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ON THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS OF SMIRNOV CLASSES IN DOUBLY-CONNECTED DOMAINS

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An overwhelming number of works are devoted to the study of the Dirichlet boundary value problem for harmonic functions under different assumptions regarding the boundary properties of unknown and given functions. The case when the function is required to be the real part of analytic functions from Smirnov classes is of special interest. The Dirichlet problem in such a statement has been investigated thoroughly in the case of simply connected domains (see, for e.g., [1–3]).

In the present work we present the results obtained by us for the Dirichlet problem in doubly-connected domains.

1^0. Let $\gamma_1 = \{z : |z| = 1\}$ and $\gamma_2 = \{z : |z| = \rho\}$, $0 < \rho < 1$ be circumferences bounding the ring $K = \{z : \rho < |z| < 1\}$, and let $\omega(z)$ $(\neq 0)$, $z \in K$ be an analytic function prescribed on K.

Definition 1. We say that a single-valued analytic in the ring K function $\phi(z)$ belongs to the Smirnov class $E^p(K;\omega)$, if

$$\sup_{\rho<r<1} \int\limits_0^{2\pi} \left|\omega(re^{i\vartheta})\phi(re^{i\vartheta})\right|^p d\vartheta < \infty. \tag{1}$$

Put

$$E^p(K) \equiv E^p(K;1),$$
$$e^p(K;\omega) = \big\{u : u = \operatorname{Re}\phi,\ \phi \in E^p(K;\omega)\big\}, \quad e^\rho(K) \equiv e^\rho(K;1). \tag{2}$$

Relying on the results from [4], we can easily state the following

Proposition 1. *Let $\phi \in E^p(K;\rho)$, then:*

(i) *if $\omega(z)$ has almost everywhere on $\gamma = \gamma_1 \cup \gamma_2$ angular boundary values ω^+, then there likewise exists $\phi^+(t)$, where $\phi^+ \in L^p(\gamma,\omega)$;*

(ii) *if $p \geq 1$, then*

$$\phi(z) = \frac{1}{\omega(z)}\big(\phi_1(z) + \phi_2(z)\big), \quad z \in K \tag{3}$$

where

$$\phi_i(z) = \frac{1}{2\pi}\int\limits_{\gamma_i} \frac{\omega(t)\phi^+(t)}{t-z}\,dt, \quad z \in K_i,$$
$$K_1 = \big\{z : |z| < 1\big\}, \quad K_2 = \big\{z : |z| > \rho\big\}.$$

(iii) *if $p > 1$ and $\frac{1}{\omega} \in E^{p'}(K)$, $p' = \frac{p}{p-1}$, then*

$$\phi(z) = \frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{\varphi_1(t)dt}{t-z} + \frac{1}{2\pi i}\int\limits_{\gamma_2} \frac{\varphi_2(t)dt}{t-z}, \quad z \in K, \quad \varphi_i \in L^p(\gamma_i;\omega_i), \tag{4}$$

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where ω_i *is the narrowing on* γ_i *of the function* ω*, i.e.,* $\omega_1(e^{i\vartheta}) = \omega^+(e^{i\vartheta})$*,* $\omega_2(e^{i\vartheta}) = \omega^+(\rho e^{i\vartheta})$*.*

Definition 2. Let Γ be a rectifiable simple curve and ω be the given function, finite almost everywhere and different from zero. We say that ω belongs to the class $W^p(\Gamma)$, if the operator

$$T_\Gamma : f \to \omega S_\Gamma \frac{f}{\omega}, \quad \text{where } \Big(\omega S_\Gamma \frac{f}{\omega}\Big)(t) = \frac{\omega(t)}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\omega(\tau)} \frac{d\tau}{\tau - t}, \quad t \in \Gamma,$$

is bounded in the space $L^p(\Gamma)$.

Assume

$$W^p \equiv W^p(\gamma_1),$$

$$W^p_E = \Big\{\omega : \omega \in \bigcup_{\delta>0} E^\delta(K),\, \omega_i \in W^p\, \omega_1 = \omega^+(e^{i\vartheta}),\, \omega_2 = \omega^+(\rho e^{i\vartheta})\Big\}.$$

Proposition 2. *If* $\omega \in W^p_E$*,* $p > 1$ *then the function*

$$F(z) = \frac{1}{2\pi i} \int\limits_{\gamma_1} \frac{f_1(t)dt}{t - z} + \frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{f_2(t)dt}{t - z}, \quad z \in K, \quad f_i \in L^p(\gamma_i; \omega_i) \tag{5}$$

belongs to the class $E^p(K; \omega)$*.*

From Propositions 1 and 2 we obtain the following

Theorem 1. *If* $\omega \in W^p_E$*, then the class* $E^p(K; \omega)$ *coincides with the class of functions* $F(z)$ *representable by formula* (5)*.*

Proposition 3. *If* $\omega \in W^p_E$ *and* $u \in e^p(K; \omega)$*,* $p > 1$ *then there exist the functions* $\mu \in L^p(I; \omega_1)$ *and* $\lambda \in L^p(I; \omega_2)$*,* $I = [0, 2\pi]$*, such that the equalities*

$$u(z) = u(re^{i\varphi}) = \frac{1}{2\pi} \int\limits_0^{2\pi} \mu(\vartheta) \frac{1 - r^2}{1 + r^2 - 2r\cos(\vartheta - \varphi)}\, d\vartheta +$$

$$+ \frac{1}{2\pi} \int\limits_0^{2\pi} \lambda(\vartheta) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\vartheta - \varphi)}\, d\vartheta, \tag{6}$$

$$\int\limits_0^{2\pi} \lambda(\vartheta)\, d\vartheta = 0, \tag{7}$$

hold.

2⁰. Consider the Dirichlet problem which is formulated as follows: Find a function u, satisfying the following conditions:

$$\begin{cases} u \in e^p(K; \omega), & p > 1, \\ u^+(e^{i\vartheta}) = f(\vartheta), & u^+(\rho e^{i\vartheta}) = g(\vartheta), \quad f \in L^p(I; \omega_1), \quad g \in L^p(I; \omega_2), \end{cases} \tag{8}$$

where $u^+(e^{i\vartheta})$ and $u^+(\rho e^{i\vartheta})$ are angular boundary values of the function $u(z)$ on γ_1 and γ_2, respectively. They are defined almost everywhere on I, and the equalities in (8) are understood almost everywhere as well.

Proceeding from equalities (6) and (7) and using Theorem 1, the problem (8) is reduced to the following system of singular integral equations considered in the class

$L^p(I;\omega_1) \times L^p(I;\omega_2)$:

$$\begin{cases} \mu(\vartheta) - \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \lambda(\alpha)\, \dfrac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\, d\alpha = f(\vartheta), \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mu(\alpha)\, \dfrac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\, d\alpha - \lambda(\vartheta) = g(\vartheta), \end{cases} \tag{9}$$

with the condition (7) which is imposed on the function λ.

Excluding λ from that system, we obtain in the class $L^p(I;\omega_1)$ the equation

$$\mu(\vartheta) - \frac{1}{2\pi} \int_0^{2\pi} \mu(\alpha) \frac{1-\rho^4}{1+\rho^4-2\rho^2\cos(\alpha-\vartheta)}\, d\alpha = \nu(\vartheta), \tag{10}$$

where

$$\nu(\vartheta) = f(\vartheta) - \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\, d\alpha. \tag{11}$$

If $\omega_2 \in W^p$, then $\frac{1}{\omega_2} \in L^{p'}(I)$. Therefore $L^p(I;\omega_2) \subset L^1(I)$, and hence the function $\frac{1}{2\pi}\int_0^{2\pi} g(\alpha) \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\, d\alpha$ is bounded on I and, consequently, $\nu \in L^p(I;\omega_1)$.

The integral equation (10) has a symmetric continuous kernel, and therefore it is of Fredholm type in the space $L^p(I;\omega_1)$. It is proved that for $\nu \equiv 0$ the solutions of that equation are only the constant functions, and hence the associated homogeneous equation in the corresponding class of functions has only constant solutions. This implies that equation (10) is solvable if and only if

$$\int_0^{2\pi} \nu(\vartheta)\, d\vartheta = 0, \tag{12}$$

and when this condition is fulfilled, all its solutions are given by the equality

$$\mu = \widetilde{\mu} + c, \tag{13}$$

where $\widetilde{\mu}$ is a particular solution of equation (10), and c is an arbitrary constant.

Taking into account condition (7), on the basis of Proposition 2 we prove the following

Theorem 2. *If $\omega \in W_E^p$, $p > 1$ then the Dirichlet problem* (8) *is solvable if and only if the condition* (12) *is fulfilled, i.e., if*

$$\int_0^{2\pi} f(\vartheta)\, d\vartheta = \int_0^{2\pi} g(\vartheta)\, d\vartheta. \tag{14}$$

When this condition is fulfilled, the solution u of the problem (8) *is given by equation* (6) *in which the function μ is defined by means of* (13)*, where $\widetilde{\mu}$ is a particular solution of equation* (10)*,*

$$c = c_0 = \frac{1}{2\pi} \int_0^{2\pi} \Big[\frac{\widetilde{\mu}(\vartheta)}{2\pi} - g(\vartheta) \Big]\, d\vartheta \tag{15}$$

and

$$\lambda(\vartheta) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\mu}(\vartheta) \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\vartheta)}\, d\alpha + c_0 - g(\vartheta). \tag{16}$$

3^0. Let D be a doubly-connected domain bounded by rectifiable Jordan curves Γ_1 and Γ_2; note that γ_2 lies in the finite domain bounded by Γ_1.

Definition 3. We say that the function $\phi(z)$, analytic in D, is of the class $E^{\rho}(D)$, $\rho > 0$ if there exists an increasing sequence of doubly-connected domains $\{D_i\}$ with rectifiable boundaries $\mathcal{L}^i$, exhausting the domain D, such that

$$\sup_i \int\limits_{\mathcal{L}^i} \big|\phi(z)\big|^p |dz| < \infty.$$

Assume

$$e^p(D) = \big\{u : u = \operatorname{Re}\phi,\ \phi \in E^p(D)\big\}, \tag{17}$$

and consider the problem: Find a harmonic in D function u satisfying the conditions

$$\begin{cases} u \in e^p(D), \quad p > 1, \\ u\big|_{\Gamma_1} = f, \quad u\big|_{\Gamma_2} = g, \quad f \in L^p(\Gamma_1), \quad g \in L^p(I_2) \end{cases} \tag{18}$$

By means of the conformal mapping $z = z(w)$, $w \in K$ of the ring K onto the domain D for which $z(\gamma_i) = \Gamma_i$, this problem is reduced to the problem (8) in which $\omega = \sqrt[p]{z'(w)}$. Using the well-known Warschawski's result on the map of a circle onto the domain bounded by a piecewise Ljapunov boundary ([5]) and easily generalized both to the case of doubly-connected domains and to the weight properties of power functions, on the basis of Theorem 2 we have

Theorem 3. *If the domain D is bounded by the piecewise Ljapunov boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ with angular points t_k, $k = \overline{1,n}$ at which the inner angles with respect to D are equal to $\mu_k \pi$, $\overline{k = 1,n}$ and*

$$0 < \mu_k < p, \tag{19}$$

then the Dirichlet problem in the class $e^p(D)$ is solvable if and only if

$$\int\limits_{\Gamma_1} f(t)\,dt = \int\limits_{\Gamma_2} g(t)\,dt, \tag{20}$$

and when this condition is fulfilled, the problem has the unique solution given by the equality $u(z) = U(w(z))$, where $U(w)$ is the solution of the problem (8) for $\omega(w) = \sqrt[p]{z'(w)}$, $f(\vartheta) = f(w(e^{i\vartheta}))$, $g(\vartheta) = g(w(re^{i\vartheta}))$.

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