcal{S}_l, \tag{3.1.19}$$

with the derivative ${}^{*}D_{ls}(S)$ defined in (3.1.13). Note that the vector $\mathcal{S}_i(x)$ defined in (3.1.14) coincides, up to divergence terms, with the spin density part of the Noetherian angular momentum, $S_i(x) := \epsilon_{ijk}A^a_j E_{ak}$, after transformation to the new variables and projection onto the constraint shell.[8] The solution $\vec{E}$ of

[7]Note that for the solution of this equation, we need to impose boundary conditions only on the physical variables S, in contrast to Eq. (3.1.6) for which boundary conditions only for the unphysical variables q_i are needed.

[8]Note that the presence of this divergence term destroys the $so(3)$ algebra of densities due to the presence of Schwinger terms

$$\{\mathcal{S}_i(x), \mathcal{S}_j(y)\} = \epsilon_{ijk}\mathcal{S}_k(x)\delta(x-y) + \epsilon_{ijs}P_{sk}(x)\partial^x_k\delta(x-y),$$

but maintains the value of spin and its algebra if one neglects the surface terms.

differential equation (3.1.19) can be expanded in a $(1/g)$-series. The zero-order term is

$$E_s^{(0)} = \gamma_{sk}^{-1}\epsilon_{klm}\,(PS)_{lm}\,, \tag{3.1.20}$$

where $\gamma_{ik} := S_{ik} - \delta_{ik}\,\mathrm{Tr}(S)$ and the first-order term is defined by the formula

$$E_s^{(1)} := \frac{1}{g}\gamma_{sl}^{-1}\left[(\mathrm{rot}\ \vec{E}^{(0)})_l - \partial_k P_{kl}\right] \tag{3.1.21}$$

from the zero-order term. The higher terms are then obtained by the simple recurrence relations

$$E_s^{(n+1)} := \frac{1}{g}\gamma_{sl}^{-1}(\mathrm{rot}\ \vec{E}^{\ (n)})_s. \tag{3.1.22}$$

One easily recognizes in these expressions the conventional definition of the covariant curl operation [86] in terms of the covariant derivative

$$\mathrm{curl}\,S(e_i,e_j) := \left\langle\nabla_{e_i}S, e_j\right\rangle - \left\langle\nabla_{e_j}S, e_i\right\rangle$$

calculated in the basis $e_i := \left(\gamma^{1/2}\right)_{ij}\partial_j$ and $\gamma_{ij} := \left\langle e_i, e_j\right\rangle$ with the corresponding connection $\nabla_{e_i}e_j = \Gamma^l_{ij}e_l$, e.g.,

$$E_{ij}^{(1)} = \mathrm{curl}\,S(e_i,e_j). \tag{3.1.23}$$

3.2. The reduced system in terms of scalar and rotational degrees of freedom. In the previous section, we have obtained the unconstrained Hamiltonian system in terms of physical fields represented by a positive-definite symmetric matrix S. The initial gauge fields A_i transform as vectors under spatial rotations. We would now like to study the transformation properties of the corresponding reduced matrix field S. For systems possessing some rigid symmetry, it is well known to be very useful for practical calculations to pass to a coordinate basis such that a subset of variables is invariant under the action of the symmetry group. In this section, we perform the explicit separation of the rotational degrees of freedom, which vary under rotations, from the scalars.

3.2.1. Transformation properties of the physical fields under space rotations. In order to search for a parametrization of the unconstrained variables in Yang–Mills theory adapted to the action of the group of spatial rotations, we study the corresponding transformation properties of the field S. The total Noetherian angular momentum vector for $SU(2)$ gluodynamics is

$$I_i = \epsilon_{ijk}\int d^3x\left(E_{aj}A_k^a + x_kE_{al}\frac{\partial A_{al}}{\partial x^j}\right). \tag{3.2.1}$$

After elimination of the gauge degrees of freedom, it reduces to

$$I_i = \int d^3x\epsilon_{ijk}\left((PS)_{jk} + x_k\,\mathrm{Tr}\left(P\partial_jS\right)\right), \tag{3.2.2}$$

where surface terms have been neglected.

Under infinitesimal rotations in 3-dimensional space, $\delta x_i = \omega_{ij}x_j$, generated by (3.2.2), the physical field S transforms as

$$\delta_\omega S_{ij} = \epsilon_{smn}\omega_{mn}\{S_{ij}, I_s\} = \omega_{mn}\left(\Sigma^{mn}S\right)_{ij} + \text{orbital part transf.} \tag{3.2.3}$$

with the matrices Σ^{mn}

$$(\Sigma)^{mn}_{(il)(sj)} := (\delta_{il}\delta^m_k\delta^n_s + \delta^m_i\delta^n_l\delta_{sj}) - (m\leftrightarrow n), \tag{3.2.4}$$

which describes the $SO(3)$ rotations of a 3-dimensional second-rank tensor field S

$$S'_{ik} = R_{il}(\omega)R_{km}(\omega)S_{lm}. \tag{3.2.5}$$

It is well known that any symmetric second-rank tensor can be decomposed into its irreducible components, one spin-0 component and five components of a spin-2 field by extraction of its trace [16]. On the other hand, it can be diagonalized via a main axis transformation, which corresponds to a separation of diagonal fields, which are invariant under rotations, from the rotational degrees of freedom. In the following sections, we investigate both representations and their relation to each other.

3.2.2. The unconstrained Hamiltonian in terms of spin-2 and spin-0 fields. As was shown in the previous section, six independent elements of the matrix field S can be represented as a mixture of fields with nonrelativistic spin 2 and spin 0. In order to put the theory into a more transparent form explicitly showing its rotational invariance, it is useful to perform a canonical transformation to the corresponding spin-2 and spin-0 fields as new variables. To achieve this, let us decompose the symmetric matrix S into the irreducible representations of the $SO(3)$ group

$$S_{ij}(x) = \frac{1}{\sqrt{2}} Y_A(x)\ T^A_{ij} + \frac{1}{\sqrt{3}} \Phi(x)\ I_{ij}, \tag{3.2.6}$$

where the field Φ is proportional to the trace of S as the spin-0 field and the 5-dimensional spin-2 vector $\mathbf{Y}(x)$ with components Y_A labeled by its value of spin along the z-axis, $A = \pm 2, \pm 1, 0$.[9] The symbol I denotes the unit (3×3)-matrix; five traceless basis (3×3)-matrices $\mathbf{T}^A$ are listed in the Appendix.

The momenta $P_A(x)$ and $P_\Phi(x)$ canonical conjugate to the fields $Y_A(x)$ and $\Phi(x)$ are the components of the corresponding expansion for the P variable

$$P_{ij}(x) = \frac{1}{\sqrt{2}} P_A(x)\ T^A_{ij} + \frac{1}{\sqrt{3}} P_\Phi(x)\ I_{ij}. \tag{3.2.7}$$

For the magnetic field B, we obtain the expansion

$$B_{ij}(x) = \frac{1}{\sqrt{2}} H_A(x)\ T^A_{ij} + \frac{1}{\sqrt{2}} h_\alpha(x)\ J^\alpha_{ij} + \frac{1}{\sqrt{3}} b(x)\ I_{ij} \tag{3.2.8}$$

with the components

$$H_A := \frac{1}{2} c^{(2)}_{A\beta B} \partial_\beta Y^B + \frac{g}{\sqrt{3}} \left(\frac{1}{\sqrt{2}} {}^*Y_A - \Phi Y_A \right), \tag{3.2.9}$$

$$h_\alpha := \frac{1}{2} d^{(1)}_{\alpha B\gamma} \partial_\gamma Y^B + \sqrt{\frac{2}{3}} \partial_\alpha\ \Phi, \tag{3.2.10}$$

$$b := \frac{g}{\sqrt{3}} (\frac{1}{2} \mathbf{Y}^2 - \Phi^2), \tag{3.2.11}$$

in terms of the structure constants $c^{(2)}_{A\beta C}$ and $d^{(1)}_{\alpha B\gamma}$ of the algebra of the spin-1 matrices J^α and the spin-2 matrices $\mathbf{T}^A$ listed in the Appendix, and another five-dimensional vector

$${}^*Y_C := d^{(2)}_{CAB} Y^A Y^B, \tag{3.2.12}$$

where the constants $d^{(2)}_{ABC}$ are explicitly given in the Appendix. Finally, we obtain the reduced Hamiltonian in terms of spin-2 and spin-0 field components

$$H[\mathbf{P}, \mathbf{Y}, P_\Phi, \Phi] := \frac{1}{2} \int d^3x\ \left(\mathbf{P}^2(x) + \vec{E}^2(x) + P^2_\Phi(x) + \mathbf{H}^2(x) + \vec{h}^2(x) + b^2(x) \right), \tag{3.2.13}$$

with expressions (3.2.9) for the magnetic-field components and the antisymmetric part $\vec{E}$ of the electric field given by (3.1.20)–(3.1.22), expressing S and P in terms of $\mathbf{Y}$, Φ and $\mathbf{P}$, P_Φ via (3.2.6) and (3.2.7).

[9] Everywhere below, 3-dimensional vectors are topped by an arrow and their Cartesian and spherical components are labeled by small Latin and Greek letters respectively, while the 5-dimensional spin-2 vectors are written in boldface and their "spherical" components labeled by capital Greek letters. For the lowering and raising of the indices of 5-dimensional vectors, the metric tensor $\eta_{AB} = (-1)^A \delta_{A,-B}$ is used.

In order to discuss the transformation properties of the spin-2 fields $\mathbf{Y}$ under spatial rotations, we rewrite angular-momentum vector (3.2.2) in terms of the fields $\mathbf{Y}, \mathbf{P}$ and Φ, P_Φ

$$I_i = S_i + \epsilon_{ijk} \int d^3x x_j (P_\Phi \partial_k \Phi + P_A \partial_k Y^A), \tag{3.2.14}$$

with the spin part

$$S_i = i(\mathbf{J}_i)_{AB} Y^A P^B, \tag{3.2.15}$$

where the three (5×5) matrices $\mathbf{J}^k$ are the elements of the $so(3)$ algebra. They are explicitly shown in the Appendix. The I_i's generate the transformation of the 5-dimensional vector Y under infinitesimal rotations $\delta x_i = \epsilon_{ijk}\omega_k x_j$ in the 3-dimensional space:

$$\delta_\omega Y^A = \omega_k \{Y^A, S_k\} = -i(\mathbf{J}^k)_{AB} Y^B. \tag{3.2.16}$$

For finite spatial rotations $R(\omega)$, we have

$$Y'_A = D_{AB}(\omega) Y_B, \tag{3.2.17}$$

where

$$D_{AB}(\omega) = \mathrm{Tr}(R(\omega)\mathbf{T}_A R^T(\omega)\mathbf{T}_B) \tag{3.2.18}$$

is the well-known 5-dimensional spin-2 D-function related to the orthogonal (3×3)-matrix $R(\omega)$ (see [16]). Transformation rule (3.2.17) is in accordance with (3.2.5).

Note that for a complete investigation of the transformation properties of the reduced matrix field S under the whole Poincaré group, one should also include the Lorentz transformations. But we restrict ourselves to the isolation of the scalars under spatial rotations and can treat S in terms of "nonrelativistic spin-0 and spin-2 fields" in accordance with the conclusions obtained in [60].

3.2.3. Separation of scalar and rotational degrees of freedom. In this section, we introduce a parametrization of the 5-dimensional $\mathbf{Y}$ field in terms of three Euler angles and two variables which are invariant under spatial rotations. Transformation property (3.2.17) prompts us with the parametrization

$$Y_A(x) = D_{AB}(\chi(x)) M^B(x) \tag{3.2.19}$$

in terms of the Euler angles $\chi_i = (\phi, \theta, \psi)$ and some 5-vector $\mathbf{M}$. The special choice

$$\mathbf{M}(\rho, \alpha) = \rho \left(-\frac{1}{\sqrt{2}} \sin\alpha,\ 0,\ \cos\alpha,\ 0,\ -\frac{1}{\sqrt{2}} \sin\alpha \right) \tag{3.2.20}$$

corresponds to the main-axis transformation of the original symmetric (3×3)-matrix field $S(x)$,

$$S(x) = R^T(\chi(x)) S_{\mathrm{diag}} \left(\phi_1(x), \phi_2(x), \phi_3(x)\right) R(\chi(x)), \tag{3.2.21}$$

where $D(\chi)$ is related with $R(\chi)$ by (3.2.18) and the rotational invariant variables Φ, ρ, and α are related with the diagonal elements ϕ_i by[10]

$$\begin{aligned}
\phi_1 &:= \frac{1}{\sqrt{3}}\Phi + \sqrt{\frac{2}{3}}\ \rho \cos\left(\alpha + \frac{2\pi}{3}\right), \\
\phi_2 &:= \frac{1}{\sqrt{3}}\Phi + \sqrt{\frac{2}{3}}\ \rho \cos\left(\alpha + \frac{4\pi}{3}\right), \\
\phi_3 &:= \frac{1}{\sqrt{3}}\Phi + \sqrt{\frac{2}{3}}\ \rho \cos\alpha.
\end{aligned} \tag{3.2.22}$$

[10]Similar variables were used as density and deformation variables in the collective model of Bohr in nuclear physics [14] and as a parametrization for the square of the eigenvalues of the rotational invariant part of the gauge field by [87] in the representation proposed in [113].

As was mentioned in the first part of the paper, the matrix S is symmetric and positive definite; therefore, the variables ϕ_i are positive:

$$\phi_i \geq 0, \quad i = 1, 2, 3, \tag{3.2.23}$$

and the domain of definition for the variables α and ρ can be taken as

$$0 \leq \rho \leq \sqrt{2}\Phi \quad \text{and} \quad \alpha \leq \frac{\pi}{3} \tag{3.2.24}$$

respectively. Therefore, the main-axis transformation of the symmetric second-rank tensor field S induces a parametrization of five spin-2 fields Y^A in terms of three rotational degrees of freedom, the Euler angles $\chi_i = (\psi, \theta, \phi)$, which describe the orientation of the "intrinsic frame," and two invariants ρ and α represented by the 5-vector $\mathbf{M}$. We can hence use either ρ, α, and the spin-0 field Φ, or the three fields ϕ_i $(i = 1, 2, 3)$ as three scalars under spatial rotations.

In what follows, we use main-axis representation (3.2.21). The momenta π_i and p_{χ_i} canonical conjugate to the diagonal elements ϕ_i and the Euler angles χ_i can easily be found using the generating function

$$F_3\left[\phi_i, \chi_i;\ P\right] := \int d^3x \, \mathrm{Tr}\,(SP) = \int d^3x \, \mathrm{Tr}\left(R^T(\chi) S_{\mathrm{diag}}(\phi) R(\chi) P\right) \tag{3.2.25}$$

as

$$\pi_i(x) = \frac{\partial F_3}{\partial \phi_i(x)} = \mathrm{Tr}\left(P R^T \overline{\alpha}_i R\right),$$
$$p_{\chi_i}(x) = \frac{\partial F_3}{\partial \chi_i(x)} = \mathrm{Tr}\left(\frac{\partial R^T}{\partial \chi_i} R\left[PS - SP\right]\right),$$

where $\overline{\alpha}_i$ are diagonal matrices with the elements $(\overline{\alpha}_i)_{lm} = \delta_{li}\delta_{mi}$. Together with the off-diagonal matrices $(\alpha_i)_{lm} = |\epsilon_{ilm}|$, they form an orthogonal basis for symmetric matrices, shown explicitly in the Appendix. The original physical momenta P_{ik} can then be expressed in terms of the new canonical variables as

$$P(x) = R^T(x)\left(\sum_{s=1}^{3} \pi_s(x)\overline{\alpha}_s + \frac{1}{\sqrt{2}}\sum_{s=1}^{3} \mathcal{P}_s(x)\alpha_s\right) R(x), \tag{3.2.26}$$

where

$$\mathcal{P}_i(x) := \frac{\xi_i(x)}{\phi_j(x) - \phi_k(x)} \quad \text{(cyclic permutation } i \neq j \neq k) \tag{3.2.27}$$

and the ξ_i's are given in terms of the Euler angles $\chi_i = (\psi, \theta, \phi)$ as

$$\xi_k(x) := \mathcal{M}(\theta, \psi)_{kl} p_{\chi_l},$$
$$\mathcal{M}(\theta, \psi) := \begin{pmatrix} \sin\psi/\sin\theta & \cos\psi & -\sin\psi\cot\theta \\ -\cos\psi/\sin\theta & \sin\psi & \cos\psi\cot\theta \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2.28}$$

Note that the ξ_i's are $SO(3)$ left-invariant Killing vectors satisfying the relations

$$\{\xi_i(x), \xi_j(y)\} = -\epsilon_{ijk}\xi_k(x)\delta(x - y).$$

The antisymmetric part $\vec{E}$ of the electric field appearing in the unconstrained Hamiltonian (3.1.17) is given by the following expansion in a $(1/g)$-series similar to (3.1.20)–(3.1.22):

$$E_i = R^T_{is} \sum_{n=0}^{\infty} \mathcal{E}_s^{(n)},$$

with the zero-order term

$$\mathcal{E}_i^{(0)} := -\frac{\xi_i}{\phi_j + \phi_k} \quad \text{(cyclic permutation } i \neq j \neq k), \tag{3.2.29}$$

the first-order term given from $\mathcal{E}^{(0)}$ via

$$\mathcal{E}_i^{(1)} := \frac{1}{g}\frac{1}{\phi_j + \phi_k}\left[\left((\nabla_{X_j}\vec{\mathcal{E}}^{(0)})_k - (\nabla_{X_k}\vec{\mathcal{E}}^{(0)})_j\right) - \Xi_i\right] \quad \text{(cyclic permutation } i \neq j \neq k), \tag{3.2.30}$$

and the higher-order terms of the expansion determined via the recurrence relations

$$\mathcal{E}_i^{(n+1)} := \frac{1}{g}\frac{1}{\phi_j + \phi_k}\left((\nabla_{X_j}\vec{\mathcal{E}}^{(n)})_k - (\nabla_{X_k}\vec{\mathcal{E}}^{(n)})_j\right). \tag{3.2.31}$$

The components of the covariant derivatives ∇_{X_k} in the direction of the vector field $X_i(x) := R_{ik}\partial_k$,

$$(\nabla_{X_i}\vec{\mathcal{E}})_b := X_i\mathcal{E}_b + \Gamma^d{}_{ib}\mathcal{E}_d, \tag{3.2.32}$$

are determined by the connection depending only on the Euler angles

$$\Gamma^b{}_{ia} := \left(RX_iR^T\right)_{ab}. \tag{3.2.33}$$

It is easy to verify that the connection $\Gamma^b{}_{ia}$ can be written in the form

$$\Gamma^b{}_{ia} = i(J^s)_{ab}(\mathcal{M}^{-1})_{sk}X_i\chi_k, \tag{3.2.34}$$

using the matrix $\mathcal{M}$ given in terms of the Euler angles $\chi_i = (\psi, \theta, \phi)$ in (3.2.28), which expresses the dual nature of the Killing vectors ξ_i in (3.2.28) and the Maurer–Cartan 1-forms ω^i defined by

$$RdR^T =: \omega^i J^i, \quad \omega^i = (\mathcal{M}^{-1})^i_k d\chi_k. \tag{3.2.35}$$

Finally, the source terms Ξ_k in (3.2.30) are given as

$$\Xi_1 = \Gamma^1{}_{22}(\pi_1 - \pi_2) + \frac{1}{2}X_1\pi_1 - \Gamma^2{}_{23}\mathcal{P}_2 - \Gamma^1{}_{23}\mathcal{P}_1 - 2\Gamma^1{}_{12}\mathcal{P}_3 + X_2\mathcal{P}_3 + (2 \leftrightarrow 3) \tag{3.2.36}$$

and its cyclic permutations Ξ_2 and Ξ_3.

Therefore, the unconstrained Hamiltonian takes the form

$$H = \frac{1}{2}\int d^3x \left(\sum_{i=1}^{3}\pi_i^2 + \frac{1}{2}\sum_{\text{cycl.}}\frac{\xi_i^2}{(\phi_j - \phi_k)^2} + \frac{1}{2}\vec{\mathcal{E}}^{\,2} + V\right), \tag{3.2.37}$$

where the potential term

$$V[\phi, \chi] = \sum_{i=1}^{3} V_i[\phi, \chi] \tag{3.2.38}$$

is the sum of

$$V_1[\phi, \chi] = \left(\Gamma^1{}_{12}\phi_{[12]} - X_2\phi_1\right)^2 + \left(\Gamma^1{}_{13}\phi_{[13]} - X_3\phi_1\right)^2 + \left(\Gamma^1{}_{23}\phi_{[13]} - \Gamma^1{}_{32}\phi_{[12]} + g\phi_2\phi_3\right)^2 \tag{3.2.39}$$

and its cyclic permutations, $\phi_{[ij]} = \phi_i - \phi_j$. We see that through the main-axis transformation of the symmetric second-rank tensor field S, the rotational degrees of freedom, the Euler angles χ, and their canonical conjugate momenta p_χ have been isolated from the scalars under spatial rotations and appear in the unconstrained Hamiltonian only via the three Killing vector fields ξ_k, the connections Γ, and the derivative vectors X_k.

4. $SU(2)$ Yang–Mills Fields and Calogero–Moser Systems

In this section, we discuss the correspondence between the dynamics of 3-particles with internal degrees interacting by pairwise $1/r^3$ forces on a line (Euler–Calogero–Moser system [45,126]) and $SU(2)$ Yang–Mills theory with spatially constant gauge fields ($SU(2)$ Yang–Mills mechanics ([3,30,54,65,89,116] and the references therein)).

The Euler–Calogero–Moser model is an extension of the famous Calogero–Sutherland–Moser models [18–20,93,118,119] (for generalizations, see [98,99,101,102]) with additional dynamical internal degrees of freedom included. It is interesting that these types of models arises in various areas of theoretical physics like the 2-dimensional Yang–Mills theory [13, 52, 75, 76, 90], black-hole physics [44], spin chain systems [56,112], generalized statistics [15,79,104], higher-spin theories [121], level dynamics for quantum systems [74], quantum Hall effect [5,64] and many others. An attractive feature of these generalizations is that they maintain the integrability property of the original Calogero–Sutherland–Moser system. For the general elliptic version of the Euler–Calogero–Moser system, action-angle-type variables were constructed and the equations of motion were solved in terms of Riemannian theta-functions [72]; the canonical symplectic form of this model is represented in terms of algebro-geometric data [6] using the general construction of Krichever and Phong [73].

In recent past years, a remarkable relation between the Calogero–Moser systems and the exact solutions of four-dimensional supersymmetric gauge theories was found [109, 110]. It was recognized that the so-called Seiberg–Witten spectral curves are identical to the spectral curves of the elliptic $SU(N)$ Calogero–Moser system [88]. Furthermore, the generalization of these relations to the $N = 2$ supersymmetric gauge theories with general Lie algebras and an adjoint representation of matter hypermultiplet were derived in [25–27] (for a review of the recent results, see, e.g., [28, 29, 59]).

Despite the existence of such a correspondence established on very general grounds, relations between gauge theories and integrable models are far from being understood.

Below, we point out a simple direct correspondence between the $SU(2)$ Yang–Mills theory and the Euler–Calogero–Moser model. This correspondence follows from the sequence of reductions of degrees of freedom thanks to different kinds of symmetries. First, based on the results obtained in the previous section, we restrict our consideration only to the zero-order derivative expansion of a nonlocal unconstrained Lagrangian of $SU(2)$ gluodynamics. We shall see that this approximation is equivalent to the supposition of the spatial homogeneity of gauge fields and reduces the Yang–Mills field theory to the 9-dimensional degenerate matrix Lagrangian model. After elimination of pure gauge degrees of freedom and rewriting the unconstrained matrix model in terms of special coordinates adapted to the action of rigid symmetry, one can arrive at the conventional form of the Euler–Calogero–Moser Hamiltonian. More precisely, we demonstrate that the unconstrained $SU(2)$ Yang–Mills mechanics represents the Euler–Calogero–Moser system of type ID_3, i.e., the inverse-square interacting 3-particle system with internal degrees of freedom related to the root system of the simple Lie algebra D_3 [98, 99, 102], and is embedded in a fourth-order external potential written in the superpotential form.

In addition, to this reduction due to the continuous symmetry of the system, we discuss another possibility of relating the Yang–Mills mechanics to higher-order matrix models using the discrete symmetries. We explore the method of constructing generalizations of the Calogero–Sutherland–Moser models elaborated recently by Polychronakos [105]. This method consists of using the appropriate reduction of the original Calogero model by a subset of its discrete symmetries to an invariant submanifold of the phase space. Representing the Euler–Calogero–Moser system with a special external potential as a symmetric (6×6)-matrix model, we show that such a matrix model, after projection onto the invariant submanifold of the phase space using a certain subset of discrete symmetries, is equivalent to the unconstrained $SU(2)$ Yang–Mills mechanics. Finally, we give a Lax-pair representation for the equations of motion of the $SU(2)$ Yang–Mills mechanics in the limit of the zero coupling constant. In this paper, we restrict our consideration to the classical level; one can find some results on quantization of Yang–Mills mechanics in [68] and the references therein.

4.1. Yang–Mills mechanics as an unconstrained matrix model. In this section, we show that the supposition of spatial homogeneity of the gauge connection $\mathbf{A}$ reproduces the results of zero-order derivative expansions for the nonlocal unconstrained Lagrangian derived in Sec. 3.

If we assume the spatial homogeneity

$$\mathcal{L}_{\partial_i}\mathbf{A} = 0 \tag{4.1.1}$$

of the gauge connection $\mathbf{A}$, then the action of the $SU(2)$ Yang–Mills theory reduces to the action for a finite-dimensional model, the so-called Yang–Mills mechanics (YMM) described by the degenerate matrix Lagrangian

$$L_{\mathrm{YMM}} = \frac{1}{2}\operatorname{tr}\left((D_tA)(D_tA)^T\right) - V(A). \tag{4.1.2}$$

The entries of the (3×3)-matrix A are nine spatial components $A_{ai} := A^a_i$ of the connection $\mathbf{A} := Y_a e_a dt + A_{ai} e_a dx^i$, where $e_a = \sigma_a/2i$, σ_a are the Pauli matrices, and D_t denotes the covariant derivative $(D_tA)_{ai} = \dot{A}_{ai} + g\varepsilon_{abc}Y_bA_{ci}$. Due to the spatial homogeneity condition (4.1.1), all dynamical variables Y_a and A_{ai} are functions only of time. The part of the Lagrangian corresponding to the self-interaction of the gauge fields is gathered in the potential $V(A)$:

$$V(A) = \frac{g^2}{4}\left(\operatorname{tr}^2(AA^T) - \operatorname{tr}(AA^T)^2\right). \tag{4.1.3}$$

To express the Yang–Mills mechanics in the Hamiltonian form, we define the phase space endowed with the canonical symplectic structure and spanned by the canonical variables (Y_a, P_{Y_a}) and (A_{ai}, E_{ai}), where

$$P_{Y_a} = \frac{\partial L}{\partial \dot{Y}_a} = 0, \quad E_{ai} = \frac{\partial L}{\partial \dot{A}_{ai}} = \dot{A}_{ai} + g\varepsilon_{abc}Y_bA_{ci}. \tag{4.1.4}$$

According to these definitions of the canonical momenta (4.1.4), the phase space is restricted to the three primary constraints

$$P^a_Y = 0 \tag{4.1.5}$$

and the evolution of the system is governed by the total Hamiltonian $H_T = H_C + u^a_Y(t)P^a_Y$, where the canonical Hamiltonian is given by the relation

$$H_C = \frac{1}{2}\operatorname{tr}(EE^T) + \frac{g^2}{4}\left(\operatorname{tr}^2(AA^T) - \operatorname{tr}(AA^T)^2\right) + gY_a\operatorname{tr}(J_aAE^T) \tag{4.1.6}$$

and the matrix $(J_a)_{bc}$ equals $(J_a)_{bc} = -\varepsilon_{abc}$. The conservation of constraints (4.1.5) in time entails the further condition on the canonical variables

$$\Phi_a = g\operatorname{tr}(J_aAE^T) = 0 \tag{4.1.7}$$

that reproduces the homogeneous part of the conventional non-Abelian Gauss-law constraints. They are the first-class constraints obeying the Poisson-bracket algebra

$$\{\Phi_a, \Phi_b\} = \varepsilon_{abc}\Phi_c. \tag{4.1.8}$$

In order to project onto the reduced phase space, we use the well-known polar decomposition

$$A_{ai}(\phi, Q) = O_{ak}(\phi)Q_{ki}, \tag{4.1.9}$$

for an arbitrary $(3{\times}3)$-matrix, where Q_{ij} is a positive-definite symmetric $(3{\times}3)$-matrix and $O(\phi_1, \phi_2, \phi_3) = e^{\phi_1 J_3}e^{\phi_2 J_1}e^{\phi_3 J_3}$ is an orthogonal matrix $O \in SO(3)$. Assuming that the matrix A_{ai} is nondegenerate, we can treat the polar decomposition as a uniquely invertible transformation from the configuration variables A_{ai} to the new set of six Lagrangian coordinates Q_{ij} and three coordinates ϕ_i. As follows from further consideration, the variables parametrizing the elements of the group $SO(3)$ (the Euler angles (ϕ_1, ϕ_2, ϕ_3)) are the pure gauge degrees of freedom.

The field strength E_{ai} in terms of the new canonical variables is

$$E_{ai} = O_{ak}(\phi)\left[P_{ki} + \varepsilon_{kil}(\gamma^{-1})_{lj}\left[\xi_j^L - S_j\right]\right], \tag{4.1.10}$$

where ξ_a^L are three left-invariant vector fields on $SO(3)$:

$$\xi_1^L = \frac{\sin\phi_3}{\sin\phi_2} P_1 + \cos\phi_3 P_2 - \cot\phi_2 \sin\phi_3 P_3,$$
$$\xi_2^L = \frac{\cos\phi_3}{\sin\phi_2} P_1 - \sin\phi_3 P_2 - \cot\phi_2 \cos\phi_3 P_3,$$
$$\xi_3^L = P_3,$$

$S_j = \varepsilon_{jmn}(PQ)_{mn}$ is the spin vector of the gauge field, and

$$\gamma_{ik} = Q_{ik} - \delta_{ik}\,\mathrm{tr}\,Q. \tag{4.1.11}$$

Reformulation of the theory in terms of these variables allows one to easily achieve the Abelianization of the secondary Gauss-law constraints. Using representations (4.1.9) and (4.1.10), one can convince oneself that the variables Q_{ij} and P_{ij} make no contribution to secondary constraints (4.1.7)

$$\Phi_a = O_{ab}(\phi)\xi_b^L = 0. \tag{4.1.12}$$

Hence, assuming that the matrix

$$M = \begin{pmatrix} \frac{\sin\phi_1}{\sin\phi_2} & \cos\phi_1 & -\sin\phi_1\cot\phi_2 \\ -\frac{\cos\phi_1}{\sin\phi_2} & \sin\phi_1 & \cos\phi_1\cot\phi_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1.13}$$

is nondegenerate, we find the set of Abelian constraints equivalent to the Gauss law (4.1.7):

$$\tilde{\Phi}_a = P_a = 0. \tag{4.1.14}$$

After having rewritten the model in this form, we are able to reduce the theory to physical phase space by a straightforward projection onto the constraint shell. The resulting unconstrained Hamiltonian defined as a projection of the total Hamiltonian onto the constraint shell

$$H_{\mathrm{YMM}} := H_C(Q_{ab}, P_{ab})\Big|_{P_a=0,\ P_Y^a=0} \tag{4.1.15}$$

can be written in terms of Q_{ab} and P_{ab} as follows:

$$H_{\mathrm{YMM}} = \frac{1}{2}\,\mathrm{tr}\,P^2 - \frac{1}{\det^2\gamma}\,\mathrm{tr}(\gamma\mathcal{M}\gamma)^2 + \frac{g^2}{4}\left(\mathrm{tr}^2 Q^2 - \mathrm{tr}\,Q^4\right), \tag{4.1.16}$$

where $\mathcal{M}_{mn} = (QP - PQ)_{mn}$ denotes the gauge-field spin tensor.

4.2. Yang–Mills mechanics as the motion of a particle on a stratified manifold. In the previous section, the unconstrained dynamics of the SU(2) Yang–Mills mechanics was identified with the dynamics of the nondegenerate matrix model (4.1.16). The configuration space $\mathcal{Q}$ of the real symmetric (3×3)-matrices can be endowed with the flat Riemannian metric

$$ds^2 = \mathrm{Tr}\left(dQ^2\right) \tag{4.2.1}$$

whose isometry group is formed by orthogonal transformations

$$Q' = RQR^T. \tag{4.2.2}$$

Since the unconstrained Hamiltonian system (4.1.16) is invariant under the action of this rigid group, we are interested in the structure of the orbit space given as a quotient $\mathcal{Q}/SO(3)$. Important information on the stratification of the space $\mathcal{Q}/SO(3)$ of orbits can be obtained from the so-called isotropy group of points of the configuration space which is defined as a subgroup of $SO(3)$ leaving point x invariant,

$RxR^T = x$. Orbits with the same isotropy group are collected into classes called *strata*. Therefore, as for the case of a symmetric matrix, the orbits are uniquely parametrized by the set of ordered eigenvalues $x_1 \leq x_2 \leq x_3$ of the matrix Q. One can classify the orbits according to the isotropy groups which are determined by the degeneracies of the matrix eigenvalues:

(1) *principal orbit-type strata*, if all eigenvalues are unequal, $x_1 < x_2 < x_3$, then the isotropy group $Z_2 \otimes Z_2$ is the smallest;
(2) *singular orbit-type strata* forming the boundaries of the orbit space with
 (a) two coinciding eigenvalues $x_1 = x_2$, $x_2 = x_3$, or $x_1 = x_3$; the isotropy group is $SO(2) \otimes Z_2$;
 (b) all three eigenvalues are equal, $x_1 = x_2 = x_3$, and the isotropy group coincides with the isometry group $SO(3)$.

In the subsequent sections, we show that the dynamics of the Yang–Mills mechanics which takes place on *principal* orbits is governed by the ID_3 Euler–Calogero-model Hamiltonian with the external potential $V^{(3)} := g^2/2 \sum\limits_{i<j} x_i^2 x_j^2$, while for singular orbits the corresponding system is either the A_2-Calogero model with the external potential $V^{(2)} := g^2/2(x^4 + 2x^2y^2)$ for *singular* orbits of type (a) or a one-dimensional system with quartic potential $V^{(1)} := 3/2g^2x^4$ for singular orbits of type (b).

4.2.1. Dynamics on principal orbit strata. To write down the Hamiltonian describing the motion on the principal orbit strata, we introduce the coordinates along the slices x_i and along the orbits χ. Namely, we decompose the nondegenerate symmetric matrix Q into the product

$$Q = \mathcal{R}^T(\chi_1, \chi_2, \chi_3)\mathcal{D}\mathcal{R}(\chi_1, \chi_2, \chi_3),$$

where $\mathcal{R}$ is an $SO(3)$-matrix parametrized by the Euler angles $\chi_i := (\chi_1, \chi_2, \chi_3)$ and $\mathcal{D} = \operatorname{diag}(x_1, x_2, x_3)$ is a diagonal matrix, and consider it as a point transformation from the physical coordinates Q_{ab} and P_{ab} to x_i, p_i and χ_i, p_{χ_i}. The Jacobian of this transformation is the relative volume of orbits

$$J := \left|\det\left\|\frac{\partial Q}{\partial x_k}, \frac{\partial Q}{\partial \chi_k}\right\|\right| = \prod_{i<k} \mid x_i - x_k \mid \tag{4.2.3}$$

and is regular for this stratum $x_1 < x_2 < x_3$.

By using the generating function

$$F\left[x_i, \chi_i;\ P\right] = \operatorname{tr}\left(QP\right) = \operatorname{tr}\left(\mathcal{R}^T(\chi)\mathcal{D}(x)\mathcal{R}(\chi)P\right), \tag{4.2.4}$$

the canonical conjugate momenta can be found in the form

$$p_i = \frac{\partial F}{\partial x_i} = \operatorname{tr}\left(P\mathcal{R}^T\overline{\alpha}_i\mathcal{R}\right), \qquad p_{\chi_i} = \frac{\partial F}{\partial \chi_i} = \operatorname{tr}\left(\mathcal{R}^T\frac{\partial \mathcal{R}}{\partial \chi_i}(PQ - QP)\right),$$

where $\overline{\alpha}_i$ are the diagonal terms of the orthogonal basis for the symmetric (3×3)-matrices used above and listed in the Appendix. Then the original physical momenta P_{ik} can be expressed in terms of the new canonical variables by the formula

$$P = \mathcal{R}^T\left(\sum_{s=1}^{3} \bar{\mathcal{P}}_s\overline{\alpha}_s + \sum_{s=1}^{3} \mathcal{P}_s\alpha_s\right)\mathcal{R}, \tag{4.2.5}$$

where $\bar{\mathcal{P}}_s = p_s$,

$$\mathcal{P}_i = -\frac{1}{2}\frac{\xi_i^R}{x_j - x_k} \quad \text{(cyclic permutation } i \neq j \neq k\text{)},$$

and the $SO(3)$ right-invariant Killing vectors are

$$\xi_1^R = p_{\chi_1},$$
$$\xi_2^R = -\sin\chi_1 \cot\chi_2\; p_{\chi_1} + \cos\chi_1\; p_{\chi_2} + \frac{\sin\chi_1}{\sin\chi_2}\; p_{\chi_3},$$
$$\xi_3^R = \cos\chi_1 \cot\chi_2\; p_{\chi_1} + \sin\chi_1\; p_{\chi_2} - \frac{\cos\chi_1}{\sin\chi_2}\; p_{\chi_3}.$$

They satisfy the Poisson-bracket algebra

$$\{\xi_a^R, \xi_b^R\} = \varepsilon_{abc}\xi_c^R. \tag{4.2.6}$$

Thus, finally, we obtain the following physical Hamiltonian defined on the unconstrained phase space:

$$H_{\mathrm{YMM}} = \frac{1}{2}\sum_{a=1}^{3} p_a^2 + \frac{1}{4}\sum_{a=1}^{3} k_a^2\xi_a^2 + V^{(3)}(x), \tag{4.2.7}$$

where

$$k_a^2 = \frac{1}{(x_b + x_c)^2} + \frac{1}{(x_b - x_c)^2} \qquad (\text{cyclic permutation } a \neq b \neq c) \tag{4.2.8}$$

and

$$V^{(3)} = \frac{g^2}{2}\sum_{a<b} x_a^2 x_b^2. \tag{4.2.9}$$

Note that the potential term in (4.2.9) has symmetry beyond the cyclic one. This fact allows us to write $V^{(3)}(x_1, x_2, x_3)$ in the form

$$V^{(3)}(x_1, x_2, x_3) = \frac{\partial W^{(3)}}{\partial x_a}\frac{\partial W^{(3)}}{\partial x_a}, \qquad a = 1, 2, 3, \tag{4.2.10}$$

with the superpotential $W^{(3)} = x_1 x_2 x_3$.

This completes our reduction of the spatially homogeneous Yang–Mills system to the equivalent unconstrained system describing the dynamics of the physical dynamical degrees of freedom. We see that the reduced Hamiltonian H_{YMM} on the principal orbit strata is exactly the Hamiltonian of the Euler–Calogero–Moser system of type ID_3, i.e., is of the inverse-square interacting three-particle system with internal degrees of freedom and related to the root system of the simple Lie algebra D_3 [98, 99, 102] embedded in the fourth-order external potential (4.2.10).

4.2.2. Dynamics on a singular stratum. Introduction of the additional constraints

$$x_1 - x_2 = 0$$

or

$$x_1 - x_2 = 0, \qquad x_1 - x_3 = 0$$

defines the invariant two- and one-dimensional strata.

One can repeat the above consideration for these singular strata and derive the following unconstrained Hamiltonians:

two-dimensional strata

$$H_{\mathrm{Sing}}^{(2)} = \frac{1}{2}p_x^2 + \frac{1}{4}p_y^2 + \frac{1}{4}\frac{l(l+1)}{(x-y)^2} + \frac{g^2}{2}(x^4 + 2x^2y^2), \tag{4.2.11}$$

where the constant $l(l+1)$ denotes the value of the square of the particle internal spin, and one-dimensional strata

$$H_{\mathrm{Sing}}^{(1)} = \frac{1}{6}p_x^2 + 3/2g^2x^4. \tag{4.2.12}$$

4.3. Yang–Mills mechanics and discrete reduction of the Euler–Calogero–Moser system. In this section, we demonstrate how the $SU(2)$ Yang–Mills mechanics arises from the higher-dimensional matrix model after projection onto a certain invariant submanifold determined by the discrete symmetries. To demonstrate this, it is useful to represent the Euler–Calogero–Moser system in the form of a nondegenerate matrix model.

4.3.1. Euler–Calogero–Moser system as a geodesic motion on symmetric matrices. Let us consider the Hamiltonian system with the phase space spanned by the $(N \times N)$-symmetric matrices X and P. The symplectic structure is given by the Poisson bracket

$$\{X_{ab}, P_{cd}\} = \frac{1}{2}\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right).$$

The Hamiltonian of the system defined as

$$H = \frac{1}{2}\operatorname{tr} P^2 \tag{4.3.1}$$

describes free motion in the matrix configuration space. The following statement holds: *The Hamiltonian* (4.3.1) *rewritten in special coordinates coincides with the Euler–Calogero–Moser Hamiltonian*

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} \frac{l_{ij}^2}{(x_i - x_j)^2} \tag{4.3.2}$$

with nonvanishing Poisson brackets for the canonical variables[11]

$$\{x_i, p_j\} = \delta_{ij} \quad \{l_{ab}, l_{cd}\} = \frac{1}{2}\left(\delta_{ac}l_{bd} - \delta_{ad}l_{bc} + \delta_{bd}l_{ac} - \delta_{bc}l_{ad}\right).$$

To find the adapted set of coordinates in which the Hamiltonian (4.3.1) coincides with the Euler–Calogero–Moser Hamiltonian (4.3.2), let us introduce new variables

$$X = O^{-1}(\theta)Q(q)O(\theta),$$

where the orthogonal matrix $O(q)$ is parametrized by $\frac{N(N-1)}{2}$ elements, e.g., the Euler angles $(\theta_1, \cdots, \theta_{\frac{N(N-1)}{2}})$, and $Q = \operatorname{diag}\|q_1, \dots, q_N\|$ denotes a diagonal matrix. This point transformation induces the canonical transformation, which we can obtain using the generating function

$$F_4 = \left[P, q_1, \dots, q_N, \theta_1, \dots, \theta_{\frac{N(N-1)}{2}}\right] = \operatorname{tr}[X(q,\theta)P]. \tag{4.3.3}$$

Using the representation

$$P = O^{-1}\left[\sum_{a=1}^{N} \bar{\alpha}_a \bar{P}_a + \sum_{i<j=1}^{\frac{N(N-1)}{2}} \alpha_{ij}P_{ij}\right]O, \tag{4.3.4}$$

where the matrices $(\bar{\alpha}_a, \alpha_{ij})$ form an orthogonal basis in the space of the symmetric $(N \times N)$-matrices under the scalar product

$$\operatorname{tr}(\bar{\alpha}_a\bar{\alpha}_b) = \delta_{ab}, \qquad \operatorname{tr}(\alpha_{ij}\alpha_{kl}) = 2\delta_{ik}\delta_{jl}, \qquad \operatorname{tr}(\alpha_a\alpha_{ij}) = 0, \tag{4.3.5}$$

[11] This system is the spin generalization of the Calogero–Moser model. Particles are described by their coordinates x_i and momenta p_i together with internal degrees of freedom of angular momentum type $l_{ij} = -l_{ji}$. A similar model was introduced in [45], where the internal degrees of freedom satisfy the following Poisson-bracket relations:

$$\{l_{ab}, l_{cd}\} = \delta_{bc}l_{ad} - \delta_{ad}l_{cb}.$$

one can obtain that $\bar{P}_a = p_a$ and the components P_{ab} are represented via the $O(N)$ right-invariant vector fields l_{ab}:

$$P_{ab} = \frac{1}{2}\frac{l_{ab}}{x_a - x_b}.$$

From this, it is clear that the Hamiltonian (4.3.1) coincides with the Euler–Calogero–Moser Hamiltonian (4.3.2).

The integration of the Hamilton equations of motion

$$\dot{X} = P, \qquad \dot{P} = 0$$

derived by using the Hamiltonian (4.3.1) gives the solution of the Euler–Calogero–Moser Hamiltonian system as follows: for the x-coordinates, we need to calculate the eigenvalues of the matrix $X = X(0) + P(0)t$, while the orthogonal matrix O, which diagonalizes X, determines the time evolution of internal variables.

Now, in order to find connections between the motion on the space of symmetric matrices and the Yang–Mills mechanics, let us consider the classical Hamiltonian system of N particles on a line with internal degrees of freedom embedded in an external field with the potential $V(x_1, x_2, \dots, x_N)$ and described by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} \frac{l_{ij}^2}{(x_i - x_j)^2} + V^{(N)}(x_1, x_2, \dots, x_N). \tag{4.3.6}$$

The particles are described by their coordinates x_i and momenta p_i together with the internal degrees of freedom of angular momentum type $l_{ij} = -l_{ji}$. The nonvanishing Poisson brackets are

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{l_{ab}, l_{cd}\} = \delta_{ac}l_{bd} - \delta_{ad}l_{bc} + \delta_{bd}l_{ac} - \delta_{bc}l_{ad}.$$

The external potential $V^{(N)}(x_1, x_2, \dots, x_N)$ is constructed in terms of the superpotential $W^{(N)}$:

$$V^{(N)}(x_1, x_2, \dots, x_N) = -\frac{1}{4}\sum_{a=1}^{N} \frac{\partial W^{(N)}}{\partial x_a}\frac{\partial W^{(N)}}{\partial x_a}, \tag{4.3.7}$$

where $W^{(N)}$ is given as[12]

$$W^{(N)} = i\sqrt{x_1 x_2 \dots x_N}. \tag{4.3.8}$$

Below it is useful to treat the internal degrees of freedom entering into the Hamiltonian (4.3.6) in the Cartesian form

$$l_{ab} = y_a \pi_b - y_b \pi_a, \tag{4.3.9}$$

where the internal variables y_a and π_a combine the canonical pairs with the canonical symplectic form. The Hamiltonian (4.3.6) has the following discrete symmetries [105]:

- parity P

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} -x_i \\ -p_i \end{pmatrix}, \qquad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} -y_i \\ -\pi_i \end{pmatrix}; \tag{4.3.10}$$

- permutation symmetry M

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} x_{M(i)} \\ p_{M(i)} \end{pmatrix}, \qquad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} y_{M(i)} \\ \pi_{M(i)} \end{pmatrix}, \tag{4.3.11}$$

where M is the element of the permutation group S_N.

[12] If we write the superpotential in the invariant form

$$W^{(N)} = i\sqrt{\det X},$$

where X is a symmetric $(N \times N)$-matrix whose eigenvalues are $x_1, x_2, \dots, x_N$, then the external potential is $V^{(N)}(x_1, x_2, \dots, x_N) = \det X\, \mathrm{tr}(X^{-2})$.

The manifold of the phase space defined as

$$x_a + x_{N-a+1} = 0, \quad p_a + p_{N-a+1} = 0, \tag{4.3.12}$$

$$y_a + y_{N-a+1} = 0, \quad \pi_a + \pi_{N-a+1} = 0 \tag{4.3.13}$$

is invariant under the action of the symmetry group $z = D(z)$, where

$$D = P \times M$$

and M is specified as $M(a) = N - a + 1$.

In order to project onto the manifold described by constraints (4.3.12)–(4.3.13), we use the Dirac method to deal with the second-class constraints. Let us introduce the Dirac brackets between the arbitrary functions F and G of all variables (x_a, p_a, y_a, π_a) as follows:

$$\{F, G\}_D = \{F, G\} - \{F, Z_a\}\{Z_a, Z_b\}^{-1}\{Z_b, G\},$$

where Z_a denote all second-class constraints $Z_a := (\chi_a, \Pi_a, \bar{\chi}_a, \bar{\Pi}_a)$, $a = 1, \cdots, N/2$:

$$\chi_a = \frac{1}{\sqrt{2}}\left(x_a + x_{N-a+1}\right), \qquad \bar{\chi}_a = \frac{1}{\sqrt{2}}\left(y_a + y_{N-a+1}\right), \tag{4.3.14}$$

$$\Pi_a = \frac{1}{\sqrt{2}}\left(p_a + p_{N-a+1}\right), \qquad \bar{\Pi}_a = \frac{1}{\sqrt{2}}\left(\pi_a + \pi_{N-a+1}\right) \tag{4.3.15}$$

with the canonical algebra

$$\{\chi_a, \bar{\chi}_b\} = \{\Pi_a, \bar{\Pi}_b\} = \{\chi_a, \bar{\Pi}_b\} = \{\bar{\chi}_a, \Pi_b\} = 0, \tag{4.3.16}$$

$$\{\chi_a, \Pi_b\} = \delta_{ab}, \qquad \{\bar{\chi}_a, \bar{\Pi}_b\} = \delta_{ab}. \tag{4.3.17}$$

Thus, the fundamental Dirac brackets are

$$\{x_a, p_b\}_D = \frac{1}{2}\delta_{ab}, \qquad \{y_a, \pi_b\}_D = \frac{1}{2}\delta_{ab}.$$

After the introduction of these new brackets, one can treat all constraints in the strong sense. Letting the constraint functions vanish, the system with Hamiltonian (4.3.6) reduces to the following:

$$H_{\text{red}} = \frac{1}{2}\sum_{a=1}^{\frac{N}{2}} p_a^2 + \frac{1}{2}\sum_{a \neq b}^{\frac{N}{2}} l_{ab}^2 k_{ab}^2 + \frac{g^2}{2}\sum_{a \neq b}^{\frac{N}{2}} x_a^2 x_b^2, \tag{4.3.18}$$

where

$$k_{ab}^2 = \frac{1}{(x_a + x_b)^2} + \frac{1}{(x_a - x_b)^2}.$$

Expression (4.3.18) for $N = 6$ coincides with the Hamiltonian of $SU(2)$ Yang–Mills mechanics after taking into account that after projection onto the constraint shell (CS) (4.3.14)–(4.3.15), potential (4.3.7) reduces to the potential of Yang–Mills mechanics

$$V^{(6)}(x_1, \cdots, x_6)\big|_{CS} = \frac{1}{2}\left(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2\right).$$

4.3.2. Lax pair for Yang–Mills mechanics in zero coupling limit. The conventional perturbative scheme of non-Abelian gauge theories starts with the zero approximation of the free theory. However, the limit of the zero coupling constant is not quite trivial. If the coupling constant in the initial Yang–Mills action vanishes, the non-Abelian gauge symmetry reduces to the $U(1) \times U(1) \times U(1)$ symmetry. In this section, we discuss this free theory limit for the case of the unconstrained Yang–Mills mechanics. The solution of the corresponding zero coupling limit of the Yang–Mills mechanics in the form of a Lax representation will be given. The relation between (4.3.6) and (4.3.18) allows one to construct the Lax pair for the free part of the Hamiltonian (4.3.18) ($g = 0$) by using the known Lax pair for the Euler–Calogero–Moser system (4.3.6) without an external potential term ($g = 0$).

According to [126], the Lax pair for the system with Hamiltonian

$$H_{\mathrm{ECM}} = \frac{1}{2}\sum_{a=1}^{N} p_a^2 + \frac{1}{2}\sum_{a\neq b}^{N} \frac{l_{ab}^2}{(x_a - x_b)^2} \tag{4.3.19}$$

is

$$\begin{aligned} L_{ab} &= p_a\delta_{ab} - (1-\delta_{ab})\frac{l_{ab}}{x_a - x_b}, \\ A_{ab} &= (1-\delta_{ab})\frac{l_{ab}}{(x_a - x_b)^2} \end{aligned}$$

and the equations of motion in the Lax form are

$$\dot{L} = [A, L], \quad \dot{l} = [A, l],$$

where the matrix $(l)_{ab} = l_{ab}$.

The introduction of Dirac brackets allows one to use the Lax pair of the higher-dimensional Euler–Calogero–Moser model (namely, A_6) for the construction of Lax pairs $(L_{\mathrm{YMM}}, A_{\mathrm{YMM}})$ of free Yang–Mills mechanics by performing the projection onto the constraint shell (4.3.14)–(4.3.15):

$$L^{\mathrm{ECM}}_{6\times 6}|_{CS} = L_{\mathrm{YMM}}, \qquad A^{\mathrm{ECM}}_{6\times 6}|_{CS} = A_{\mathrm{YMM}}.$$

Thus, the explicit form of the Lax pair matrices for free $SU(2)$ Yang–Mills mechanics is given by the following (6×6)-matrices:

$$L_{\mathrm{YMM}} = \left(\begin{array}{ccc|ccc} p_1 & -\frac{l_{12}}{x_1-x_2} & -\frac{l_{13}}{x_1-x_3} & \frac{l_{13}}{x_1+x_3} & \frac{l_{12}}{x_1+x_2} & 0 \\ -\frac{l_{12}}{x_1-x_2} & p_2 & -\frac{l_{23}}{x_2-x_3} & \frac{l_{23}}{x_2+x_3} & 0 & -\frac{l_{12}}{x_1+x_2} \\ -\frac{l_{13}}{x_1-x_3} & -\frac{l_{23}}{x_2-x_3} & p_3 & 0 & -\frac{l_{23}}{x_2+x_3} & -\frac{l_{13}}{x_1+x_3} \\ \hline \frac{l_{13}}{x_1+x_3} & \frac{l_{23}}{x_1+x_2} & 0 & -p_3 & -\frac{l_{23}}{x_2-x_3} & -\frac{l_{13}}{x_1-x_3} \\ \frac{l_{12}}{x_1+x_2} & 0 & -\frac{l_{23}}{x_2+x_3} & -\frac{l_{23}}{x_2-x_3} & -p_2 & -\frac{l_{12}}{x_1-x_2} \\ 0 & -\frac{l_{12}}{x_1+x_2} & -\frac{l_{13}}{x_1+x_3} & -\frac{l_{13}}{x_1-x_3} & -\frac{l_{12}}{x_1-x_2} & -p_1 \end{array}\right) \tag{4.3.20}$$

and

$$A_{\mathrm{YMM}} = \left(\begin{array}{ccc|ccc} 0 & \frac{l_{12}}{(x_1-x_2)^2} & \frac{l_{13}}{(x_1-x_3)^2} & -\frac{l_{13}}{(x_1+x_3)^2} & -\frac{l_{12}}{(x_1+x_2)^2} & 0 \\ -\frac{l_{12}}{(x_1-x_2)^2} & 0 & \frac{l_{23}}{(x_2-x_3)^2} & -\frac{l_{23}}{(x_2+x_3)^2} & 0 & \frac{l_{12}}{(x_1+x_2)^2} \\ -\frac{l_{13}}{(x_1-x_3)^2} & -\frac{l_{23}}{(x_2-x_3)^2} & 0 & 0 & \frac{l_{23}}{(x_2+x_3)^2} & \frac{l_{13}}{(x_1+x_3)^2} \\ \hline \frac{l_{13}}{(x_1+x_3)^2} & \frac{l_{23}}{(x_1+x_2)^2} & 0 & 0 & -\frac{l_{23}}{(x_2-x_3)^2} & -\frac{l_{13}}{(x_1-x_3)^2} \\ \frac{l_{12}}{(x_1+x_2)^2} & 0 & -\frac{l_{23}}{(x_2+x_3)^2} & \frac{l_{23}}{(x_2-x_3)^2} & 0 & -\frac{l_{12}}{(x_1-x_2)^2} \\ 0 & -\frac{l_{12}}{(x_1+x_2)^2} & -\frac{l_{13}}{(x_1+x_3)^2} & \frac{l_{13}}{(x_1-x_3)^2} & \frac{l_{12}}{(x_1-x_2)^2} & 0 \end{array}\right). \tag{4.3.21}$$

The equations of motion for $SU(2)$ Yang–Mills mechanics in the zero constant coupling limit read in a Lax form as follows:

$$\dot{L}_{\mathrm{YMM}} = [A_{\mathrm{YMM}}, L_{\mathrm{YMM}}], \qquad \dot{l}_{\mathrm{YMM}} = [A_{\mathrm{YMM}}, l_{\mathrm{YMM}}],$$

where

$$l_{\mathrm{YMM}} = \left(\begin{array}{ccc|ccc} 0 & l_{12} & l_{13} & -l_{13} & -l_{12} & 0 \\ -l_{12} & 0 & l_{23} & -l_{23} & 0 & l_{12} \\ -l_{13} & -l_{23} & 0 & 0 & l_{23} & l_{13} \\ \hline l_{13} & l_{23} & 0 & 0 & -l_{23} & -l_{13} \\ l_{12} & 0 & -l_{23} & l_{23} & 0 & -l_{12} \\ 0 & -l_{12} & -l_{13} & l_{13} & l_{12} & 0 \end{array}\right). \tag{4.3.22}$$

5. Unconstrained $SU(2)$ Yang–Mills Theory with Theta Angle

The gauge- and Poincaré-invariant action of Yang–Mills theory depends on two parameters, the coupling constant g and so-called θ-angle, as coefficients of the CP even part $S^{(+)}$

$$S^{(+)} = \frac{1}{2g^2} \int d^4x \, \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}, \tag{5.0.23}$$

and the CP odd part $S^{(-)}$

$$S^{(-)} = \frac{\theta}{32\pi^2} \int d^4x \, \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{5.0.24}$$

respectively. At the classical level, neither the value of the coupling constant nor that of the theta angle affect the observables, since the complete information for the description of the classical behavior of the gauge fields is coded entirely in the extremum of the action. If all components of the gauge potential entering the action are varied as independent variables, then the topological charge density term $Q(x) = \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$ can be discarded as a total divergence

$$\mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} = \partial_\mu K^\mu, \tag{5.0.25}$$

where

$$K^\mu = \varepsilon^{\mu\alpha\beta\gamma} \, \mathrm{tr} \left(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma \right) \tag{5.0.26}$$

is the Chern–Simons current and thus the extremal curves are independent of both the coupling constant and the theta angle.

Passing to the quantum theory, it is generally believed [17, 32, 61, 62] that the physical observables become dependent on theta. Although in perturbative calculations all diagrams with vertex $Q(x)$ vanish, nonperturbative phenomena such as tunneling between the above topologically distinct classical vacua, labeled by the integer value of the winding number functional

$$W[A] = \int d^3x K^0, \tag{5.0.27}$$

lead to the appearance of theta-vacua. Configurations with different winding number are related to each other by large gauge transformations reflecting the fact that the topological current K_μ is not gauge invariant.

Therefore, we pose here the question whether it is possible to express the Chern–Simons term in the classical action as a total divergence of a gauge invariant current using the unconstrained formulation of gauge theories. In the hope of obtaining such a representation of the Chern–Simons term, we would like to generalize the Hamiltonian reduction of classical $SU(2)$ Yang–Mills field theory to an arbitrary theta angle by including the CP odd part (5.0.24) of the action. We shall reformulate the original degenerate

Yang–Mills theory as an unconstrained nonlocal theory of self-interacting second-rank symmetric tensor fields.

Performing such a reduction in the presence of a total divergence term in the action, one can encounter the so-called *"divergence problem"* specific for the field theory with constraints, which has no analogues for finite-dimensional mechanical systems. This problem was first formulated explicitly in the context of the canonical reduction of general relativity.[13] Forty years later, Arnowitt, Deser, and Misner [2] gave a clear and vivid formulation of the phenomenon: *"a term which in the original Lagrangian (or Hamiltonian) is a pure divergence may cease to be a divergence upon elimination of the redundant variables and hence may contribute to the equations of motion obtained from the reduced Lagrangian (Hamiltonian)."* A simple ad hoc example from [2] explains the idea of this statement. Consider a theory where among the variables, there is a redundant variable satisfying the constraint

$$\nabla^2\Phi = \chi^2. \tag{5.0.28}$$

The term $\nabla^2\Phi$ added to the degenerate Lagrangian, being a divergence, has no influence on the classical equation of motion, while after projection onto the constraint shell, it appears as χ^2 and would contribute to equations of motion.

We demonstrate that the Hamiltonian reduction of $SU(2)$ Yang–Mills gauge theory is free of the above-mentioned "divergence problem" due to the Bianchi identities. Equivalence of constrained and unconstrained formulations of gauge theories on the classical level requires the demonstration of the agreement between reduced and original non-Abelian Lagrangian equations of motion. We explicitly construct the canonical transformation, well-defined on the reduced phase space, that eliminates the theta dependence of the classical equations of motion for the unconstrained variables.

5.1. Theta independence on the constrained level. First, let us review the case of the original constrained theory and demonstrate that under the special boundary conditions for the fields at spatial infinity (see Eq. (5.1.5) below), there exists a canonical transformation which completely eliminates the theta dependence from the classical degenerate theory.

5.1.1. Hamiltonian formulation of the constrained theory. Inclusion of the CP odd part of the action $S^{(-)}$ leads to the modification of the canonical momenta

$$\Pi_a = \frac{\partial L}{\partial \dot{A}_{a0}} = 0,$$
$$\Pi_{ai} = \frac{\partial L}{\partial \dot{A}_{ai}} = \frac{1}{g^2}\left(\dot{A}_{ai} - (D_i(A))_{ac}\, A_{c0}\right) - \frac{\theta}{8\pi^2}B_{ai},$$

where the covariant derivative D_i is

$$(D_i(A))_{mn} = \delta_{mn}\ \partial_i + (J^c)_{mn}\ \ A_{ci}, \tag{5.1.1}$$

where $(J_s)_{mn} := \epsilon_{msn}$ are (3×3)-matrix generators of the group $SO(3)$ and

$$B_{ai} = \varepsilon_{ijk}\left(\partial_j A_{ak} + \frac{1}{2}\epsilon_{abc}A_{bj}A_{ck}\right)$$

are non-Abelian magnetic fields. Independently of this modification, the phase space spanned by the variables (A_{a0}, Π_a) and (A_{ai}, Π_{ai}) is restricted to three primary constraints $\Pi_a(x) = 0$.

The canonical Hamiltonian is

$$H_C = \int d^3x \left[\frac{g^2}{2}\left(\Pi_{ai} + \frac{\theta}{8\pi^2}B_{ai}\right)^2 + \frac{1}{2g^2}B^2_{ai} + \Pi_{ai}\left(D_iA_0\right)_a\right], \tag{5.1.2}$$

[13] Presumably, the idea of the importance of the careful consideration of terms which are total spatial divergences goes back to Dirac in 1959, when he constructed the reduced Hamiltonian in general relativity as a certain surface integral at spatial infinity [34, 36, 37].

where we have used the fact that the topological charge density $Q(x) = \operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}$ can be rewritten in terms of the non-Abelian electric and magnetic fields as

$$Q = -\frac{1}{2\pi} E_{ai} B_{ai}. \tag{5.1.3}$$

The standard way in the Hamiltonian approach to proceed further is to perform a partial integration in the last term in expression (5.1.2) for the canonical Hamiltonian

$$\int_{V_R} d^3x \Pi_{ai} \left(D_i A_0\right)_a = -\int_{V_R} d^3x A_{a0} \left(D_i \Pi_i\right)_a + \oint_{\Sigma_R} d^2\sigma_i A_{a0} \Pi_{ai}, \tag{5.1.4}$$

where, according to the Gauss theorem, the surface integral is taken over the two-dimensional closed surface covering the three-dimensional volume V_R (for simplicity, we assume that it is a ball with radius R). Assuming that

$$\lim_{R\to\infty} \oint_{\Sigma_R} d^2\sigma_i A_{a0} \Pi_{ai} = 0, \tag{5.1.5}$$

we obtain the non-Abelian Gauss-law constraint

$$\left(D_i\right)_{ac} \Pi_{ic} = 0 \tag{5.1.6}$$

as the condition to maintain the primary constraints $\Pi_a = 0$ during the evolution. According to the Dirac prescription, the generator of time translation is the total Hamiltonian

$$H_T = \int d^3x \left[\frac{g^2}{2}\left(\Pi_{ai} + \frac{\theta}{8\pi^2} B_{ai}\right)^2 + \frac{1}{2g^2} B_{ai}^2 - A_{a0} D_i \Pi_{ai} + \lambda_a \Pi_a\right] \tag{5.1.7}$$

depending on three arbitrary functions $\lambda_a(x)$, and the Poisson bracket has the canonical structure

$$\begin{aligned} \{A_{ai}(\vec{x},t), \Pi_{bj}(\vec{y},t)\} &= \delta^{ab}\delta_{ij}\delta^3(\vec{x}-\vec{y}), \\ \{A_{a0}(\vec{x},t), \Pi_{b}(\vec{y},t)\} &= \delta^{ab}\delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{5.1.8}$$

5.1.2. Canonical transformation to constrained theory with $\theta = 0$. Based on representation (5.1.7) for the total Hamiltonian, one can immediately verify the equivalence of classical theories with different values of the parameter θ. To this end, let us perform the transformation to new coordinates A_{ai} and E_{bj}:

$$\begin{aligned} A_{ai}(x) &\to A_{ai}(x) = A_{ai}(x), \\ \Pi_{bj}(x) &\to E_{bj} = \Pi_{bj}(x) + \frac{\theta}{8\pi^2} B_{bj}(x). \end{aligned} \tag{5.1.9}$$

One can easily verify that this transformation is canonical, the new coordinates A_{ai} and E_{ai} satisfy the same canonical Poisson-bracket relations (5.1.8) as the original one. By virtue of the Bianchi identity

$$\epsilon^{\mu\nu\lambda\rho} D_\nu F_{\lambda\rho} = 0,$$

one can conclude that the θ-dependence completely disappears from Hamiltonian (5.1.7).

Note that canonical transformation (5.1.9) can be represented in the form

$$E_{ai} = \Pi_{ai} - \frac{\theta}{8\pi^2} \frac{\delta}{\delta A_{ai}} W[A], \tag{5.1.10}$$

where $W[A]$ denotes the winding number functional (5.0.27).

5.2. Theta independence on the unconstrained level. Now we derive the unconstrained version of Yang–Mills theory with theta angle and then give the analogue of transformation (5.1.9) after projection to the reduced phase space, thus checking the consistency of the unconstrained canonical formulation of Yang–Mills theory.

5.2.1. Hamiltonian formulation of the unconstrained theory. Now we follow the method developed in Sec. 3 for the CP even part of the gluodynamcs action and reduce the CP odd part similarly.

Let us perform the following point transformation to the new set of Lagrangian coordinates q_j $(j = 1, 2, 3)$ and six elements $S_{ik} = S_{ki}$ $(i, k = 1, 2, 3)$ of the positive-definite symmetric (3×3)-matrix S:

$$A_{ai}\left(q, S\right) = O_{ak}\left(q\right) S_{ki} - \frac{1}{2}\epsilon_{abc}\left(O\left(q\right)\partial_i O^T\left(q\right)\right)_{bc}, \tag{5.2.1}$$

where $O(q)$ is an orthogonal (3×3)-matrix parametrized by three fields q_i.

Again, as in Sec. 3, using the corresponding generating functional

$$F_3\left[\Pi; q, S\right] = \int d^3z\ \Pi_{ai}(z) A_{ai}\left(q(z), S(z)\right),$$

one can obtain the transformation to the new canonical momenta p_i and P_{ik}:

$$p_j(x) = \frac{\delta F_3}{\delta q_j(x)} = -\Omega_{jr}\left(D_i(Q) S^T \Pi\right)_{ri}, \tag{5.2.2}$$

$$P_{ik}(x) = \frac{\delta F_3}{\delta S_{ik}(x)} = \frac{1}{2}\left(\Pi^T O + O^T \Pi\right)_{ik}, \tag{5.2.3}$$

where

$$\Omega_{ji}(q) := -\frac{1}{2}\,\mathrm{Tr}\left(O^T\left(q\right)\frac{\partial O\left(q\right)}{\partial q_j} J_i\right). \tag{5.2.4}$$

The symplectic structure of new variables is encoded in the fundamental Poisson brackets[14]

$$\{S_{ij}(x), P_{kl}(y)\} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\delta^{(3)}(x - y). \tag{5.2.5}$$

From Eqs. (5.2.2) and (5.2.3), follows the expression for the old momenta Π_{ai} in terms of the new canonical variables:

$$\Pi_{ai} = O_{ak}\left(q\right)\Big[P_{\ ki} + \epsilon_{kis} P_s\Big]. \tag{5.2.6}$$

The vector P_s is a solution to the system of first-order partial differential equations

$$^*D_{ks}(S) P_s = -s_k(x) + \Omega^{-1}_{kl} p_l. \tag{5.2.7}$$

In (5.2.7), $^*D$ denotes the matrix operator

$$^*D_{ik}(S) = -i\left(J^m D_m(S)\right)_{ik} \tag{5.2.8}$$

and

$$s_k(x) = (D_i(S))_{lk} P_{il}.$$

Again, as in Sec. 3, we find that the new variables S and P make no contribution to the Gauss-law constraints (5.1.6)

$$O_{as}(q)\Omega^{-1}_{sj}(q) p_j = 0,$$

and the equivalent set of Abelian constraints is

$$p_a = 0.$$

The reduced Hamiltonian is defined as the projection of the total Hamiltonian to the constraint shell $p_a = 0$. In terms of the unconstrained canonical variables S and P, it reads

$$H^* = \int d^3x\left[\frac{g^2}{2}\left(P_{ai} + \frac{\theta}{8\pi^2} B_{(ai)}\right)^2 + \frac{g^2}{2}\left(P_a + \frac{\theta}{8\pi^2} B_a\right)^2 + \frac{1}{2g^2} B^2_{ai}\right], \tag{5.2.9}$$

[14] These new brackets take into account the symmetry constraints $S_{ij} = S_{ji}$ and $P_{kl} = P_{lk}$ and rigorously speaking are the Dirac brackets.

where $B_{(ai)}$ and B_a denote the symmetric tensor $B_{(ai)} = 1/2(B_{ai} + B_{ia})$ and vector $B_a = 1/2\epsilon_{abc}B_{bc}$ constructed from chromomagnetic field

$$B_{sk} = \epsilon_{klm}\left(\partial_l S_{sm} + \frac{1}{2}\epsilon_{sbc}S_{bl}S_{cm}\right).$$

The vector P_a representing the nonlocal term in Hamiltonian (5.2.9) is given as the solution to the system of differential equations

$$^*D_{ks}(S)P_s = -s_k(x), \tag{5.2.10}$$

which is the projection of Eqs. (5.2.7) to the constraint surface $p_a = 0$.

5.2.2. Canonical transformation to the unconstrained theory with $\theta = 0$. For the original degenerate action in terms of the fields A_μ, the equivalence of classical theories with arbitrary value of theta angle was reviewed in Sec. 5.1. Now let us examine the same problem for the derived unconstrained theory considering the analogue of canonical transformation (5.1.9) after projection onto the constraint surface:

$$\begin{aligned} S_{ai}(x) &\to S_{ai}(x) = S_{ai}(x), \\ P_{bj}(x) &\to E_{bj}(x) = P_{bj}(x) + \frac{\theta}{8\pi^2}B_{(bj)}(x). \end{aligned} \tag{5.2.11}$$

First, one can easily verify that this transformation to new variables S_{ai} and E_{bj} is canonical with respect to the Dirac brackets (5.2.5). Hamiltonian (5.2.9) in terms of the new variables S_{ai} and E_{bj} is therefore θ-independent. It has the form

$$H^* = \int d^3x\left[\frac{g^2}{2}E_{ai}^2 + \frac{g^2}{2}E_a^2 + \frac{1}{2g^2}B_{ai}^2\right], \tag{5.2.12}$$

where E_a is a solution to Eq. (5.2.10) with the replacement $P_{ai} \to E_{ai}$. This follows from the observation that if P_a is a solution to Eq. (5.2.10), then the expression

$$E_a = P_a + \frac{\theta}{8\pi^2}B_a$$

is a solution to the same equation with the replacement $P_{ai} \to E_{ai}$. This is indeed valid since the field B_{ai} satisfies the identity

$$^*D_{ks}(S)B_s = (D_i(S))_{kl}B_{(li)}. \tag{5.2.13}$$

Equation (5.2.13) is the Bianchi identity $(D_i)_{ab}B_{bi} = 0$ rewritten in terms of the symmetric $B_{(ai)}$ and antisymmetric B_a parts of the chromomagnetic field strength.

The reduced form of generating functional (5.0.27) corresponding to transformation (5.2.11) is the same functional W evaluated for the symmetric tensor S_{ik}. One can verify that the symmetric part of the magnetic field $B_{(ij)}(S)$ can be written as the functional derivative of this functional $W[S]$

$$\frac{\delta}{\delta S_{ij}(x)}W[S] = B_{(ij)}(x), \tag{5.2.14}$$

and thus the canonical transformation that eliminates the theta-dependence from the Hamiltonian can be represented in the same form as (5.1.10) with nine gauge fields A replaced by the six unconstrained fields $S_{ik}(x)$.

We have explored the question of theta-independence of classical unconstrained $SU(2)$ gluodynamics in order to construct the basis for passing to the quantum level. We showed that the exact projection of $SU(2)$ gluodynamics to the reduced phase space leads to an unconstrained system whose classical equations of motion are consistent with the original degenerate theory in the sense that they are theta-independent. The crucial point is that the fulfillment of this condition is due to properly taking into account the Bianchi identity for the magnetic field. As a consequence of the independence of the classical equations of motion of the gauge-invariant local fields, the parity odd term in the Yang–Mills action is a total divergence of some gauge-invariant current, in contrast to the original unconstrained theory, where it was the total divergence of the gauge-invariant Chern–Simons current K_μ. To deal practically with such a complicated nonlocal Hamiltonian as (5.2.9), one would have to use some approximation, since the exact solution to Eq. (5.2.10) is unknown. Implementing one approximating solution or another, it is desirable to be consistent with the theta-independence of classical theory.

6. The Infrared Limit of Unconstrained $SU(2)$ Gluodynamics

We obtain an effective low-energy theory involving only two of three rotational fields and one of tree scalar fields, and discuss its possible relation to the effective soliton Lagrangian proposed recently in [41].

6.1. The strong coupling limit of the theory. From expression (3.2.37) for the unconstrained Hamiltonian, one can analyze the classical system in the strong coupling limit up to order $O(1/g)$. Using the leading order (3.2.30) of $\vec{\mathcal{E}}$, we obtain the Hamiltonian

$$H_S = \frac{1}{2}\int d^3x \left(\sum_{i=1}^{3} \pi_i^2 + \sum_{\text{cycl.}} \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + V[\phi, \chi] \right). \tag{6.1.1}$$

For spatially constant fields, the integrand of this expression reduces to the Hamiltonian of $SU(2)$ Yang–Mills mechanics. For further investigation of the low-energy properties of $SU(2)$ field theory, a thorough understanding of the properties of the leading-order g^2 term in (3.2.38), containing no derivatives,

$$V_{\text{hom}}[\phi_i] = g^2[\phi_1^2\phi_2^2 + \phi_2^2\phi_3^2 + \phi_3^2\phi_1^2], \tag{6.1.2}$$

is crucial. The stationary points of potential term (6.1.2) are

$$\phi_1 = \phi_2 = 0, \quad \phi_3 \text{ is arbitrary}, \tag{6.1.3}$$

and its cyclic permutations. Analyzing the second-order derivatives of the potential at the stationary points, one can conclude that they form a continuous line of degenerate absolute minima at zero energy. In other words, the potential has a "valley" of zero-energy minima along the line $\phi_1 = \phi_2 = 0$. They are the unconstrained analogues of the toron solutions [83] representing constant Abelian field configurations with vanishing magnetic field in the strong coupling limit. The special point $\phi_1 = \phi_2 = \phi_3 = 0$ corresponds to the ordinary perturbative minimum.

In terms of the variables ρ, Φ, and α, homogeneous potential (6.1.2) becomes

$$V_{\text{hom}} := \frac{g^2}{3}\left(\Phi^4 + \frac{3}{4}\rho^4 - \sqrt{2}\Phi\rho^3\cos 3\alpha\right); \tag{6.1.4}$$

this shows that α parametrizes the strength of the coupling between the spin-0 and spin-2 fields. The valley of minima is given by $\rho = \sqrt{2}\Phi$, $\alpha = 0$, Φ arbitrary, and the perturbative vacuum by $\rho = \Phi = \alpha = 0$.

For the investigation of configurations of higher energy, it is necessary to include the part of the kinetic term in (6.1.1) containing the angular momentum variables ξ_i. Since the singular points of this term just correspond to the absolute minima of the potential, there will be a competition between an attractive and a repulsive force. At the balance point, we have a local minimum corresponding to a classical configuration with higher energy.

6.2. Nonlinear sigma-type effective model of $SU(2)$ gluodynamics. In this section, we describe the effective classical field theory to which the unconstrained theory reduces in the limit of infinite coupling constant g if we assume that the classical system spontaneously chooses one of the classical zero-energy minima of the leading order g^2 part (6.1.2) of the potential. As was discussed in the preceding section, these classical minima include, apart from the perturbative vacuum, where all fields vanish, also field configurations with one scalar field attaining arbitrary values. Without loss of generality, we set (explicitly breaking the cyclic symmetry)

$$\phi_1 = \phi_2 = 0, \quad \phi_3 \text{ is arbitrary},$$

such that potential (6.1.2) vanishes. In this case, the part of potential (3.2.38) containing derivatives takes the form

$$\begin{aligned} V_{\text{inhom}} = {} & \phi_3(x)^2 \left[(\Gamma^2{}_{13}(x))^2 + (\Gamma^2{}_{23}(x))^2 + (\Gamma^2{}_{33}(x))^2 + (\Gamma^3{}_{11}(x))^2 + (\Gamma^3{}_{21}(x))^2 + (\Gamma^3{}_{31}(x))^2 \right] \\ & + \left[(X_1\phi_3)^2 + (X_2\phi_3)^2 \right] + 2\phi_3(x) \left[\Gamma^3{}_{31}(x) X_1\phi_3 + \Gamma^3{}_{32}(x) X_2\phi_3 \right]. \end{aligned}$$

Introducing the unit vector

$$n_i(\phi, \theta) := R_{3i}(\phi, \theta)$$

directed along the 3-axis of the "intrinsic frame," one can write

$$V_{\text{inhom}} = \phi_3(x)^2 \, (\partial_i \vec{n})^2 + (\partial_i \phi_3)^2 - (n_i \partial_i \phi_3)^2 - (n_i \partial_i n_j) \partial_j (\phi_3^2).$$

Concerning the contribution from the nonlocal term in this phase, we obtain for the leading part of the electric fields

$$\mathcal{E}_1^{(0)} = -\xi_1/\phi_3, \quad \mathcal{E}_2^{(0)} = -\xi_2/\phi_3. \tag{6.2.1}$$

Since the third component $\mathcal{E}_3^{(0)}$ and $\mathcal{P}_3$ are singular in the limit $\phi_1, \phi_2 \to 0$, we necessarily have $\xi_3 \to 0$.

Hence we obtain the following effective Hamiltonian:

$$H_{\text{Eff}}^{(2)} = \frac{1}{2} \int d^3x \left[\pi^2 + \frac{1}{\phi^2}(\xi_i^2) + (\partial_i \phi)^2 + \phi^2 [(\partial_i \vec{n})^2 + (\vec{n} \cdot \text{rot } \vec{n})^2] - (n_i \partial_i \phi)^2 - (n_i \partial_i n_j) \partial_j (\phi^2) \right].$$

After the inverse Legendre transformation, we obtain the corresponding nonlinear sigma-model-type effective Lagrangian for the unit vector $\vec{n}(x)$ coupled to the scalar field $\phi(x)$:

$$L_{\text{Eff}}^{(2)}[\phi, \vec{n}] = \frac{1}{2} \int d^3x \left[(\partial_\mu \phi)^2 + \phi^2 \left((\partial_\mu \vec{n})^2 - (\vec{n} \cdot \text{rot } \vec{n})^2 \right) + (n_i \partial_i \phi)^2 + n_i (\partial_i n_j) \partial_j (\phi^2) \right]. \tag{6.2.2}$$

Thus, we reduce the $SU(2)$ Yang–Mills theory to an effective classical field theory involving only one scalar field and two of the three rotational fields χ_i summarized in the unit vector $\vec{n}$.

Note that this nonlinear sigma-model-type Lagrangian admits singular hedgehog configurations of the unit vector field $\vec{n}$. Due to the absence of a scale at the classical level, however, these are unstable. Consider, for example, the case of one static monopole placed at the origin:

$$n_i := x_i/r, \quad \phi_3 = \phi_3(r), \quad r := \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Minimizing its total energy

$$E[\phi_3] = 4\pi \int dr \phi_3^2(r)$$

with respect to $\phi_3(r)$, we find the classical solution $\phi_3(r) \equiv 0$. There is no scale in the classical theory. Only in a quantum investigation may a mass scale such as a nonvanishing value for the condensate $\langle 0|\hat{\phi}_3^2|0\rangle$ appear, which might be related to the string tension of flux tubes directed along the unit-vector field $\vec{n}(t, \vec{x})$. The singular hedgehog configurations of such string-like directed flux tubes might then be associated with the glueballs. Note that for the case of a spatially constant condensate,

$$\langle 0|\hat{\phi}_3^2|0\rangle =: 2m^2 = \text{const}, \tag{6.2.3}$$

the quantum effective action corresponding to (6.2.2) should reduce to the lowest-order term of the effective soliton Lagrangian discussed by Faddeev and Niemi

$$L_{\text{eff}}[\vec{n}] = m^2 \int d^3x (\partial_\mu \vec{n})^2 \tag{6.2.4}$$

(see [41]). As was discussed in [41], for the stability of these knots furthermore a higher-order skyrmion-like term in the derivative expansion of the unit-vector field $\vec{n}(t, \vec{x})$ is necessary.

7. Remarks on Quantization of the Unconstrained Theory

In this section, we discuss some relations between the quantization of the unconstrained Hamiltonian system and the corresponding extended system. We analyze the well-known exact, but nonnormalizable, solution [82] of the functional Schrödinger equation with zero energy in the framework of the unconstrained formulation of $SU(2)$ Yang–Mills theory.

7.1. Zero-energy state of a constrained $SU(2)$ gluodynamic. For the original constrained system of $SU(2)$ gluodynamics in terms of the gauge fields $A_i^a(x)$ with the Hamiltonian

$$\mathcal{H}(A) := \frac{1}{2}\int d^3x \, \left(- \left(\frac{\delta}{\delta A_i^a(x)} \right)^2 + B^2(x) \right)$$

and the Gauss-law operators

$$\mathcal{G}^a(x) := \left(\partial_i \delta_b^a - g\epsilon^{abc} A_i^c(x) \right) \frac{\delta}{\delta A_i^b(x)}$$

in the Schrödinger functional formalism, a physical state has to satisfy both the functional Schrödinger equation and the Gauss-law constraints:

$$\mathcal{H}\Psi[A] = E\Psi[A], \qquad \mathcal{G}^a(x)\Psi[A] = 0. \tag{7.1.1}$$

Remarkably, an exact solution for the wave functional $\Psi[A]$

$$\Psi[A] = \exp\left(-8\pi^2 W[A]\right) \tag{7.1.2}$$

(see [82]) can be represented in terms of the "winding number functional" $W[A]$ defined as the integral

$$W[A] := \int d^3x \ K_0(x)$$

of the zero component of the Chern–Simons secondary characteristic class vector (5.0.26) over the 3-space. Since $W[A]$ obeys the functional differential equation

$$\frac{\delta}{\delta A_i^a(x)} W[A] = B_i^a(x),$$

wave functional (7.1.2) satisfies the above Schrödinger equation. However, note that this exact solution for the functional Schrödinger equation with zero energy is known to be nonnormalizable and hence does not seem to have a physical meaning [61].

7.2. Zero-energy wave functional of an unconstrained system. Quantizing the variables S and P of the unconstrained Hamiltonian (5.2.9) similarly to A_i^a above,[15] we have

$$\mathcal{H} = \frac{1}{2}\int d^3x \, \left(- \left(\frac{\delta}{\delta S_{ij}(x)} \right)^2 + B^2(x) + \frac{1}{2}\vec{E}^2(S, \frac{\delta}{\delta S}) \right),$$

and hence the functional Schrödinger equation

$$\mathcal{H}\Psi[S] = E\Psi[S]. \tag{7.2.1}$$

The Gauss law has already been implemented by the reduction to the physical variables.

A corresponding exact zero-energy solution can indeed be found for our reduced Schrödinger equation (7.2.1). For this, we note the following two important properties of the potential terms in the Schrödinger equation (7.2.1). First, the reduced magnetic field $B_{ij}(S)$ can be written as the functional derivative

$$\frac{\delta}{\delta S_{ij}(x)} \mathcal{W}[S] = B_{(ij)}(x)$$

[15] Note that due to the positive-definiteness of the elements of the matrix field S, we have to solve the Schrödinger equation in a restricted domain of the functional space. Special boundary conditions have to be imposed on the wave functional such that all operators are well defined (e.g., the hermicity of the Hamiltonian).

of the functional $\mathcal{W}[S]$:

$$\mathcal{W}[S] := \frac{1}{32\pi^2}\int d^3x \left[\operatorname{Tr}(BS) - \frac{1}{12}g\left((\operatorname{Tr}(S^3) + \operatorname{Tr}^3(S) - 2\operatorname{Tr}(S)\operatorname{Tr}(S^2)\right)\right]. \tag{7.2.2}$$

Furthermore, the nonlocal term in the Schrödinger equation (7.2.1) annihilates $\mathcal{W}[S]$:

$$\vec{E}^2[S, \frac{\delta}{\delta S_{ij}(x)}]\mathcal{W}[S] = 0. \tag{7.2.3}$$

The last equation can easily be found to hold if one takes into account that the magnetic field $B_i = {}^*F_{0i}$ satisfies the Bianchi identity $D_i{}^*F_{0i} = 0$.

Thus, the corresponding ground-state wave-functional solution for the unconstrained Hamiltonian is

$$\Psi[S] = \exp\left(-8\pi^2\mathcal{W}[S]\right). \tag{7.2.4}$$

In order to investigate the relation of $\mathcal{W}[S]$ to the above winding number functional $W[A]$, we write the zero component of the Chern–Simons secondary characteristic class vector K^μ given in (5.0.26) in terms of the new variables S and q_i:

$$K^0(S,q) = \mathcal{K}^0(S) - \frac{1}{24\pi^2}\epsilon^{ijk}\left[\frac{2}{3}g\operatorname{Tr}\left(\Omega_i\Omega_j\Omega_k\right) - \partial_i\operatorname{Tr}\left(S_j\Omega_k\right)\right]. \tag{7.2.5}$$

The first term

$$\mathcal{K}^0(S) := -\frac{1}{16\pi^2}\epsilon^{ijk}\operatorname{Tr}\left(F_{ij}S_k - \frac{2}{3}gS_iS_jS_k\right)$$

is a functional only of the physical fiekd S of a form similiar to that of the original Chern–Simons secondary characteristic class vector. Here we have introduced the $SU(2)$ matrices $S_l := S_{li}\tau_i$, where τ_i are the Pauli matrices, and

$$\Omega_i(q) := \frac{1}{g}U^{-1}(q)\partial_i U(q) = \frac{1}{g}\Omega_{ls}(q)\tau^s\left(\frac{\partial q_l}{\partial x_i}\right),$$

where the $SU(2)$ matrices $U(q)$ are related with the orthogonal (3×3)-matrix $O(q)$ by the formula

$$O_{ab}(q) = \frac{1}{2}\operatorname{Tr}(O(q)\tau_a O^T(q)\tau_b)$$

and the (3×3)-matrix Ω_{ij} defined in (5.2.4).

We observe that the space integral of the first term coincides with the above functional $\mathcal{W}[S]$ of (7.2.2):

$$\int d^3x\mathcal{K}^0(S) = \mathcal{W}[S]. \tag{7.2.6}$$

Using the usual boundary condition[16]

$$U(q) \longrightarrow \pm I, \tag{7.2.7}$$

we see that the space integral of the second term is proportional to the natural number n representing the winding of the mapping of the compactified 3-space into $SU(2)$:

$$\frac{g^3}{24\pi^2}\int d^3x\ \epsilon^{ijk}\operatorname{Tr}(\Omega_i\Omega_j\Omega_k) = n.$$

Assume that the physical field S vanishes at spatial infinity; then there is no contribution from the third term. Hence we obtain the relation

$$\Psi[A] = \exp[-\frac{8\pi^2}{g^2}n]\Psi[S]$$

between the ground-state wave functional (7.1.2) of the extended quantization scheme and the reduced (7.2.4). We find that the winding number of the original gauge field A appears only as an unphysical

[16] Note that we have no information about the behavior of the unphysical variables q_i. For example, the requirement of the finiteness of the action usually used to fix the behavior of the physical fields does not apply for the unphysical field q_i.

normalization prefactor originating from the second term in (7.2.5), which depends only on the unphysical variables q_i. Furthermore, we note that the power $8\pi^2/g^2 n$ is the classical Euclidean action of $SU(2)$ Yang–Mills theory of self-dual fields [11] with winding number n.

On the other hand, the physical part $\Psi[S]$ of the wave function has the same unpleasant property as (7.1.2) in that it is nonnormalizable.

8. Concluding Remarks

Following the Dirac formalism for constrained Hamiltonian systems, several representations for the classical $SU(2)$ Yang–Mills gauge theory were considered entirely in terms of unconstrained gauge-invariant local fields. All used transformations, canonical transformations, and the Abelianization of the constraints maintain the canonical structures of the generalized Hamiltonian dynamics. The unconstrained field was identified with a symmetric, positive-definite, second-rank tensor field under spatial rotations. Its decomposition into irreducible representations under spatial rotations leads to two fields, a five-dimensional vector field $\mathbf{Y}(x)$ and a scalar field $\Phi(x)$. Their dynamics is governed by an explicitly rotational invariant nonlocal Hamiltonian. It is distinct from the local Hamiltonian obtained by Goldstone and Jackiw [51] and by Izergin et al. [60]. They used the so-called electric-field representation with vanishing antisymmetric part of the electric field. A representation for the Hamiltonian with a nonlocal interaction of the unconstrained variables similar to ours was derived in [113] based on another separation of scalar and rotational degrees of freedom. However, the present separation of the unconstrained fields into scalars under spatial rotations and rotational degrees of freedom leads to a simpler form of the Hamiltonian, which, in particular, is free of operator-ordering ambiguities in the strong coupling limit. Our unconstrained representation of the Hamiltonian allows us to derive an effective low-energy Lagrangian for the rotational degrees of freedom coupled to one of the scalar fields suggested by the form of the classical potential in the strong coupling limit. The dynamics of the rotational variables in this limit is summarized by the unit vector describing the orientation of the intrinsic frame. Due to the absence of a scale in the classical theory, the singular hedgehog configuration of the unit vector field is found to be unstable classically. In order to obtain a nonvanishing value for the vacuum expectation value for one of the three scalar-field operators, which would set a scale, a quantum treatment at least to one loop order is necessary and is under present investigation. For the case of a spatially constant scalar quantum condensate, we expect to obtain the first term of a derivative expansion proposed recently by Faddeev and Niemi [41]. As was shown in their work, such a soliton Lagragian allows for stable massive knotlike configurations, which might be related to glueballs. For the stability of the knots, higher-order terms in the derivative expansion, such as the Skyrme-type fourth-order term in [41], are necessary.

In conclusion, it is also necessary to emphasize that the presented approach for studying the low-energy aspects of non-Abelian gauge theories directly in terms of the physical unconstrained fields offers an alternative to the variational calculations based on the gauge-projection method [71].

The reason for trying to construct the physical variables entirely in internal terms without the use of any gauge fixing is the aspiration to maintain all local and global properties of the initial gauge theory.[17] Several questions in connection with the global aspects of the reduction procedure are arising at this point. In the paper, we describe how to project $SU(2)$ Yang–Mills theory onto the constraint shell defined by the Gauss law. It is well known that the exponentiation of infinitesimal transformations generated by the Gauss-law operator can lead only to homotopically trivial gauge transformations continuously deformable to unity. However, the initial classical action is invariant under all gauge transformations, including the homotopically nontrivial transformations. What trace does the existence of large gauge transformations leave on the unconstrained system?

[17]A discussion of rich local and global geometric structures in gauge theories is beyond the scope of this review; for an introduction, the reader can see [4, 7, 46, 78, 91, 92, 95, 114, 115] and the references therein.

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Appendix A. Notation and Some Formulas

A.1. $SU(2)$ **gluodynamics conventions and notations.** We define $SU(2)$ gluodynamics in the Minkowski space-time as the gauge theory using the field strength 2-form

$$F = dA + A \wedge A,$$

in terms of the antihermitian $su(2)$-valued one-form $A = A^a \tau^a / 2i$, where τ_a are standard Pauli matrices satisfying the commutation relations $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$, ϵ_{abc} are $SU(2)$ structure constants, and $\mathrm{Tr}\,(\tau_a \tau_b) = 2\delta_{ab}$.

In the coordinate basis, the components of non-Abelian field strength are

$$F^{a}_{\cdot\ \mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu, \tag{A.1.1}$$

The notation for the electric and magnetic field strength is

$$E^a_{.\,i} = F^a_{.\,0i}, \qquad B^a_{.\,i} = \frac{1}{2}\varepsilon_{ijk} F^a_{.\,jk}.$$

The dual field strength tensor is normalized as

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.$$

In the text, we often use the following matrix notation:

$$A_{ai} := A^a_{.\,i}, \quad E_{ai} := E^a_{.\,i}, \quad B_{ai} := B^a_{.\,i}.$$

The expression for the covariant derivative in adjoint representation and the dual derivative are as follows:

$$(D_i(S))_{ac} = \delta_{ac}\partial_i + \varepsilon_{abc} S_{bi}, \qquad (*D(S))_{ma} := \varepsilon_{ain}\,(D_i(S))_{mn}\,.$$

A.2. Pontryagin's invariant. The topological charge density $Q(x) = \mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$ expressed through the Chern–Simons current $Q(x) = \partial_\mu K^\mu$ can also be represented as the exterior derivative

$$Q = dC$$

of a certain so-called Chern 3-form C defined by the gauge 1-form A:

$$C = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

In the coordinate basis, its components $C = \frac{1}{3!} C_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$ are

$$C_{\alpha\beta\gamma} = \frac{1}{16\pi^2}\left(A_\alpha \overset{\leftrightarrow}{\partial_\beta} A_\gamma - A_\beta \overset{\leftrightarrow}{\partial_\alpha} A_\gamma + A_\gamma \overset{\leftrightarrow}{\partial_\beta} A_\alpha + 2\epsilon_{abc} A^a_\alpha A^b_\beta A^c_\gamma\right).$$

The Chern–Simons current K_μ is Hodge-dual to C:

$$K^\mu = \frac{1}{3!}\epsilon^{\mu\nu\rho\sigma} C_{\nu\rho\sigma}$$

Thus, the CP odd part of the action is given as the integral of the 3-form C over the boundary of the manifold:

$$S^{(-)} = \int_{\partial M} C$$

A.3. Spin-1 matrices and eigenvectors. For generators of spin-1 matrices obeying the algebra $[J_i, J_j] = i\epsilon_{ijk}\ J_k$, we use the following matrix realizations:

$$J_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, the representation of rotations $\mathrm{R}(\chi)$ in terms of the Euler angles $\chi = (\theta, \psi, \phi)$ is

$$R(\psi, \theta, \phi) = e^{-i\psi J_3} e^{-i\theta J_1} e^{-i\phi J_3}.$$

The eigenfunctions of J_2 and J_3 are

$$\vec{e}_{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \qquad \vec{e}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{e}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix};$$

these vectors are orthogonal with respect to the metric $\eta_{\alpha\beta} := (-1)^\alpha \delta_{\alpha,-\beta}$,

$$(\vec{e}_\alpha \cdot \vec{e}_\beta) = \eta_{\alpha\beta},$$

and satisfy the completeness condition

$$e^i_\alpha e^j_\beta \eta^{\alpha\beta} = \delta^{ij}.$$

A.4. Spin-0, spin-1, and spin-2 tensor basis. To obtain a matrix representation for spin-0, spin-1, and spin-2 basis matrices, we use the Clebsch–Gordan decomposition for the direct product of spin-1 eigenvectors e^α_i into irreducible components: $3 \otimes 3 = 0 \oplus 1 \oplus 2$. To distinguish between the matrices corresponding to the different spins, we use boldface notation for spin 2.

We have for spin-0

$$I_0 := \frac{1}{\sqrt{3}}\left(\vec{e}_0 \otimes \vec{e}_0 - \vec{e}_1 \otimes \vec{e}_{-1} - \vec{e}_{-1}\ \otimes \vec{e}_1\right) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for spin-1

$$J^+ := (\vec{e}_0\ \otimes \vec{e}_{+1} - \vec{e}_{+1} \otimes \vec{e}_0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix},$$

$$J^- := (\vec{e}_{-1}\ \otimes \vec{e}_0 - \vec{e}_0 \otimes \vec{e}_{-1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix},$$

$$J^0 := (\vec{e}_{-1}\ \otimes \vec{e}_1 - \vec{e}_1 \otimes \vec{e}_{-1}) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and for spin-2

$$\mathbf{T}_{+2} = \sqrt{2}\,(\vec{e}_{+1}\otimes\vec{e}_{+1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{T}_{-2} = \sqrt{2}\,(\vec{e}_{-1}\otimes\vec{e}_{-1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{T}_{+1} := (\vec{e}_{+1}\otimes\vec{e}_{0} + \vec{e}_{0}\otimes\vec{e}_{+1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ -1 & -i & 0 \end{pmatrix},$$

$$\mathbf{T}_{-1} := (\vec{e}_{-1}\otimes\vec{e}_{0} + \vec{e}_{0}\ \otimes\vec{e}_{-1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix},$$

$$\mathbf{T}_{0} := \frac{1}{\sqrt{3}}\,(\vec{e}_{+1}\otimes\vec{e}_{-1} + 2\vec{e}_{0}\otimes\vec{e}_{0} + \vec{e}_{-1}\ \otimes\vec{e}_{+1}) = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

These matrices satisfy the orthonormality relations

$$\mathrm{Tr}(\mathbf{T}_A\mathbf{T}_B) = 2\eta_{AB}, \qquad \mathrm{Tr}(\mathbf{T}_A J_\alpha) = 0, \qquad \mathrm{Tr}(J_\alpha J_\beta) = 2\eta_{\alpha\beta},$$

the completeness condition

$$\frac{1}{10}\sum_A (\mathbf{T}_A)_{il}(\mathbf{T}_A)_{km} + (I_0)_{il}(I_0)_{km} = \frac{1}{4}\left(\delta_{im}\delta_{lk} + \delta_{il}\delta_{mk}\right),$$

and the commutation and anticommutation relations

$$[\mathbf{T}_A, \mathbf{T}_B]_+ = \frac{4}{\sqrt{3}}\eta_{AB} I_0 + \frac{2}{\sqrt{3}} d^{(2)}_{ABC}\mathbf{T}^C, \qquad [\mathbf{T}_A, \mathbf{T}_B]_- = c^{(2)}_{AB\gamma} J^\gamma;$$

$$[J_\alpha, J_\beta]_+ = \frac{4}{\sqrt{3}}\eta_{\alpha\beta} I_0 + d^{(1)}_{\alpha\beta C}\mathbf{T}^C, \qquad [J_\alpha, J_\beta]_- = c^{(1)}_{\alpha\beta\gamma} J^\gamma;$$

$$[J_\alpha, \mathbf{T}_B]_+ = d^{(1)}_{\alpha\gamma B} J^\gamma, \qquad [J_\alpha, \mathbf{T}_B]_- = c^{(2)}_{BD\alpha}\mathbf{T}^D.$$

The coefficients $c^{(1)}_{\alpha\beta\gamma}$ are totally antisymmetric and $c^{(1)}_{-+0} = 1$. The coefficients $d^{(1)}_{\alpha\beta C}$, $d^{(2)}_{ABC}$, and $c^{(2)}_{AB\gamma}$ are given in the following tables:

A	-2	-2	-2	-1	-1	-1	-1	0	0	0	0	0
B	2	1	0	2	1	0	-1	2	1	0	-1	-2
C	0	1	2	-1	0	1	2	-2	-1	0	1	2
$d^{(2)}_{ABC}$	-1	$\sqrt{\frac{3}{2}}$	-1	$\sqrt{\frac{3}{2}}$	$-1/2$	$-1/2$	$\sqrt{\frac{3}{2}}$	-1	$-\frac{1}{2}$	1	$-\frac{1}{2}$	-1

A	1	1	1	1	2	2	2
B	1	0	-1	-2	0	-1	-2
C	-2	-1	0	1	-2	-1	0
$d^{(2)}_{ABC}$	$\sqrt{\frac{3}{2}}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\sqrt{\frac{3}{2}}$	-1	$\sqrt{\frac{3}{2}}$	-1

A	-2	-2	-1	-1	-1	0	0	0	1	1	1	2	2
B	2	1	2	1	0	1	0	-1	0	-1	-2	-1	-2
γ	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0
$c^{(2)}_{AB\gamma}$	-2	$-\sqrt{2}$	$-\sqrt{2}$	1	$-\sqrt{3}$	$-\sqrt{3}$	0	$\sqrt{3}$	$\sqrt{3}$	-1	$\sqrt{2}$	$\sqrt{2}$	2

α	-1	-1	-1	0	0	0	1	1	1
β	1	0	-1	1	0	-1	1	0	-1
C	0	1	2	-1	0	1	-2	-1	0
$d^{(1)}_{\alpha\beta C}$	$-1/\sqrt{3}$	1	$-\sqrt{2}$	1	$-2/\sqrt{3}$	1	$-\sqrt{2}$	1	$-1/\sqrt{3}$

Note that

$$d^{(1)}_{abA} = (\mathbf{T}_A)^{\alpha\beta}\, e^\alpha_a e^\alpha_b.$$

A.5. Generators for D-functions. In the text, the following five-dimensional spin matrices were used:

$$\left(\mathbf{J}^+\right)^B_A = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \left(\mathbf{J}^-\right)^B_A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 \end{pmatrix},$$

$$\left(\mathbf{J}^0\right)^B_A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

The corresponding Cartesian components $\left(\mathbf{J}^i\right)^B_A := \eta^{\alpha\beta} e^i_\alpha \left(\mathbf{J}_\beta\right)^B_A$ are

$$\left(\mathbf{J}^1\right)^B_A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \left(\mathbf{J}^2\right)^B_A = i \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & -\sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & -\sqrt{3/2} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\left(\mathbf{J}^3\right)^B_A = \left(\mathbf{J}^0\right)^B_A;$$

they compose the algebra $SO(3)$

$$[\mathbf{J}_a, \mathbf{J}_b] = i\epsilon_{abc}\mathbf{J}_c.$$

Note that

$$c^{(2)}_{ABc} = i(\mathbf{J}^c)_{AB}.$$

We use D-functions as representation of rotations in 3-space defined in terms of the Euler angles $\chi = (\theta, \psi, \phi)$:

$$D(\psi, \theta, \phi) = e^{-i\psi\mathbf{J}_3} e^{-i\theta\mathbf{J}_1} e^{-i\phi\mathbf{J}_3}.$$

They can be obtained from the corresponding 3-dimensional representation (see [16]) by the formula

$$D(\chi)_{AB} = \frac{1}{2}\operatorname{Tr}\left(R(\chi)\mathbf{T}_A R^T(\chi)\mathbf{T}_B\right).$$

A.6. Basis for symmetric matrices. We use the orthogonal basis $\alpha_A = (\overline{\alpha}_i, \alpha^i)$ for symmetric matrices. They have the form

$$\overline{\alpha}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \overline{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \overline{\alpha}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\alpha^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \alpha^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \alpha^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and satisfy the orthonormality relations

$$\mathrm{tr}\,(\overline{\alpha}_i\overline{\alpha}_j) = \delta_{ij}, \qquad \mathrm{tr}\,(\alpha_i\alpha_j) = 2\delta_{ij}, \qquad \mathrm{tr}\,(\overline{\alpha}_i\alpha_j) = 0.$$

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