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ZAREMBA'S PROBLEM IN SMIRNOV CLASS OF HARMONIC FUNCTIONS IN DOMAINS WITH PIECEWISE-LYAPUNOV BOUNDARIES

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Abstract. Zaremba's problem is studied in a weighted Smirnov class of harmonic functions in domains bounded by piecewise-Lyapunov curves. The conditions of solvability are obtained and the solutions are constructed in quadratures. In the case of domains with Lyapunov boundaries, the same problem is investigated under weaker assumptions regarding boundary data.

რეზიუმე. ზარემბას სასაზღვრო ამოცანა შესწავლილია ჰარმონიულ ფუნქციათა სმირნოვის წონიან კლასებში უბან-უბან ლიაპუნოვის წირებით შემოსაზღვრულ არეში. მიღებულია ამოხსნადობის პირობები და აგებულია ამონახსნები კვადრატურებში. ლიაპუნოვის წირებით შემოსაზღვრული არეების შემთხვევაში იგივე ამოცანა გამოკვლეულია სასაზღვრო მონაცემთა მიმართ უფრო სუსტ მოთხოვნებში.

In [1–3], by analogy with the classes of analytic functions E^p introduced by V. I. Smirnov ([4], see also [5] Ch. X), we defined weighted classes of harmonic functions $e\big(L_{1p}(\rho_1), L'_{2q}(\rho_2)\big)$, investigated some of their properties and studied mixed boundary value problem, when values of an unknown function are given on one part of the boundary, and those of its derivative to the interior normal are given on the other part of the boundary ([6], Zaremba's problem). As for the domain in which we consider the problem, it assumed to be bounded by a simple Lyapunov curve.

In the present work we continue our investigation of Zaremba's problem, but this time for domains which are bounded by piecewise-Lyapunov curves. Moreover, for the domains bounded by Lyapunov curves, this problem is considered in a new statement.

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1^0. Definitions, Notation and Auxiliary Statements

1.1. Let D be a simply connected finite domain bounded by a simple rectifiable curve L and let $\mathcal{L}_k=(A_k,B_k)$, $k=\overline{1,m}$ be the arcs lying separately on that curve (the points $A_1,B_1,A_2,B_2,\dots,A_m,B_m$ follow each other in the positive direction). By $C_1,C_2,\dots,C_{2m}$ we denote the points taken arbitrarily. In a plane, cut along a set of curves $L_1=\bigcup\limits_{k=1}^{m}\mathcal{L}_k$, we consider analytic functions

$$\Pi_1(z)=\sqrt{\prod_{k=1}^{m_1}(z-C_k)},\quad \Pi_2(z)=\sqrt{\prod_{k=m_1+1}^{2m}(z-C_k)},\tag{1}$$

where m_1 is an integer from the interval $[0;2m]$; we take any branch of the first function, while the second one we choose such that the function

$$R(z)=\Pi_1(z)\Big[\Pi_2(z)\Big]^{-1}\tag{2}$$

decomposes in the neighborhood of the point at infinity as follows:

$$R(z)=z^{m_1-m}+a_1\,z^{m_1-m-1}+\cdots$$

(see [7], p. 277). For $z=t\in L$, under $\Pi_1(t)$, $\Pi_2(t)$, $R(t)$ we mean a value which takes the function on the left side of L.

1.2. Let $p\geq 1$, and $\rho(t)$ be a measurable function on L, almost everywhere different from zero. By $L_p(L;\rho)$ we denote a set of measurable on L functions f (in the arc measure ds) for which $|f\rho|^p$ is Lebesgue measurable.

1.3. Let $[A_k',B_k']$ be the arcs lying on $\mathcal{L}_k$ (the points B_k' follow A_k' in the direction of $\mathcal{L}_k$). Denote

$$L_1=\bigcup_{k=1}^{m}\mathcal{L}_k,\quad \widetilde{L}=\bigcup_{k=1}^{m}\big[A_k;A_k'\big]\ \bigcup_{k=1}^{m}\big[B_k';B_k\big],\quad L_2=L\backslash L_1.\tag{3}$$

1.4. Suppose $U=\{w:|w|\leq 1\}$, and let $z=z(w)$ be the conformal mapping of U onto D, and $w=w(z)$ be an inverse mapping. Denote

$$\begin{cases}\Gamma_1=w(L_1),\quad \widetilde{\gamma}=w(\widetilde{L}),\quad \Gamma_2=w(L_2),\\ \quad a_k=w(A_k),\quad b_k=w(B_k),\\ \Gamma_j(r)=\Big\{w:w=re^{i\zeta},\zeta\in\Theta(\Gamma_j)\Big\},\quad L_j(r)=z\big(\Gamma_j(r)\big),\end{cases}\tag{4}$$

where $\Theta(\Gamma)=\big\{\zeta:0\leq\zeta\leq 2\pi,\ e^{i\zeta}\in\Gamma\big\}$, $\Gamma\subseteq\gamma=\{\tau:|\tau|=1\}$.

1.5. Let the points $D_1,D_2,\dots,D_n$ belong to L and be different from C_k; $D_1,D_2,\dots,D_{N_1}$ lie on L_1, while $D_{n_1+1},\dots,D_n$ on L_2. Let $p>1$, $q>1$;

assume

$$\rho_1(z)=\prod_{k=1}^{n_1}|z-D_k|^{\alpha_k}, \tag{5}$$

$$\rho_2(z)=\prod_{k=1}^{m_1}|z-C_k|^{\nu_k}\prod_{k=m_1+1}^{2m}|z-C_k|^{\lambda_k}\prod_{k=n_1+1}^{n}|z-D_k|^{\beta_k}, \tag{6}$$

where

$$-\frac{1}{p}<\alpha_k<\frac{1}{p'}\,,\quad -\frac{1}{q}<\beta_k<\frac{1}{q'}\,,\quad p'=\frac{p}{p-1}\,,\quad q'=\frac{q}{q-1}\,, \tag{7}$$

$$-\frac{1}{q}<\nu_k<0,\quad 0\le\lambda_k<\frac{1}{q'}\,. \tag{8}$$

1.6. We say that the harmonic in D function $u(z)$, $z=x+iy=re^{i\zeta}$ belongs to the class $e\big(L_{1p}(\rho_1),L'_{2q}(\rho_2)\big)$, if

$$\sup_{r<1}\Bigg[\int\limits_{L_1(r)}\big|u(z)\rho_1(z)\big|^p|dz|+\int\limits_{L_2(r)}\bigg(\Big|\frac{\partial u}{\partial x}\Big|^q+\Big|\frac{\partial u}{\partial y}\Big|^q\bigg)\rho_2^q(z)|dz|\Bigg]<\infty. \tag{9}$$

When $D=U$ (and hence $L_k=\Gamma_k$, $k=1,2$), we will write $h\big(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2)\big)$ instead of $e\big(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2)\big)$. If $\Gamma_1=\gamma$, $\rho_1\equiv 1$, then this class coincides with the well-known class h_p (see [5], Ch.IX).

1.7. If $f(z)$ is the function defined on the set of E-finite union of closed arcs lying on L, and $t=t(s)$, $0\le s\le l$ is the equation of the curve L with respect to the arc abscissa, then we say that f is absolutely continuous on E and write $f\in A(E)$: if, the function $f(t(s))$ is absolutely continuous on the set $\{s: t(s)\in E\}$, i.e. for any $\varepsilon>0$ there exists a number $\delta>0$, such that for any intervals $[t(s_k),t(\sigma_k)]$, lying on E with the condition $\sum|(\sigma_k-s_k)_{\bmod l}|<\delta$, we have $\sum|f(t(\sigma_k)-f(t(s_k))|<\varepsilon$.

It can be easily proved that if $f(z)\in A(E)$, and $z(\tau)$ is the restriction on γ of the conformal mapping $z=z(w)$, then the function $f(z(\tau))\in A(e)$, $e=w(E)$; and vice versa, if $\varphi\in A(e)$, where e is a finite union of closed arcs on γ, then $\varphi(w(t))\in A(E)$, $E=z(e)$ ([2], Lemma 9).

1.8. Below we will present some properties of functions from $e\big(L_{1p}(\rho_1),L'_{2q}(\rho_2)\big)$.

Statement 1 ([2], Lemmas 2–4)**.** *If the functions ρ_1,ρ_2 are given on γ by the equalities* (5)–(6)*, the conditions* (7)–(8) *are fulfilled, and $u(z)\in h\big(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2)\big)$, then:*

(i) *there exists $\sigma>1$, such that $u\in h_\sigma$;*

(ii) *if v is the function which is harmonically conjugate to u, then $v\in h\big(\Gamma_{1p_1}(\rho_1),\Gamma'_{2q}(\rho_2)\big)$, where $p_1=\frac{p\sigma}{p+\sigma}$;*

(iii) *if* $u\in e\big(L_{1p}(\rho_1),L'_{2q}(\rho_2)\big)$*, then the function* $U(w)=u(z(w))$ *belongs to the class* $h\big(\Gamma_{1p}(\omega_1(w)),\Gamma'_{2q}(\omega_2(w))\big)$*, where*

$$\omega_1(w)=\rho_1(z(w))\sqrt[p]{|z'(w)|}\,,\quad \omega_2(w)=\rho_2(z(w))\sqrt[q]{|z'(w)|}\,.$$

Statement 2 ([3])**.** *If* $u\in h\big(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2)\big)$*, then* $u(z)$ *is continuously extendable on every closed arc lying on* Γ_2*, and the boundary function* $u^+(t)$ *is such that there exist the limits*

$$u(a_k-)=\lim_{t\to a_k-}u^+(t),\quad u(b_k+)=\lim_{t\to b_k+}u^+(t),$$

thus the obtained on $\overline{\Gamma}_2$ *function* $u^+(t)$ *is absolutely continuous on* $\overline{\Gamma}_2$ *and* $\frac{\partial u^+}{\partial\zeta}\in L^q(\Gamma_2;\rho_2)$*.*

1.9. It follows from Statement 1 of item (i) that if u belongs to $h\big(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2)\big)$, then there exist angular boundary values $u^+(t)$ almost everywhere on γ, and $u(re^{i\varphi})$ is representable by the Poisson integral with density u^+ ([5], Ch. IX).

2^0. Zaremba's Problem in a Domain with Lyapunov Boundary

Let us consider the boundary value problem: find a function u satisfying the following conditions

$$\begin{cases}\Delta u=0,\quad u\in e\big(L_{1p}(\rho_1),L'_{2q}(\rho_2)\big),\quad p>1,\quad q>1,\\ u^+\big|_{L_1\backslash\widetilde{L}}=F,\quad F\in L^p(L_1\backslash\widetilde{L},\rho_1),\quad u^+\in A(L_1\cup\widetilde{L}),\\ u^+\big|_{\widetilde{L}}=\Psi,\ \ \Psi'\in L^q(\widetilde{L};\rho_2),\ \ \Big(\dfrac{\partial u}{\partial n}\Big)^+\Big|_{L_2}=G,\quad G\in L^q(L_2;\rho_2).\end{cases}\tag{10}$$

Using the results obtained in [8]–[9] and Statements 1–2 in [2], the following theorem is proved.

Theorem A. *Let* D *be a finite simply connected domain bounded by a simple closed Lyapunov curve* L*;* ρ_1 *and* ρ_2 *are defined by the equalities* (5)–(7)*, and moreover,*

$$-\frac{1}{q}<\nu_k<\min\Big(0;\frac{1}{q'}-\frac{1}{2}\Big),\quad \max\Big(0;\frac{1}{2}-\frac{1}{q}\Big)\le\lambda_k<\frac{1}{q'}\,.\tag{11}$$

Then for the problem (10) *to be solvable:*

I. *For* $m_1\le m$*, it is necessary and sufficient that the conditions*

$$\int\limits_{\varphi_k}^{\zeta_{k+1}}\mathrm{Re}\bigg[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Theta(\Gamma_2)}\frac{i\mu(\tau)+a}{R(\tau)}\,\frac{d\tau}{\tau-z(e^{i\alpha})}\bigg]d\alpha= \\ =\Psi(A_{k+1})-\Psi(B_k),\quad k=\overline{1,m}\tag{12}$$

be fulfilled; here R is the function given by the equality (2) *and it is assumed that*

$$e^{i\zeta_k} = w(A_k),\;\; e^{i\varphi_k} = w(B_k),\;\; \zeta_k,\varphi_k \in [0,2\pi],\;\; \zeta_{m+1}=\zeta_1,\;\; A_{m+1}=A_1,$$

$$\mu(\tau) = -\,G(z(\tau)) + \frac{1}{2\pi}\sum_{k=1}^{m}\Big[\Psi(A_{k+1})\operatorname{ctg}\frac{\zeta_{k+1}-\varphi}{2} - \Psi(B_k)\operatorname{ctg}\frac{\varphi_k-\varphi}{2}\Big] - $$
$$- \frac{1}{2}\int\limits_{\Theta(\widetilde{\gamma})}\Psi(z(e^{i\zeta}))\frac{d\zeta}{2\sin^2\frac{\zeta-\varphi}{2}} - \frac{1}{2\pi}\int\limits_{\Theta(\Gamma_1\setminus\widetilde{\gamma})}F(z(e^{i\zeta}))\frac{d\zeta}{2\sin^2\frac{\zeta-\varphi}{2}}\,,$$

where

$$\tau = e^{i\varphi}\in\Gamma_2,\quad a = \frac{1}{2\pi}\sum_{k=1}^{m}\big[\Psi(A_{k+1}) - \Psi(B_k)\big].$$

II. *For $m_1 > m$, it is necessary and sufficient that the conditions* (12) *and also*

$$\int\limits_{L_2}\overline{\frac{i\mu(w(t))+a}{R(w(t))}}\,w^k(t)\,w'(t)\,dt = 0,\;\; k=\overline{0,l-1},\;\; l=m_1-m \tag{13}$$

be fulfilled.

III. *If the above conditions are fulfilled, then the solution of the problem* (10) *is given by the equality*

$$u(z) = u^*(z) + u_0(z),$$

where

$$u^*(z) = \frac{1}{2\pi}\int\limits_{\Theta(\widetilde{\gamma})}\Psi(z(e^{i\zeta}))P(r,\zeta-\varphi)d\zeta + \frac{1}{2\pi}\int\limits_{\Theta(\Gamma_1\setminus\widetilde{\gamma})}F(z(e^{i\zeta}))P(r,\zeta-\varphi)d\zeta +$$
$$+ \frac{1}{2\pi}\int\limits_{\Theta(\Gamma_2)}W_{\Gamma_2}(\zeta)P(r,\zeta-\varphi)d\zeta, \tag{14}$$

in which $P(r,x) = \frac{1-r^2}{1+r^2-2r\cos x}$

$$W_{\Gamma_2}(\zeta) = \int\limits_{\varphi_1}^{\zeta}\chi_{\Theta(\Gamma_2)}(\alpha)\operatorname{Re}\Big[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Theta(\Gamma_2)}\frac{i\mu(\tau)+a}{R(\tau)}\,\frac{d\tau}{\tau-e^{i\alpha}}\Big]d\alpha + M_k, \tag{15}$$

χ_E *denotes the characteristic function of the set E,*

$$M_k = \Psi(A_{k+1}) - \int\limits_{\varphi_1}^{\zeta_{k+1}}\chi_{\Theta(\Gamma_2)}(\alpha)\operatorname{Re}\Big[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Theta(\Gamma_2)}\frac{i\mu(\tau)+a}{R(\tau)(\tau-e^{i\alpha})}\,d\tau\Big]d\alpha, \tag{16}$$

and

$$u_0(z)=\begin{cases} 0, \quad \textit{for} \quad m_1>m, \\ \dfrac{1}{2\pi}\displaystyle\int\limits_0^{2\pi} W^*_{\Gamma_2}(\zeta)P(r,\zeta-\varphi)d\zeta, \;\; W^*_{\Gamma_2}(\zeta)= \\ \qquad =\displaystyle\int\limits_{\beta_1}^{\zeta}\chi_{\Theta(\Gamma_2)}(\alpha)\,\mathrm{Re}\big[R(e^{i\alpha})P_{r-1}(e^{i\alpha})\big]d\alpha+N_k, \\ \textit{for} \quad m_1\le m. \end{cases} \tag{17}$$

$$N_k=-\int\limits_{\varphi_k}^{\zeta_{k+1}}\mathrm{Re}\big[R(e^{i\alpha})P_{r-1}(e^{i\alpha})\big]d\alpha, \quad r=m-m_1.$$

Here $P_{r-1}(e^{i\zeta})=0$*, if* $r-1<0$*, but if* $m_1\le m$*, then* $P_{r-1}(e^{i\zeta})=\sum\limits_{j=0}^{r-1}(x_j+iy_j)e^{ij\zeta}$ *is the polynomial in which* $(x_0,y_0,x_1,y_1,\dots,x_{r-1},y_{r-1})$ *is a solution of the system*

$$\begin{gathered} \int\limits_{\zeta_k}^{\varphi_k}\sum_{j=0}^{r-1}\big[x_jR_1(e^{i\zeta})\cos j\zeta-y_jR_2(e^{i\zeta})\sin j\zeta\big]d\zeta=0, \\ k=\overline{1,m}, \quad r=m-m_1, \\ \int\limits_{\zeta_k}^{\varphi_k}\sum_{j=0}^{r-1}\big[x_jR_2(e^{i\zeta})\cos j\zeta-y_jR_1(e^{i\zeta})\sin j\zeta\big]d\zeta=0, \end{gathered} \tag{18}$$

where $R_1(t)=\mathrm{Re}\,R(t)$*,* $R_2(t)=\mathrm{Im}\,R(t)$*.*

If ν *is the rank of the matrix of the system* (18)*,* $(\nu\le 2r)$*, then a solution of the homogeneous problem* $u_0(z)$ *contains* $2(m-m_1)-\nu$ *real parameters.*

3^0. The Mixed Problem in Domains with Piecewise Lyapunov Curves

The theorem given shows that the mixed boundary value problem (10) is solvable in Smirnov classes, when the domain D is bounded by a simple Lyapunov curve. Below we will try to find those Smirnov classes (i.e., those values p,q and admissible weights ρ_1,ρ_2) for which in case of the domains with piecewise-Lyapunov curves one can apply to the investigation of the problem (10) the method used in [2] and obtain an analogue of Theorem A.

Let D be the finite domain bounded by a simple piecewise-Lyapunov curve L with angular points $t_1,t_2,\dots,t_s$. We assume that the sizes of angles at these points which are interior with respect to the domain D, are equal to $\mu_k\pi$, $0<\mu_k\le 2$. A set of such curves we denote by $C^1(t_1,\dots,t_s;\mu_1,\dots,\mu_s)$.

Let $L \subset C^1(t_1, t_2, \dots, t_s; \mu_1, \mu_2, \dots, \mu_s)$, $L_1, L_2, \widetilde{L}$ are defined by virtue of (3), and the functions ρ_1, ρ_2 are given by the equalities (5)–(6). Consider the problem (10). Using item (iii) of Statement 1, we reduce it by means of the conformal mapping to the problem of the same type, but now for the circle. Undoubtedly, the weighted functions in this case vary.

Since $L \subset C^1(t_1, t_2, \dots, t_s; \mu_1, \mu_2, \dots, \mu_s)$, therefore as is known ([10]; see also [11], [12]),

$$\begin{aligned} z'(w) &= \prod_{k=1}^{s} (w - \tau_k)^{\mu_k - 1} z_0(w), \quad \tau_k = w(t_k), \\ w'(z) &= \prod_{k=1}^{s} (z - t_k)^{\frac{1}{\mu_k} - 1} w_0(z), \end{aligned} \tag{19}$$

$$z(w) = \prod_{k=1}^{s} (w - \tau_k)^{\mu_k} z_1(w), \quad w(z) = \prod_{k=1}^{s} (z - t_k)^{\frac{1}{\mu_k}} w_1(z), \tag{20}$$

where $z_0(w)$, $z_1(w)$ are other than zero Hölder class functions in the closed circle $\overline{U}$, and $w_0(z), w_1(z)$ are the same functions in $\overline{D}$.

We divide the set of angular points $\{t_1, t_2, \dots, t_s\}$ into four parts and denote by $t_1, t_2, \dots, t_{s_1}$ those which are contained in the product Π_1; by $t_{s_1+1}, \dots, t_{\sigma_1}$ we denote the points in Π_2, and the remaining ones will be inserted in the set of points $\{D_1, D_2, \dots, D_n\}$. In this case, let $t_{\sigma_1+1}, \dots, t_{\sigma_2}$ lie on L_1, and $t_{\sigma_2+1}, \dots, t_s$ on L_2. Assume $t_k = C_k$, $k = \overline{1, s_1}$, $t_{s_1+k} = C_{m_1+k}$, $k = \overline{1, \sigma_1 - s_1}$, $t_{\sigma_1+k} = D_k$, $k = \overline{1, \sigma_2 - \sigma_1}$, $t_{\sigma_2+k} = D_{n_1+k}$, $k = \overline{1, s - \sigma_2}$. In this connection, we write the weights ρ_1, ρ_2 in the form

$$\rho_1(z) = \prod_{k=\sigma_1+1}^{\sigma_2} |z - t_k|^{\alpha_k} \prod_{k=\sigma_2+1}^{n_1} |z - D_k|^{\alpha_k}, \tag{21}$$

$$\begin{aligned} \rho_2(z) = {} & \prod_{k=1}^{s_1} |z - t_k|^{\nu_k} \prod_{k=s_1+1}^{m_1} |z - C_k|^{\nu_k} \prod_{k=s_1+1}^{\sigma_1} |z - t_k|^{\lambda_k} \cdot \\ & \cdot \prod_{k=m_1+\sigma_1+1}^{2m} |z - C_k|^{\lambda_k} \prod_{k=\sigma_2+1}^{s} |z - t_k|^{\beta_k} \prod_{k=n_1+s-\sigma_2+1}^{n} |z - D_k|^{\beta_k}. \end{aligned} \tag{22}$$

Introduce the notation

$$c_k = w(C_k), \quad \tau_k = w(t_k), \quad d_k = w(D_k). \tag{23}$$

Taking into account the equalities (19)–(20) (and the fact that at the points $D_k \neq t_j$ we have $\mu_k = 1$) and item (iii) of Statement 1, the classes $L^p(\rho_i)$,

$i=1,2$, transform into the classes $L^p(\omega_i)$ in which

$$\omega_1(w)=\prod_{k=\sigma_1+1}^{\sigma_2}|w-c_k|^{\alpha_k\mu_k+\frac{\mu_k-1}{p}}\prod_{k=\sigma_2+1}^{n_1}|w-d_k|^{\alpha_k}, \tag{24}$$

$$\begin{aligned}\omega_2(w)=&\prod_{k=1}^{s_1}|w-\tau_k|^{\mu_k\nu_k+\frac{\mu_k-1}{q}}\prod_{k=s_1+1}^{m_1}|w-c_k|^{\nu_k}\cdot\\ &\cdot\prod_{k=s_1+1}^{\sigma_1}|w-\tau_k|^{\lambda_k\mu_k+\frac{\mu_k-1}{q}}\prod_{k=m_1+\sigma_1+1}^{2m}|w-c_k|^{\lambda_k}\cdot\\ &\cdot\prod_{k=\sigma_2+1}^{s}|w-\tau_k|^{\beta_k\mu_k+\frac{\mu_k-1}{q}}\prod_{k=n_1+s-\sigma_2+1}^{n}|w-d_k|^{\beta_k}.\end{aligned} \tag{25}$$

Denote

$$U(w)=u(z(w)),\quad f(\tau)=F(z(\tau)),\quad \psi(\tau)=\Psi(z(\tau)),\quad g(\tau)=G(z(\tau)). \tag{26}$$

Then the function $U(w)$ will be of the class $h\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2)\big)$ which, according to Statements 1, 2 and the conclusion of point 1.7 in Section 1^0, belongs to $A(\Gamma_2\cup\widetilde{\gamma})$, where $\Gamma_1,\widetilde{\gamma},\Gamma_2$ are defined by virtue of (4). Consequently, $U(w)$ is a solution of the problem

$$\begin{cases}\Delta U=0,\ \ U\in h\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2)\big),\ \ p>1,\ \ q>1,\\ \qquad U^+\in A(\Gamma_2\cup\widetilde{\gamma}),\\ U^+\big|_{\Gamma_1}=f,\ \ f\in L^p(\Gamma_1\backslash\widetilde{\gamma},\omega_1), U^+\big|_{\widetilde{\gamma}}=\psi,\ \ \psi'\in L^q(\widetilde{\gamma},\omega_2),\\ \left(\frac{\partial U}{\partial n}\right)^+\Big|_{\Gamma_2}=g,\quad g\in L^q(\Gamma_2;\omega_2).\end{cases} \tag{27}$$

Relying on Statements 1, 2 and equalities (19)–(20), it is not difficult to show that if $U(w)$ is a solution of the problem (27), then the function $u(z)=U(w(z))$ will be a solution of the problem (10).

Thus the problem (10) under consideration with the curve $L\subset C^1(t_1,\dots,t_s;\mu_1,\mu_2,\dots,\mu_s)$ has been reduced equivalently to the problem (27). Theorem A can likewise be applied to the same problem if for the weights ω_1 and ω_2 will be fulfiled the conditions (11), i.e., if for the exponents in the equalities (24)–(25) the following relations will be fulfilled:

$$\begin{aligned}&\begin{cases}-\dfrac{1}{p}<\alpha_k<\dfrac{1}{p'}\,,\\ -\dfrac{1}{p}<\alpha_k\mu_k+\dfrac{\mu_k-1}{p}<\dfrac{1}{p'}\,,\end{cases}\qquad k=\overline{\sigma_1+1,\sigma_2},\\ &\begin{cases}-\dfrac{1}{q}<\beta_k<\dfrac{1}{q'}\,,\\ -\dfrac{1}{q}<\beta_k\mu_k+\dfrac{\mu_k-1}{q}<\dfrac{1}{q'}\,,\end{cases}\qquad k=\overline{\sigma_2+1,s},\end{aligned} \tag{28}$$

$$\begin{cases} -\dfrac{1}{q} < \nu_k < 0, \\ -\dfrac{1}{q} < \mu_k\nu_k + \dfrac{\mu_k - 1}{q} < \min\Big(0; \dfrac{1}{q'} - \dfrac{1}{2}\Big), \end{cases} \quad k = \overline{1, s_1},$$
$$\begin{cases} 0 \le \lambda_k < \dfrac{1}{q'}, \\ \max\Big(0; \dfrac{1}{2} - \dfrac{1}{q}\Big) \le \mu_k\lambda_k + \dfrac{\mu_k - 1}{q} < \dfrac{1}{q'}, \end{cases} \quad k = \overline{s_1 + 1, \sigma_1}, \tag{29}$$

$$\begin{cases} -\dfrac{1}{q} < \nu_k < \min\Big(0; \dfrac{1}{q'} - \dfrac{1}{2}\Big), \qquad \max\Big(0; \dfrac{1}{2} - \dfrac{1}{q}\Big) \le \lambda_k < \dfrac{1}{q'}, \\ k = \overline{s_1 + 1, m_1}, \qquad\qquad k = \overline{m_1 + \sigma_1 - s_1 + 1, 2m}, \end{cases} \tag{30}$$

$$\begin{cases} -\dfrac{1}{p} < \alpha_k < \dfrac{1}{p'}, \quad k = \overline{\sigma_2 + 1, m}, \\ -\frac{1}{q} < \beta_k < \frac{1}{q'}, \quad k = \overline{n_1 + s - \sigma_2 + 1, n}. \end{cases} \tag{31}$$

The solution of the systems (28)–(29) yields

$$\begin{cases} -\dfrac{1}{p} < \alpha_k < \min\Big(\dfrac{1}{p'}; \dfrac{1}{\mu_k} - \dfrac{1}{p}\Big), \quad -\dfrac{1}{q} < \beta_k < \min\Big(\dfrac{1}{q'}; \dfrac{1}{\mu_k} - \dfrac{1}{q}\Big), \\ \qquad k = \overline{\sigma_1 + 1, \sigma_2}, \qquad\qquad k = \overline{\sigma_2 + 1, s}, \end{cases} \tag{28_1}$$

$$\begin{cases} -\dfrac{1}{q} < \nu_k < \dfrac{1}{\mu_k} \min\Big(0; \dfrac{1 - \mu_k}{q}; \dfrac{\frac{q}{2} - \mu_k}{q}\Big), \quad k = \overline{1, s_1}, \\ \dfrac{1}{\mu_k} \max\Big(0; \dfrac{1 - \mu_k}{q}; \dfrac{\frac{q}{2} - \mu_k}{q}\Big) \le \lambda_k < \min\Big(\dfrac{1}{q'}; \dfrac{1}{\mu_k}\Big(1 - \dfrac{\mu_k}{q}\Big)\Big), \end{cases} \tag{29_1}$$

Thus the following theorem is valid.

Theorem 1. *Let* $L \subset C^1(t_1, t_2, \dots, t_s; \mu_1, \mu_2, \dots, \mu_s)$*,* $L_1, \widetilde{L}, L_2$ *be defined by virtue of* (3)*, and the weights* ρ_1, ρ_2 *be given by the equalities* (21)–(22) *in which the exponents* $\alpha_k, \beta_k, \nu_k, \lambda_k$ *satisfy the relations* (28_1)*,* (29_1)*,* (30)*,* (31)*, then for the problem* (10) *the Statements* I-III *of Theorem* A *are valid.*

4^0. Discussion of Results of Theorem 1

Theorem 1 allows one to define a family of those Smirnov weight spaces for which a character of solvability of the mixed problem (10) with domains containing piecewise-Lyapunov boundaries, remains the same as for domains containing Lyapunov boundaries; that is, the defining point for the solvability of the problem is the choice of number m_1 of those arc ends $\mathcal{L}_k$ at which the solution is required to fulfil the following comparatively hard condition B, i.e., "to be integrable with power weight concentrated at the given end with a negative exponent". For the Lyapunov boundaries, the Lebesgue exponents p and q are arbitrary numbers from the interval $(1; +\infty)$, and the exponents of the weights ρ_1 and ρ_2 satisfy the natural conditions (11) ensuring the summability of the boundary function of a solution (and necessary for the summability).

Theorem 1 states that for a sufficiently wide set of Smirnov classes the choice of number m_1 of ends at which the requirement B is fulfilled is defining for the problem (10) to be solved in them, and the solutions are being constructed explicitly for the curves from the class $C^1(t_1,t_2,\dots,t_s;\mu_1,\mu_2,\dots,\mu_s)$ with the condition $0<\mu_k\le 2$, $k=\overline{1,s}$. But this is not the case in all classes with Lyapunov boundaries. The conditions (28)–(31) show what parameters of Smirnov classes and within what limits (depending on the geometry of the boundary) one can take for which Statements I–III of Theorem A hold for the solution of problem (10).

It is not difficult to show that a set of spaces in which the abovementioned situation is realized is not empty, i.e., for the given $\mu_1,\mu_2,\dots,\mu_s$, $0<\mu_k\le 2$ there exist families $(p,q,\alpha_k,\beta_k,\nu_k,\lambda_k)$ satisfying the system (28)–(31); note that p and q are, as a rule, taken arbitrarily from the intervals $(1;+\infty)$ and $(2;+\infty)$, respectively, and $\alpha_k,\beta_k,\nu_k,\lambda_k$ belong to certain admissible intervals. A number of such families depend on those angular points and angle values at them which turn out to be the ends of the arcs of $\mathcal{L}_k$.

First of all, it should be noted that the fulfilment of the inequalities (30) and (31) and of the first inequalities of (28_1) and (29_1), is the necessary condition in the case of Lyapunov boundaries, as well (see Section 2^0). Hence when considering nonsmooth curves, the second inequalities of the systems (28_1) and (29_1) turn out to be supplementary ones.

It is seen from (28_1) and (29_1) that: if $\frac{1}{p'}\ge\frac{1}{\mu_k}-\frac{1}{p}$ (i.e. for $\mu_k>1$), we have $-\frac{1}{p}<\alpha_k<\frac{1}{\mu_k}-\frac{1}{p}$, but if $\frac{1}{p'}<\frac{1}{\mu_k}-\frac{1}{p}$ (i.e. for $\mu_k<1$), then $-\frac{1}{p}<\alpha_k<\frac{1}{p'}$.

Analogously, for $\mu_k>1$ we have $-\frac{1}{q}<\beta_k<\frac{1}{\mu_k}-\frac{1}{q}$, and for $\mu_k<1$, $-\frac{1}{q}<\beta_k<\frac{1}{q'}$.

This implies that for any given $p>1$, $q>1$ we can choose α_k,β_k, such that the conditions (28_1) are satisfied.

As for ν_k and λ_k, since $\frac{1}{q'}-\frac{1}{\mu_k}\big(1-\frac{\mu_k}{q}\big)=\frac{\mu_k-1}{\mu_k}$, we have:

if $q\ge 2$, $0<\mu_k<1$, then
$$-\frac{1}{q}<\nu_k<0,\quad \frac{q-2\mu_k}{2\mu_k q}\le\lambda_k<\frac{1}{q'}\,;$$
if $q>2$, $1<\mu_k\le 2$, then
$$-\frac{1}{q}<\nu_k<\frac{1-\mu_k}{q\mu_k}\,,\quad \max\Big(0;\frac{q-2\mu_k}{2\mu_k q}\Big)\le\lambda_k<\frac{q-\mu_k}{q\mu_k}\,;$$
if $q\le 2$, $0<\mu_k<1$, then

$$-\frac{1}{q}<\nu_k<\min\Big(0;\frac{q-2\mu_k}{2q\mu_k}\Big),\quad \frac{1-\mu_k}{\mu_k q}\le\lambda_k<\frac{1}{q'}\,;$$

if $q<2$, $1<\mu_k\le 2$, then

$$-\frac{1}{q}<\nu_k<\frac{q-2\mu_k}{2q\mu_k}\,,\quad 0\le\lambda_k<\frac{q-\mu_k}{q\mu_k}\,.$$

Obviously, the systems (28) and (29) are unsolvable for $q\le 2$, $0<\mu_k<1$ and $\mu_k q<1$, or when $q<2$, $1<\mu_k\le 2$ and $\mu_k>q$ (in both cases it is impossible to determine λ_k).

This means that if we take $q<2$, then at the angle points which are the arc ends of $\mathcal{L}_k$ for which either $\mu_k q<1$ or $\mu_k>q$, the weight multiplier $|t-t_k|$ should be with negative degree.

The same conditions (28)–(31) can also be interpreted as follows: if there is a class of unknown functions, i.e., p,q,ρ_1,ρ_2 are the given functions, then there arises the question: what kind is the set of piecewise-Lyapunov curves (that is, what kind is the set of admissible values for μ_k) for which Theorem 1 holds.

This question can be answered analogously to the above one, and we do not dwell on it.

5^0. The Classes of Harmonic Functions $e\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2),H(\widetilde{\gamma};\delta)\big)$

According to Statement 2, the equality

$$h\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2)\big)= \\ =h\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2)\big)\cap\Big\{u:u^+\in A(\overline{\Gamma}_2),\ \frac{\partial u^+}{\partial\zeta}\in L^q(\Gamma_2;\omega_2)\Big\}$$

holds.

We consider the boundary value problem (10) in somewhat narrower class

$$h\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2)\big)\cap \\ \cap\Big\{u:u^+\in A(\Gamma_2\cup\widetilde{\gamma}_2),\ \Big(\frac{\partial u}{\partial n}\Big)^+\in L^q(\Gamma_2;\omega_2)\Big\}. \tag{32}$$

Along with the classes (9) and (32), let us consider the class

$$h\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2),H(\widetilde{\gamma},\delta)\big)=h\big(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2)\big)\cap \\ \cap\Big\{u:u^+\in H(\widetilde{\gamma};\delta),\ \Big(\frac{\partial u}{\partial n}\Big)^+\in L^q(\Gamma_2;\omega_2)\Big\}. \tag{33}$$

where by $H(\widetilde{\gamma};\delta)$ we denote a set of functions satisfying the Hölder condition with exponent δ_k on that segment of $\widetilde{\gamma}$ which contains the end c_k, $k=\overline{1,2m}$, and $\delta=(\delta_1,\delta_2,\dots,\delta_{2m})$, $\delta_k\in(0,1]$. The only difference between the classes of functions defined by (32) and (33) is that in the first case u^+ is required to be absolutely continuous on $\widetilde{\gamma}$, and this together with the condition $\frac{\partial u^+}{\partial\zeta}\in L^q(\Gamma_2;\omega_2)$ implies that u^+ belongs to the fixed Hölder class $H(\widetilde{\gamma};\delta)$, where

$\delta = (\delta_1, \dots, \delta_{2m})$, $\delta_k = \frac{1}{q'}$, $k = \overline{1, m}$, $\delta_k = \frac{\lambda_k q}{q'}$, $k = \overline{m+1, 2m}$; in the second case, u^+ is required belong to any given class of functions. Thus the family of the classes (33), i.e., their union with respect to δ, is wider than the class (32).

In the sequel, we will need one property of the Poisson integral. This property is, probably, known, but we will cite it together with a rather simple proof.

6^0. On a Derivative of the Poisson Integral

Let there be given on $[0, 2\pi]$ the real periodic function $\psi(\zeta)$ satisfying the Hölder condition on the interval $[\alpha, \beta] \subset [0, 2\pi]$, with the exponent δ, $\delta \in (0, 1]$, while on the remaining part from $[0, 2\pi]$ equal to zero, and

$$u(r\,e^{i\varphi}) = \frac{1}{2\pi} \int\limits_{\alpha}^{\beta} \psi(\zeta) \frac{1 - r^2}{1 + r^2 - 2r\cos(\zeta - \varphi)}\, d\zeta. \tag{34}$$

We investigate the function $\frac{\partial u}{\partial r}$ in the neighborhood of the points $a = e^{i\alpha}$ and $b = e^{i\beta}$. We have

$$\frac{\partial u}{\partial r} = \frac{1}{2\pi} \int\limits_{\alpha}^{\beta} \psi(\zeta) \frac{-\varphi r + (2r^2 + 2)\cos(\zeta - \varphi)}{[1 + r^2 - 2r\cos(\zeta - \varphi)]^2}\, d\zeta. \tag{35}$$

If $\varphi < \alpha$, then we write $\frac{\partial u}{\partial r}$ in the form

$$\begin{aligned}\frac{\partial u}{\partial r} =& \frac{1}{2\pi} \int\limits_{\alpha}^{\beta} [\psi(\zeta) - \psi(\alpha)] \frac{-\varphi r + (2r^2 + 2)\cos(\zeta - \varphi)}{[1 + r^2 - 2r\cos(\zeta - \varphi)]^2}\, d\zeta + \\ &+ \frac{\psi(\alpha)}{2\pi} \int\limits_{\alpha}^{\beta} \frac{-\varphi r + (2r^2 + 2)\cos(\zeta - \varphi)}{[1 + r^2 - 2r\cos(\zeta - \varphi)]^2}\, d\zeta = u_1(r\,e^{i\varphi}) + u_2(r\,e^{i\varphi}).\end{aligned}$$

Passing to the limit, as $r \to 1$, and assuming $\varphi < \alpha$, we obtain

$$\lim_{\substack{r \to 1 \\ (\varphi < \alpha)}} u_2(r\,e^{i\varphi}) = \frac{\psi(\alpha)}{2\pi} \int\limits_{\alpha}^{\beta} \frac{d\zeta}{2\sin^2 \frac{\zeta - \varphi}{2}} = \frac{\psi(\alpha)}{2\pi} \Big[\operatorname{ctg} \frac{\alpha - \varphi}{2} - \operatorname{ctg} \frac{\beta - \varphi}{2} \Big]. \tag{36}$$

As for $u_1(r\,e^{i\varphi})$, we have

$$\lim_{r \to 1} u_1(r\,e^{i\varphi}) = -\frac{1}{2\pi} \int\limits_{\alpha}^{\beta} \frac{\psi(\zeta) - \psi(\alpha)}{2\sin^2 \frac{\zeta - \varphi}{2}}\, d\zeta = u_1^+(e^{i\varphi}).$$

Therefore

$$\begin{aligned}\big|u_1^+(e^{i\varphi})\big| &\le M_1\int\limits_\alpha^\beta \frac{|\zeta-\alpha|^\delta}{\sin^2\frac{\zeta-\varphi}{2}}\,d\zeta=\\ &=M_1\int\limits_\alpha^\beta \frac{|\zeta-\alpha|^\delta-|\zeta-\varphi|^\delta}{\sin^2\frac{\zeta-\varphi}{2}}\,d\zeta+M_1\int\limits_\alpha^\beta \frac{|\zeta-\varphi|^\delta}{\sin^2\frac{\zeta-\varphi}{2}}\,d\zeta\le\\ &\le M_1|\alpha-\varphi|^\delta\int\limits_\alpha^\beta \frac{d\zeta}{\sin^2\frac{\zeta-\alpha}{2}}\,d\zeta+M_2\int\limits_\alpha^\beta\frac{d\zeta}{|\zeta-\varphi|^{2-\delta}}=\\ &=M_1|\alpha-\varphi|^\delta\Big(\operatorname{ctg}\frac{\alpha-\varphi}{2}-\operatorname{ctg}\frac{\beta-\varphi}{2}\Big)+M_3\,K(\varphi),\end{aligned}$$

where $K(\varphi)=\frac{1}{|\alpha-\varphi|^{1-\delta}}$ if $\delta<1$, and $K(\varphi)=-\ln|\alpha-\varphi|$ if $\delta=1$.

It follows from the above estimate that in the left neighborhood of the point α

$$\big|u_1^+(e^{i\varphi})\big|<M\Big(\frac{1}{|\alpha-\varphi|^{1-\delta}}+\big|\ln|\alpha-\varphi|\big|\Big). \tag{37_1}$$

Analogously, in the right neighborhood of the point β we have

$$\big|u_1^+(e^{i\varphi})\big|<M\Big(\frac{1}{|\beta-\varphi|^{1-\delta}}+\big|\ln|\beta-\varphi|\big|\Big). \tag{37_2}$$

Consequently, if $\Gamma_a=\gamma_{a'a}$, where a' is the point on γ preceding a and lying near it, then u_1^+ belongs to $L^q(\Gamma_a;|\zeta-\alpha|^x)$, where $x>(1-\delta)-\frac1q=\frac{1}{q'}-\delta$.

Analogously, if Γ_b is a small right neighborhood of the point b, then $u_1^+\in L^q(\Gamma_b;|\zeta-\beta|^x)$ for the same values x.

Note here that $\frac{\partial u}{\partial n}=-\frac{\partial u}{\partial r}$, and the relations of the type (36), (37_1) and (37_2) are valid for $\frac{\partial u}{\partial n}$, as well.

Thus we have proved the following

Lemma 1. *If u is the harmonic function given by the equality* (34)*, where $\psi\in H([\alpha,\beta];\delta)$, $\delta=(\delta_\alpha,\delta_\beta)$, then for $e^{i\varphi}\in(\Gamma_a\cup\Gamma_b)$ we have*

$$\Big(\frac{\partial u}{\partial n}\Big)^+(e^{i\varphi})=\frac{\psi(\alpha)}{2\pi}\operatorname{ctg}\frac{\alpha-\varphi}{2}-\frac{\psi(\beta)}{2\pi}\operatorname{ctg}\frac{\beta-\varphi}{2}+\psi_1(\varphi), \tag{37}$$

in addition, $\psi_1\in L^q([\Gamma_a\cup\Gamma_b],|\varphi-\alpha|^x|\varphi-\beta|^y)$, where $x>-\frac1q+(1-\delta_\alpha)$, $y>-\frac1q+(1-\delta_\beta)$.

Lemma 2. *If $u\in h(\Gamma_{1p}(\rho_1),\Gamma'_{2q}(\rho_2))\cap H(\widetilde{\gamma};\delta)$, $\delta=(\delta_1,\dots,\delta_{2m})$, then for $e^{i\varphi}\in\Gamma_2$ we have*

$$\Big(\frac{\partial u}{\partial n}\Big)^+(e^{i\varphi})=\sum_{k=1}^{2m}h_k\operatorname{ctg}\frac{\alpha_k-\varphi}{2}+u_1(\varphi)+u_2(\varphi), \tag{38}$$

where $h_k = (-1)^{n(c_k)}\frac{u(c_k-)}{2\pi}$, *in which* $n(c_k) = 0$ *if* $c_k \in \{a_1, \dots, a_{2m}\}$, *and* $n(c_k) = 1$ *if* $c_k \in \{b_1, \dots, b_m\}$; $u_1 \in L^q(\widetilde{\gamma}; \rho)$, $\rho = \prod\limits_{k=1}^{2m} |e^{i\varphi} - c_k|^{x_k}$, $x_k > \frac{1}{q'} - \delta_k$, $u_2 \in L^q(\widetilde{\gamma}; \rho)$.

Proof. It follows from Statement 1 that u is representable by the Poisson integral (see 1.8), i.e.,

$$u(re^{i\varphi}) = \Big(\int\limits_{\Gamma_1\setminus\widetilde{\gamma}} + \int\limits_{\widetilde{\gamma}} + \int\limits_{\Gamma_2}\Big)\frac{1}{2\pi}\, u^+(e^{i\zeta})\, P(r, \zeta - \varphi) d\zeta. \tag{39}$$

It is evident that the character of behavior of the function $\left(\frac{\partial u}{\partial n}\right)^+$ on $\widetilde{\gamma}$ is defined by that of the second and third summands of the given sum. The character if the summand $\frac{1}{2\pi}\int\limits_{\widetilde{\gamma}} u^+(e^{i\zeta}) P(r, \zeta - \varphi) d\zeta$ is given by Lemma 1, since $u^+ \in H(\widetilde{\gamma}; \delta)$. Density $u^+(e^{i\zeta}) = \chi_{\Theta(\Gamma_2)}(\zeta) u^+(e^{i\zeta})$ in the third integral belongs, by Statement 2, to $A(\overline{\Gamma}_2)$ and $\frac{\partial u}{\partial \zeta} \in L^q(\Gamma_2; \rho_2)$, therefore $\chi_{\Theta(\Gamma_2)}(\zeta) u^+(e^{i\zeta})$ belongs to $L^q(\gamma; \rho_2)$. The above facts and equality (37) allow us to conclude that the representation (38) is valid. □

Lemma 3. *For the boundary function* u^+ *of the function* u *from the class* $h(\Gamma_{1p}(\rho_1), \Gamma'_{2q}(\rho_2))$ *to belong to the set*

$$h\big(\Gamma_{1p}(\rho_1), \Gamma'_{2q}(\rho_2)\big) \cap \Big\{u : u^+ \in H(\widetilde{\gamma}; \delta), \Big(\frac{\partial u}{\partial n}\Big)^+ \in L(\Gamma_2; \rho_2)\Big\},$$

it is necessary and sufficient for u^+ *to be continuous at the points* c_k *and the inclusion* $L^q(\Gamma_2; \rho) \subset L^q(\Gamma_2; \rho_2)$, $\rho = \prod\limits_{k=1}^{2m} |e^{i\varphi} - c_k|^{x_k}$, *i.e.*

$$x_k \geq \nu_k, \quad k = \overline{1, m_1}, \quad x_k \leq \lambda_k, \quad k = \overline{m_1 + 1, 2m} \tag{40}$$

to take place.

Proof. If we write u in the form

$$\begin{aligned} u(re^{i\varphi}) = {} & \frac{1}{2\pi}\int\limits_{\Theta(\gamma_{ac_k})} \big[u^+(e^{i\zeta}) - u(c_k-)\big] P(r, \zeta - \varphi) d\zeta + \\ & + \frac{1}{2\pi}\int\limits_{\Theta(\gamma_{c_k a})} \big[u^+(e^{i\zeta}) - u(c_k+)\big] P(r, \zeta - \varphi) d\zeta + \\ & + \frac{u(c_k-)}{2\pi}\int\limits_{\Theta(\gamma_{ac_k})} P(r, \zeta - \varphi) d\zeta + \frac{u(c_k+)}{2\pi}\int\limits_{\Theta(\gamma_{c_k a})} P(r, \zeta - \varphi) d\zeta, \end{aligned}$$

where $a \in \gamma$, and γ_{ac_k}, $\gamma_{c_k a}$ are two mutually disjoint arcs of the circumference γ, then using Lemma 1, we can easily obtain statements of the above lemma. □

7^0. Zaremba's Problem in the Class $h\big(L_{1p}(\rho_1), L'_{2q}(\rho_2); H(\widetilde{L};\delta)\big)$

Let D be a simply connected domain bounded by a simple Lyapunov curve L; $L_1, L_2, \widetilde{L}$ are the sets, and ρ_1 and ρ_2 are the weighted functions defined in Section 1^0. Suppose

$$e\big(L_{1p}(\rho_1), L'_{2q}(\rho_2), H(\widetilde{L};\delta)\big) = e\big(L_{1p}(\rho_1), L'_{2q}(\rho_2)\big)\cap \\ \cap\Big\{u : u^+ \in H(\widetilde{L};\delta), \Big(\frac{\partial u}{\partial n}\Big)^+\Big|_{L_2} \in L^q(L_2;\rho_2)\Big\}$$

and consider the mixed boundary value problem: Find the function u satisfying the conditions

$$\begin{cases} \Delta u = 0, \;\; u \in e\big(L_{1p}(\rho_1), L'_{2q}(\rho_2) H(\widetilde{L};\delta)\big), \\ u^+\big|_{L_1\setminus\widetilde{L}} = F, \;\; F \in L^p\big(L_1\setminus\widetilde{L};\rho_1\big), \;\; u^+\big|_{\widetilde{L}} = \Psi, \;\; \Psi \in H(\widetilde{L};\delta), \\ \Big(\frac{\partial u}{\partial n}\Big)^+\Big|_{L_2} = G, \;\; G \in L^q(L_2;\rho_2). \end{cases} \tag{41}$$

Using the conformal mapping, we reduce this problem to a circle. For the existence of a solution of the obtained problem, it suffices to fulfil the conditions (40) of Lemma 3, or the conditions

$$\delta_k \geq \frac{1}{q'} - \nu_k, \quad k = \overline{1, m_1}, \quad \delta_k \leq \frac{1}{q'} - \lambda_k, \quad k = \overline{m_1 + 1, 2m}, \tag{42}$$

with regard of the inequalities $x_k > \frac{1}{q'} - \delta$.

Moreover, to apply the method suggested in [2], we regard the conditions (11) are fulfilled. In obtaining the integral equation with respect to $\frac{\partial u^+}{\partial \zeta}$ (see Section 3^0 of [2]) we have instead of an absolute continuity of u on $\widetilde{\gamma} = w(\widetilde{L})$ to take advantage of the fact that $u \in H(\widetilde{L};\delta)$ and apply Lemmas 2 and 3.

It is not difficult to verify that the Poisson integral constructed by means of its solution, satisfies all requirements of (41). Thus we arrive at the following

Theorem 2. *If the conditions* (11) *and* (42) *are fulfilled, then Statements* I–III *of Theorem* A *are valid for the problem* (41).

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