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ZAREMBA'S PROBLEM FOR HARMONIC FUNCTIONS FROM THE SMIRNOV'S WEIGHTED CLASSES IN DOMAINS WITH PIECEWISE LYAPUNOV BOUNDARIES

(Reported on 28.06.2006)

In [1-3], by analogy with the classes of analytic functions introduced by V.I. Smirnov (see [4] and also [5], Ch.X), we defined weighted classes of harmonic functions, investigated their properties, and in these classes we studied the mixed boundary value problem, when values of an unknown function are given on one part of the boundary and those of its derivative in the direction of the normal are given on the supplementary portion of the boundary (Zaremba's problem [6]). Regarding the domain in which we considered the problem, it was assumed that the domain was bounded by a simple Ljapunov curve.

Here we continue investigation of Zaremba's problem for domains which are bounded by piecewise Lyapunov curves.

1^0. Let D be a simply connected finite domain bounded by a simple curve L, and let $\mathcal{L}_k = (A_k, B_k)$, $k = \overline{1,m}$ be the arcs lying on that curve separately. Denote by $C_1, C_2, \dots, C_{2m}$ the ends of these arcs taken arbitrarily. Consider in a plane, cut along $L_1 = \bigcup_{k=1}^m \mathcal{L}_k$, the analytic functions

$$\Pi_1(z) = \sqrt{\prod_{k=1}^{m_1} (z - C_k)}, \quad \Pi_2(z) = \sqrt{\prod_{k=m_1+1}^{2m} (z - C_k)}, \tag{1}$$

where m_1 is an integer, $0 \le m_1 \le 2m$, and let

$$R(z) = \Pi_1(z)[\Pi_2(z)]^{-1}. \tag{2}$$

Let $p \ge 1$, and $\rho(t)$ be a measurable on L_1 function, different almost everywhere from zero. By $L^p(\Gamma;\rho)$ we denote a set of functions f for which $|f\rho|^p$ is Lebesgue summable.

Next, let $[A'_k, B'_k]$ be the arcs lying on $\mathcal{L}_k$. Denote $L_1 = \bigcup_{k=1}^m \mathcal{L}_k$, $\widetilde{L} = \bigcup_{k=1}^m [A_k, A'_k] \cup [B'_k, B_k]$, $L_2 = L \backslash L_1$.

By $z = z(w)$ we denote conformal mapping of the unit circle onto D, and let $w = w(z)$ be its inverse function. Suppose

$$\begin{cases} \Gamma_1 = w(L_1),\ (\widetilde{\gamma}) = w(\widetilde{L}),\ \Gamma_2 = w(L_2),\ a_k = w(A_k),\ b_k = w(B_k), \\ \Gamma_j(r) = \{w : w = re^{i\theta},\ \theta \in \Theta(\Gamma_j)\},\ L_j(r) = z(\Gamma_j(r)), \end{cases} \tag{3}$$

where $\Theta(\Gamma) = \{\theta : 0 \le \theta \le 2\pi,\ e^{i\theta} \in \Gamma\}$, $\Gamma \subset \gamma = \{\tau : |\tau| = 1\}$.

$A(E)$ will denote a class of absolutely continuous on E functions.

Let the points $D_1, \dots, D_n$ lie on L and be different from C_k. The points $D_1, D_s, \dots, D_{n_1}$ lie on L_1 and the points $D_{n_1+1}, \dots, D_n$ on L_2.

2000 *Mathematics Subject Classification:* 35105, 36125.

Key words and phrases. Harmonic function of Smirnov's weighted classes, Zaremba's problem, mixed problem.

Consider the functions

$$\rho_1(z)=\prod_{k=1}^{n_1}|z-D_k|^{\alpha_k},\ \rho_2(z)=\prod_{k=1}^{m_1}|z-C_k|^{\nu_k}\prod_{k=m_1+1}^{2m}|z-C_k|^{\lambda_k}\prod_{k=n_1+1}^{n}|z-D_k|^{\beta_k}. \quad (4)$$

We say that the harmonic in the domain D function $u(z)$, $z=x+iy=r\exp i\theta$ belongs to the class $e(L_{1p}(\rho_1),L'_{2q}(\rho_2))$, $p>1$, $q>1$ if

$$\sup_{r<1}\Big[\int\limits_{L_1(r)}|u(z)\rho_1(z)|^p|dz|+\int\limits_{L_2(r)}\Big(\Big|\frac{\partial u}{\partial x}\Big|^q+\Big|\frac{\partial u}{\partial y}\Big|^q\Big)\rho_2^q(z)|dz|\Big]<\infty. \quad (5)$$

2^0. Let D be the domain bounded by a simple piecewise-Ljapunov curve L with angular points $t_1,t_2,\dots,t_s$. We assume that the angle sizes at these points are equal to $\pi\mu_k$, $0<\mu_k\le 2$. A set of such curves we denote by $C^1(t_1,t_2,\dots,t_s;\mu_1,\mu_2,\dots,\mu_s)$. Let L be a curve from that class and L_1, L_2, $\widetilde{L}$, ρ_1, ρ_2 be the sets and functions defined above.

We divide the set $\{t_1,t_2,\dots,t_s\}$ into four parts. Denote by $t_1,t_2,\dots,t_{s_1}$ those which are contained in the product Π_1 (in the capacity of C_k), and by $t_{s_1+1},\dots,t_{\sigma_1}$ those which are contained in Π_2. The rest points we insert into the set of points $\{D_1,D_2,\dots,D_n\}$. Moreover, let $t_{\sigma_1},\dots,t_{\sigma_2}$ lie on L_1, and $t_{\sigma_2+1},\dots t_s$ on L_2. We assume that $t_k=C_k$, $k=\overline{1,s_1}$, $t_{s_1+k}=C_{m_1+k}$, $k=\overline{1,\sigma_1-s_1}$, $t_{\sigma_1+k}=D_k$, $k=\overline{1,\sigma_2-\sigma_1}$, $t_{\sigma_2+k}=D_{n_1+k}$, $k=\overline{1,s-\sigma_2}$ and write the weights ρ_1 and ρ_2 in the form

$$\rho_1(z)=\prod_{k=\sigma_1+1}^{\sigma_2}|z-t_k|^{\alpha_k}\prod_{k=\sigma_2+1}^{n_1}|z-D_k|^{\alpha_k}, \quad (6)$$

$$\rho_2(z)=\prod_{k=1}^{s_1}|z-t_k|^{\nu_k}\prod_{k=s_1+1}^{m_1}|z-C_k|^{\nu_k}\prod_{k=s+1}^{\sigma_1}|z-t_k|^{\lambda_k}\times$$
$$\times\prod_{k=m_1+\sigma_1+1}^{2m}|z-C_k|^{\lambda_k}\prod_{k=\sigma_2+1}^{s}|z-t_k|^{\beta_k}\prod_{k=n_1+s-\sigma_2+1}^{n}|z-D_k|^{\beta_k}. \quad (7)$$

Consider the boundary value problem: Find a function u, satisfying the conditions

$$\begin{cases}\Delta u=0,\ u\in e(L_{1p}(\rho_1),L'_{2q}(\rho_2)),\ p>1,q>1,\\ u^+|_{L_1\backslash\widetilde{L}}=F,\ F\in L^p(L_1\backslash\widetilde{L};\rho_1),\ u^+\in A(L_2\cup\widetilde{L}),\\ u^+|_{\widetilde{L}}=\Psi,\ \Psi'\in L^q(\widetilde{L},\rho_2),\ \left(\frac{\partial u}{\partial n}\right)^+|_{L_2}=G,\ G\in L^q(L_2;\rho_2).\end{cases} \quad (8)$$

3^0.

Theorem. Let $L\in C^1(t_1,\dots,t_s;\mu_1,\mu_2,\dots,\mu_s)$, $\rho_1(z)$ and $\rho_2(z)$ be given by equalities (6) and (7), where

$$-\frac{1}{q}<\nu_k<\min\Big(0;\frac{1}{q'}-\frac{1}{2}\Big),\quad \max\Big(0;\frac{1}{2}-\frac{1}{q}\Big)\le\lambda_k<\frac{1}{q'} \quad (9)$$
$$k=\overline{s_1+1,m_1},\qquad k=\overline{m_1+\sigma_1-s_1+1,2m}$$

$$-\frac{1}{p}<\alpha_k<\frac{1}{p'},\ k=\overline{\sigma_2+1,m},\ -\frac{1}{q}<\beta_k<\frac{1}{q'},\ k=\overline{n_1+s-\sigma_2+1,n} \quad (10)$$

$$-\frac{1}{p}<\alpha_k<\min\Big(\frac{1}{p'};\frac{1}{\mu_k}-\frac{1}{p}\Big),\ -\frac{1}{q}<\beta_k<\min\Big(\frac{1}{q'};\frac{1}{\mu_k}-\frac{1}{q}\Big) \quad (11)$$
$$k=\overline{\sigma_1+1,\sigma_2}\qquad k=\overline{\sigma_2+1,s}$$

$$-\frac{1}{q} < \nu_k < \frac{1}{\mu_k} \min\Big(0; \frac{1-\mu_k}{q}; \frac{\frac{q}{2}-\mu_k}{q}\Big), \quad k = \overline{1, s_1}$$
$$\frac{1}{\mu_k} \max\Big(0; \frac{1-\mu_k}{q}; \frac{\frac{q}{2}-\mu_k}{q}\Big) \le \lambda_k \le \min \frac{1}{\mu_k}\Big(\frac{1}{q'}, (1 - \frac{\mu_k}{q})\Big), \quad k = \overline{s_1 + 1, \sigma_1} \tag{12}$$

Then for the problem (8) to be solvable it is necessary and sufficient that:

(a) for $m_1 \le m$, the conditions

$$\int\limits_{\varphi_k}^{\theta_k+1} \operatorname{Re}\Big[\frac{R(e^{i\alpha})}{\pi i} \int\limits_{\Theta(\Gamma_2)} \frac{i\mu(\tau)+a}{R(\tau)} \frac{d\tau}{\tau - z(e^{i\alpha})}\Big] d\alpha =$$
$$= \Psi(A_{k+1}) - \Psi(B_k), \quad k = \overline{1, m}, \tag{13}$$

where R is the function given by equality (2) and it is assumed that $\rho^{i\theta_k} = w(A_k)$

$$e^{i\varphi_k} = w(B_k), \theta_k, \varphi_k \in [0, 2\pi], \theta_{m+1} = \theta_1, A_{m+1} = A_1,$$
$$\mu(\tau) = -G(z(\tau)) + \frac{1}{2\pi} \sum_{k=1}^{m} \Big[\Psi(A_{k+1}) \operatorname{ctg} \frac{\theta_{k+1} - \varphi}{2} - \Psi(B_k) \operatorname{ctg} \frac{\varphi_k - \varphi}{2}\Big] -$$
$$-\frac{1}{2\pi} \int\limits_{\Theta(\tilde\gamma)} \Psi(z(e^{i\theta})) \frac{d\theta}{2\sin^2 \frac{\theta-\varphi}{2}} - \frac{1}{2\pi} \int\limits_{\Theta(\Gamma_1 \setminus \tilde\gamma)} F(z(e^{i\theta})) \frac{d\theta}{2\sin^2 \frac{\theta-\varphi}{2}},$$

where $\tau = e^{i\varphi}$, $a = \frac{1}{2\pi} \sum_{k=1}^{m} [\Psi(A_{k+1}) - \Psi(B_k)]$;

(b) for $m_1 > m$, the conditions (13) and also the conditions

$$\int\limits_{L_2} \frac{i\mu(W(t)) + a}{R(W(t))} w^k(t) W'(t) dt = 0, \quad k = \overline{0, l-1}, \quad l = m_1 - m. \tag{14}$$

be fulfilled.

(c) If the above-mentioned conditions are fulfilled, then a solution of the problem (8) is given by the equality

$$u(z) = u^*(z) + u_0(z),$$

where

$$u^*(z) = \frac{1}{2\pi} \int\limits_{\Theta(\tilde\gamma)} \Psi(z(e^{i\theta})) P(r, \theta - \varphi) d\theta + \frac{1}{2\pi} \int\limits_{\Theta(\Gamma_1 \setminus \tilde\gamma)} F(z(e^{i\theta})) P(r, \theta - \varphi) d\theta +$$
$$+ \frac{1}{2\pi} \int\limits_{\theta(\Gamma_2)} W_{\Gamma_2}(\theta) P(r, \theta - \varphi) d\theta, \tag{15}$$

in which $P(r, x) = \frac{1 - r^2}{1 + r^2 - 2r\cos x}$,

$$W_{\Gamma_2}(\theta) = \int\limits_{\varphi_1}^{\theta} \chi_{\Theta_2(\Gamma)}(\alpha) \operatorname{Re}\Big[\frac{R(e^{i\alpha})}{\pi i} \int\limits_{\Theta(\Gamma_2)} \frac{i\mu(\tau)+a}{R(\tau)} - \frac{d\tau}{\tau - e^{i\gamma}}\Big] d\alpha + M_k. \tag{16}$$

χ_E denotes the characteristic function of the set E,

$$M_k = \Psi(A_{k+1}) - \int\limits_{\varphi_1}^{\theta_{k+1}} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re}\Big[\frac{R(e^{i\alpha})}{\pi i} \int\limits_{\Theta(\Gamma_2)} \frac{i\mu(\tau)+a}{R(\tau)(\tau - e^{i\alpha})} d\tau\Big] d\alpha \tag{17}$$

and

$$u_0(z)=\begin{cases}0, & \text{for} \quad m_1>m,\\ \dfrac{1}{2\pi}\int\limits_0^{2\pi} W^*_{\Gamma_2}(\theta)P(r,\theta-\varphi)d\theta, & \\ \qquad W^*_{\Gamma_2}(\theta)=\int\limits_{\beta_1}^{\theta}\chi_{\Theta(\Gamma_2)}(\alpha)\,\mathrm{Re}[R(e^{i\alpha})P_{r-1}(e^{i\alpha})]d\alpha+N_k, & \end{cases} \tag{18}$$

$$N_k=-\int\limits_{\varphi_k}^{\theta_{k+1}}\mathrm{Re}[R(e^{i\alpha})P_{r-1}(e^{i\alpha})]d\alpha, \quad r=m-m_1.$$

Here $P_{r-1}(e^{i\alpha})=0$, if $r-1=m-m_1-1<0$; however, if $m_1<m$, then $P_{r-1}(e^{i\theta})=\sum_{j=0}^{r-1}(x_j+iy_j)e^{ij\theta}$ is the polynomial in which $(x_0,y_0,\dots,x_{r-1},y_{r-1})$ is the solution of the system

$$\begin{gathered}\int\limits_{\theta_k}^{\varphi_k}\sum_{j=0}^{r-1}[x_jR_1(e^{i\theta})\cos j\theta-y_jR_2(e^{i\theta})\sin j\theta]d\theta=0,\\ k-\overline{1,m}, \quad r=m-m_1,\\ \int\limits_{\theta_k}^{\varphi_k}\sum_{j=0}^{r-1}[x_jR_2(e^{i\theta})\cos j\theta-y_jR_1(e^{i\theta})\sin j\theta]d\theta=0,\end{gathered} \tag{19}$$

where $R_1(t)=\mathrm{Re}\,R(t)$, $R_2(t)=\mathrm{Im}\,R(t)$.

If ν is a rang of the matrix of the system (19), $(\nu\le 2r)$, then the solution of the homogeneous problem $u_0(z)$ contains $2(m-m_1)-\nu$ arbitrary real parameters.

4^0. The conditions (9)–(12) show what parameters (depending on the boundary's geometry) can be taken from Smirnov classes under which the above theorem is valid for the problem (8). It is not difficult to show that a set of parameters under which the situation under consideration is realizable is not empty, i.e., for the given $\mu_1,\dots,\mu_k\le 2$, there exist the collections $(p;q;\alpha_k,\beta_k,\nu_k,\lambda_k)$ satisfying the system (9)–(12), where p and q can be taken arbitrarily from the intervals $(1,\infty)$ and $(2;\infty)$, respectively, and α_k, β_k, ν_k, λ_k belong to certain admissible intervals. A set of such collections depend on a number of angular points (and angle sizes) which turn out to be the ends of $\mathcal{L}_k$.

It should be first of all noted that the fulfilment of inequalities (9) and (10) and of the first inequalities of (11) and (12) is necessary in the case of Ljapunov boundaries, as well (see [2]). Thus in considering non-smooth boundaries the second inequalities in (11) and (12) turn out to be additional ones. These inequalities show that if $\mu_k>1$, then it is necessary to take $-\frac{1}{p}<\alpha_k<\frac{1}{\mu_k}-\frac{1}{p}$, $-\frac{1}{q}<\beta_k<\frac{1}{\mu_k}-\frac{1}{q}$ while if $\mu_k<1$, then we take $-\frac{1}{p}<\alpha_k<\frac{1}{p'}$, $-\frac{1}{q}<\beta_k<\frac{1}{q'}$.

The system (11)–(12) is unsolvable when either $q\le 2$, $0<\mu_k<1$ and $\mu_kq<1$, or $q<2$, $1<\mu_k\le 2$ and $\mu_k>q$ (in both cases it is impossible to define λ_k). This means that if we take $q<2$, then at the points which are the ends of the arcs $\mathcal{L}_k$ for which either $\mu_kq_k<1$, or $\mu_k>q$, the weighted multiplier $|t-t_k|$ should be taken with negative degree.

The conditions (9)–(12) can also be understood as follows: if we have a class of unknown functions, then what must be the set of piecewise-Ljapunov curves for which the above theorem holds.

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