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ON THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS FROM SMIRNOV CLASSES IN DOUBLY-CONNECTED DOMAINS

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Abstract. In the present paper, in a circular ring we investigate the Dirichlet problem for harmonic functions which are real parts of analytic functions from the weighted Smirnov class. The condition of solvability is pointed out and the unique solvability is proved. One representation of a derivative of conformal mapping of a circular ring onto a doubly-connected domain, bounded by a piecewise smooth curve, is given. On the basis of the obtained result we reveal the domains with piecewise smooth boundaries for which the unique solvability of the Dirichlet problem remains valid.

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There is a vast literature devoted to the investigation of the Dirichlet problem for harmonic functions in various assumptions regarding the given and unknown functions. Of special interest for consideration of the problem in the case in which the unknown function is required to be a real part of an analytic function from the Smirnov class. The Dirichlet problem in such a statement has been investigated thoroughly for a singly-connected domain (see, e.g., [1]–[3], [4], Ch. IV).

In the present paper we investigate the case of a doubly-connected domain. First, we investigate the Dirichlet problem in a circular ring for harmonic functions from the Smirnov weighted class. Under certain assumptions regarding weighted functions the condition of solvability is found and the unique solvability of the problem is proved (Sect.1). Then using the results obtained in [5] and [6], we present one representation of a derivative of conformal mapping of a circular ring onto a doubly-connected

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domain which is bounded by a piecewise smooth curve (Sect. 2). Relying on that result, we consider in such domains the Dirichlet problem in Smirnov weighted classes (Sect.3). Using the conformal mapping, the problem is reduced to the case of a circular ring but now in a new weighted class with weight depending both on the given weight and on the derivative of the conformal mapping. As far as in the framework of our investigation in Section 1 the weight function should satisfy certain conditions, there arise additional restrictions on the boundary curves. If these conditions are fulfilled, the problem turns out to be uniquely solvable.

1. The Dirichket Problem in the Smirnov Weighted Class in a Circular Ring

1.1. **Smirnov Classes and Some Properties of Functions from These Classes.** Let $\gamma_1 = \{z : |z| = 1\}$ and $\gamma_2 = \{z : |z| = \rho\}$, $\rho < 1$ be circumferences bounding the ring $K = \{z : \rho < |z| < 1\}$, $\gamma = \gamma_1 \cup \gamma_2$ and $\omega(z) \neq 0$, $z \in K$ be an analytic in K function.

Definition 1. We say that the one-valued analytic in the ring K function $\Phi(z)$ belongs to the Smirnov class $E^p(K;\omega)$, $p > 0$, if

$$\sup_{\rho<r<1} \int_0^{2\pi} \left|\omega(re^{i\zeta})\Phi(re^{i\zeta}\right|^p d\zeta < \infty. \tag{1}$$

Suppose $E^p(K) = E^p(K;1)$. Then the condition $\Phi \in E^p(K;\omega)$ is equivalent to the condition $\omega\Phi \in E^p(K)$.

Let K_i, $i = 1, 2$, be one of the domains bounded by the circumference γ_i containing K. For analytic in K_i functions Φ we assume

$$E^p(K_1;\omega) = \left\{\Phi : \sup_{0\le r<1} \int_0^{2\pi} \left|\omega(re^{i\zeta})\, \Phi(re^{i\zeta})\right|^p d\zeta < \infty\right\},$$

$$E^p(K_2;\omega) = \left\{\Phi : \sup_{r>\rho} \int_0^{2\pi} \left|\omega(re^{i\zeta})\, \Phi(re^{i\zeta})\right|^p d\zeta < \infty\right\},$$

$$E^p(K_i) = E^p(K_i;1).$$

Statement 1 ([7], p. 66)**.** *If $\Phi \in E^p(K)$, then $\Phi = \Phi_1 + \Phi_2$, where $\Phi \in E^p(K_i)$; and vice versa, if $\Phi = \Phi_1+\Phi_2$, $\Phi_i \in E^p(K_i)$, then $\Phi \in E^p(K)$.*

This implies that the class $E^p(K;\omega)$ coincides with the class of functions Φ, representable in the form

$$\Phi = \Phi_1 + \Phi_2, \quad \Phi_i \in E^p(K_i;\omega). \tag{2}$$

Taking now into account the well-known properties of the functions from the Smirnov classes for singly-connected domains (see, for e.g., [8], Ch. III, or [9], Ch.X), we obtain:

(i) every function $\Phi \in E^p(K)$ has almost everywhere on $\gamma = \gamma_1 \cup \gamma_2$ angular boundary values $\Phi^+(t)$, $t \in \gamma$, where $\Phi^+ \in L^p(\gamma)$; if, however, $\Phi \in E^p(K;\omega)$, and $\omega(z)$ has angular boundary values $\omega^+(t)$, $t \in \gamma$, then there likewise exists Φ^+, and $\Phi^+ \in L^p(\gamma;\omega)$;

(ii) the class $E^1(K;\omega)$ coincides with the class of functions, representable in K by the Cauchy integral, and in this case in the representation (2) we can take

$$\Phi_i(z) = (K_{\gamma_i}\Phi)(z) = \frac{1}{2\pi i}\int\limits_{\gamma_i} \frac{\Phi^+(t)dt}{t-z}\,, \quad z \in K_i, \tag{3}$$

(here the direction on γ_i is chosen in such a way that when moving in that direction the domain K remains on the left).

Statement 2. *If $\Phi \in E^p(K;\omega)$, $p \geq 1$, then*

$$\Phi(z) = \frac{1}{\omega(z)}\big(\Phi_1(z) + \Phi_2(z)\big), \quad z \in K, \tag{4}$$

where

$$\Phi_i(z) = \frac{1}{2\pi i}\int\limits_{\gamma_i} \frac{\omega^+(t)\Phi^+(t)dt}{t-z} = \frac{1}{2\pi i}\int\limits_{\gamma_i} \frac{f_i dt}{t-z}\,, \quad z \in K_i, \quad f_i \in L^p(\gamma_i). \tag{5}$$

Further, if $p > 1$, $\frac{1}{\omega} \in E^{p'}(K)$, $p' = \frac{p}{p-1}$, *then*

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{\varphi_1(t)dt}{t-z} + \frac{1}{2\pi i}\int\limits_{\gamma_2} \frac{\varphi_2 dt}{t-z}\,, \quad z \in K, \tag{6}$$

where $\varphi_i \in L^p(I;\omega_i)$, $i = 1,2$, $I = [0,2\pi]$, and ω_i is the narrowing on γ_i of the function ω, i.e., $\omega_1(e^{i\zeta}) = \omega^+(e^{i\zeta})$, $\omega_2(e^{i\zeta}) = \omega^+(\rho e^{i\zeta})$.

Proof. Equality (5) is easily obtained from Statement 1, since in the case under consideration $\omega\Phi \in E^p(K) \subseteq E^1(K)$, and hence $\omega\Phi = \Phi_1 + \Phi_2$, $\Phi_i \in E^p(K_i)$, where according to (3), $\Phi_i = K_{\gamma_i}(\omega^+\Phi^+)$, $i = 1,2$.

Next, when $\frac{1}{\omega} \in E^{p'}(K)$, then taking into account the fact that the Cauchy type integral with density from $L^p(\gamma_i)$, $p > 1$ belongs to $E^p(K^i)$ (see, for e.g., [10]), equality (4) allows us to conclude that $\frac{1}{\omega}\Phi_i \in E^1(K_i)$, and hence $\Phi \in E^1(K)$, where $(\omega\Phi)^+ \in L^p(\gamma)$. Therefore the narrowing on γ_i of the function Φ^+ belongs to $L^p(\gamma_i;\omega_i)$, $i = 1,2$. By virtue of (2)–(3) this implies that equality (6) is valid. □

Definition 2. Let Γ be a simple rectifiable curve. We say that the other than zero function ω almost everywhere on Γ belongs to the class $W^p(\Gamma)$, if the operator

$$T_\Gamma : f \to \omega\, S_\Gamma\, \frac{f}{\omega},$$

where

$$\Big(\omega\, S_\Gamma\, \frac{f}{\omega}\Big)(t) = \frac{\omega(t)}{2\pi i} \int\limits_\Gamma \frac{f(\tau)}{\omega(\tau)}\, \frac{d\tau}{\tau - t}\,, \quad t \in \Gamma,$$

is bounded in $L^p(\Gamma)$.

Assume $W^p = W^p(\gamma_1)$.

Statement 3. *If* $\omega \in E^\delta(K)$, $\delta > 0$, *and the functions* ω_0 *belong to the class* W^p, *then the function*

$$\begin{aligned} F(z) &= \frac{1}{2\pi i} \int\limits_{\gamma_1} \frac{f_1(t)dt}{t-z} + \frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{f_2(t)dt}{t-z} = \\ &= F_1(z) + F_2(z), \quad z \in K, \quad f_i \in L^p(\gamma_i, \omega_i) \end{aligned} \tag{7}$$

belongs to the class $E^p(K;\omega)$.

Proof. It suffices to establish that $F_i \in E^p(K_i;\omega)$. First of all, let us prove that the well-known Smirnov's theorem which says: if $\Phi \in E^p(K_i)$ and $\Phi^+ \in L^{p_1}(\gamma_1)$, $p_1 > p$, then $\Phi \in E^{p_1}(K_i)$ ([8], p. 116), can easily be extended to the doubly-connected domain K.

Thus let $\Phi \in E^p(K)$ and $\Phi^+ \in L^{p_1}(\gamma)$, $p_1 > p$. Let us prove that $\Phi \in E^{p_1}(K)$.

Since $\Phi \in E^p(K)$, therefore by Statement 2, we have $\Phi = \Phi_1 + \Phi_2$, $\Phi \in E^p(K_i)$ and it follows from the condition $\Phi^+ \in L^{p_1}(\gamma)$ that $\Phi_i^+ \in L^{p_1}(\gamma_i)$, $i = 1, 2$. Then by the above-mentioned Smirnov's theorem, we have $\Phi_i \in E^{p_1}(K_i)$, and since $\Phi = \Phi_1 + \Phi_2$, by virtue of Statement 1, we can conclude that $\Phi \in E^{p_1}(K)$.

Let us now prove that the functions F_i from equality (7) belong to $E^p(K_i;\omega)$. Since $f_i \in L^p(\gamma_i;\omega_i)$, therefore $f_i = \varphi_i \omega_i^{-1}$, where $\varphi_i \in L^p(\gamma_i)$. As far as $\omega_i \in W^p$, there exists $\varepsilon > 0$, such that $\omega_i \in L^{p+\varepsilon}(\gamma_i)$, $\omega_i^{-1} \in L^{p'+\varepsilon}(\gamma_i)$ (see, for e.g., [1]), and hence $\varphi_i \omega_i^{-1} \in L^{1+\delta}(\gamma_i)$, $\delta > 0$. Then $F_i(z) \in E^{1+\delta}(K)$, and since $\omega(z) \in E^\delta(K)$, $M(z) = \omega(z)F(z)$ belongs to $E^\eta(K)$ for some $\eta > 0$. Consider the functions $M_i^+(e^{i\zeta})$, $i = 1, 2$. We have

$$M_i(e^{i\zeta}) = \omega_i(e^{i\zeta}) \bigg[\frac{\varphi_i(e^{i\zeta})}{2\omega(e^{i\zeta})} + \frac{1}{2\pi i} \int\limits_{\gamma_i} \frac{\varphi_i(\tau)}{\omega_i(\tau)}\, \frac{d\tau}{\tau - e^{i\zeta}} \bigg],$$

and it follows from the condition $\omega_i \in W^p$ that $M_i^+ \in L^p(\gamma_i)$. Then $M(t) = \omega(t)F(t)$ belongs to $L^p(\gamma)$, and hence $M(z) \in E^p(K)$ (owing of the just now proven Smirnov's theorem for the ring). This implies that $F(z) \in E^p(K;\omega)$. $\square$

In what follows, we will need the following class of weighted functions:

$$W_E^p=\Big\{\omega:\omega^{\pm1}\in\bigcup_{\delta>0}E^{\delta}(K),\quad \omega_i\in W^p\Big\}.$$

If $\omega\in W_E^p$, then $\frac{1}{\omega}\in E^{\eta}(K)$, $\eta>0$, and $\frac{1}{\omega^+}\in L^{p'+\varepsilon}(\gamma)$, hence $\frac{1}{\omega}\in E^{p'+\varepsilon}(K)$. Consequently, the condition from Statement 2 is fulfilled. Thus from Statements 2 and 3 we have

Theorem 1. *If $\omega\in W_E^p$, $p>1$, then the class $E^p(K;\omega)$ coincides with the class of functions Φ, representable in the form*

$$\Phi(z)=\frac{1}{2\pi i}\int_{\gamma_1}\frac{\varphi_1(t)dt}{t-z}+\frac{1}{2\pi i}\int_{\gamma_2}\frac{\varphi_2(t)dt}{t-z}=$$
$$=\Phi_1(z)+\Phi_2(z),\quad z\in K,\quad \varphi_i\in L^p(\gamma_i;\omega_i).$$

Definition 3. The harmonic in K function u belongs to the class $e^p(K;\omega)$, if $u=\operatorname{Re}\Phi$, where $\Phi\in E^p(K;\omega)$.

Statement 4. *If $u\in e^p(K;\omega)$, $\omega\in W_E^p$, then there exist the functions $\mu\in L^p(I;\omega_1)$ and $\lambda\in L^p(I;\omega_2)$, $I=[0,2\pi]$, such that the equalities*

$$u(z)=u(re^{i\varphi})=\frac{1}{2\pi}\int_0^{2\pi}\mu(\zeta)\frac{1-r^2}{1+r^2-2r\cos(\zeta-\varphi)}\,d\zeta+$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\lambda(\zeta)\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\zeta-\varphi)}\,d\zeta, \tag{8}$$

$$\int_0^{2\pi}\lambda(\zeta)d\zeta=0 \tag{9}$$

hold.

Indeed, since $u\in e^p(K;\omega)$, $\omega\in W_E^p$, therefore $u=\operatorname{Re}(\Phi_1+\Phi_2)$, where Φ_i are defined by equality (6). Note here that $\varphi_i\in L^{1+\varepsilon}(\gamma_1)$, $\varepsilon>0$. Hence $\Phi_i\in E^1(K)$. But the functions from $E^1(K_i)$ are representable by the Schwarz integrals (see, for e.g., [12], p. 84)

$$\Phi_1(z)=\frac{1}{4\pi}\int_0^{2\pi}\operatorname{Re}\Phi_1^+(e^{i\zeta})\frac{e^{i\zeta}+z}{e^{i\zeta}-z}\,d\zeta+\operatorname{Im}\Phi_1(0),\quad z\in K_1, \tag{10}$$

$$\Phi_2(z)=\frac{1}{4\pi}\int_0^{2\pi}\operatorname{Re}\Phi_2^+(\rho e^{i\zeta})\frac{\rho e^{i\zeta}+z}{\rho e^{i\zeta}-z}\,d\zeta,\quad \Phi_2(\infty)=0,\quad z\in K_2, \tag{11}$$

and hence

$$\int_0^{2\pi} \operatorname{Re}\Phi_2 d\zeta = 0. \tag{12}$$

Denote $\mu(\zeta) = \operatorname{Re}\Phi_1^+(e^{i\zeta})$, $\lambda(\zeta) = \operatorname{Re}\Phi_2^+(\rho e^{i\zeta})$. Condition $\omega \in W_E^p$ implies that $\mu \in L^p(I;\omega_1)$, $\lambda \in L^q(I;\omega_2)$. From the equality $u = \operatorname{Re}(\Phi_1 + \Phi_2)$ we now obtain equality (8), and equality (12) is transformed into equality (9).

1.2. **The Dirichlet Problem in the Class $e^p(K;\omega)$.** Consider the Dirichlet problem in the following statement: find the function u by the conditions

$$\begin{cases} u \in e^p(K;\omega), \quad p > 1, \\ u^+(e^{i\zeta}) = f(\zeta), \ u^+(\rho e^{i\zeta}) = g(\zeta), \ f \in L^p(I;\omega_1), \ g \in L^p(I;\omega_2), \end{cases} \tag{13}$$

where $u^+(e^{i\zeta})$, $u^+(\rho e^{i\zeta})$ are the angular boundary values $u(z)$ on γ_1 and γ_2, respectively. They are defined almost everywhere on $[0, 2\pi]$, and the equalities in (13) are likewise understood almost everywhere.

A solution of the problem (13) will be sought in the form (8).

Taking into account the boundary conditions (13), the properties of the Poisson integral and the condition (9), we obtain the following system of integral equations:

$$\begin{cases} \mu(\zeta) - \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \lambda(\alpha) \dfrac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\alpha-\zeta)}\, d\alpha = f(\zeta), \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mu(\alpha) \dfrac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\alpha-\zeta)}\, d\alpha - \lambda(\zeta) = g(\zeta), \end{cases} \tag{14}$$

with the supplementary condition (9) with respect to λ.

From the second equality of (14) we define the function λ and substituting it into the first one. We obtain

$$\begin{aligned} \mu(\zeta) - \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2\pi} \int_0^{2\pi} \mu(\beta) \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\beta-\alpha)}\, d\beta \right] \times \\ \times \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\alpha-\zeta)}\, d\alpha = \nu(\zeta), \end{aligned} \tag{15}$$

where

$$\nu(\zeta) = f(\zeta) - \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\alpha-\zeta)}\, d\alpha. \tag{16}$$

Obviously, $\nu \in L^p(I;\omega_1)$.

Consider the function

$$V(r,\zeta)=\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-(\rho r)^2}{1+(\rho r)^2-2\rho r\cos(\alpha-\zeta)}\,d\alpha, \tag{17}$$
$$0\le r<1,\quad 0<\rho<1.$$

This function is harmonic in K_1 and continuous in $\overline{K}_1$. Therefore

$$V(1,\zeta)=\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\zeta)}\,d\alpha. \tag{18}$$

Consequently,

$$\begin{aligned}V(\rho,\zeta)&=\frac{1}{2\pi}\int\limits_0^{2\pi}V(1,\alpha)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\zeta)}\,d\alpha=\\&=\frac{1}{2\pi}\int\limits_0^{2\pi}\left[\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\beta)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\beta-\alpha)}\,d\beta\right]\times\\&\times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\zeta)}\,d\alpha,\end{aligned} \tag{19}$$

and we can rewrite equality (15) as

$$\mu(\zeta)-V(\rho;\zeta)=\nu(\zeta). \tag{20}$$

But according to (17),

$$\begin{aligned}V(\rho,\zeta)&=\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-(\rho^2)^2}{1+(\rho^2)^2-2\rho^2\cos(\alpha-\zeta)}\,d\alpha=\\&=\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^4}{1+\rho^4-2\rho^2\cos(\alpha-\zeta)}\,d\alpha.\end{aligned}$$

Now, equality (15) with regard for (20) takes the form

$$\mu(\zeta)-\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^4}{1+\rho^4-2\rho^2\cos(\alpha-\zeta)}\,d\alpha=\nu(\zeta). \tag{21}$$

Assume

$$K(\alpha,\zeta)=-\frac{1}{2\pi}\,\frac{1-\rho^4}{1+\rho^4-2\rho^2\cos(\alpha-\zeta)}\,.$$

Then equality (21) is has the form

$$\mu(\zeta)+\int_0^{2\pi}\gamma(\alpha)\,K(\alpha,\zeta)\,d\alpha=\nu(\zeta), \tag{22}$$

in the class $L^p(I;\omega_1)$, in which $K(\alpha,\zeta)$ is continuous on the square $I\times I$, and $K(\zeta,\alpha)=K(\alpha,\zeta)$. By virtue of the above-said, (22) is the Fredholm equality in $L^p(I;\omega)$, and its conjugate equation has the form

$$\gamma(\zeta)+\int_0^{2\pi}\gamma(\alpha)\,K(\zeta,\alpha)\,d\alpha=m(\zeta),$$

which is considered in the class $L^{p'}(I;\omega_1^{-1})$.

Consider the case $\nu(\zeta)\equiv 0$, i.e., the equation

$$\mu(\zeta)-\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^4}{1+\rho^4-2\rho^2\cos(\alpha-\zeta)}\,d\alpha=0. \tag{23}$$

Let us prove by induction that for every natural n a solution μ of equation (23) satisfies the equality

$$\mu(\zeta)-\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n}}{1+\rho^{4n}-2\rho^{2n}\cos(\zeta-\alpha)}\,d\alpha=0. \tag{24}$$

Thus we assume that for some n the expression (24) is valid.

Multiplying equality (24) by $\frac{1}{2\pi}\,\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}$ and integrating it with respect to ζ, we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\mu(\zeta)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}\,d\zeta-\\&\quad-\frac{1}{2\pi}\int_0^{2\pi}\bigg[\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n}}{1+\rho^{4n}-2\rho^{2n}\cos(\zeta-\alpha)}\,d\alpha\bigg]\times\\&\quad\times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}\,d\zeta=0\end{aligned} \tag{25}$$

Consider the harmonic in K_1 and continuous in $\overline{K}_1$ function

$$V_{1,\rho,n}(r,\zeta)=\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-(\rho^{2n}r)^2}{1+(\rho^{2n}r)^2-2\rho^{2n}r\cos(\zeta-\alpha)}\,d\alpha,\quad r\le 1. \tag{26}$$

This function is representable by the Poisson integral, hence

$$V_{1,\rho,n}(\rho,\varphi)=\frac{1}{2\pi}\int\limits_0^{2\pi}V_{1,\rho,n}(1,\zeta)\frac{1-\rho^2}{1+\rho^2-2\rho r\cos(\zeta-\varphi)}\,d\zeta=$$
$$=\frac{1}{2\pi}\int\limits_0^{2\pi}\left[\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n}}{1+\rho^{4n}-2\rho^{2n}\cos(\zeta-\alpha)}\,d\alpha\right]\times$$
$$\times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}\,d\zeta. \tag{27}$$

But according to (26), we have

$$V_{1,\rho,n}(\rho,\varphi)=\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+2}}{1+\rho^{4n+2}-2\rho^{2n+1}\cos(\alpha-\varphi)}\,d\alpha. \tag{28}$$

By equalities (27)–(28), equality (25) takes the form

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\zeta)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}\,d\zeta-$$
$$-\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+2}}{1+\rho^{4n+2}-2\rho^{2n+1}\cos(\alpha-\varphi)}\,d\alpha. \tag{29}$$

Consider now the function

$$V_{2,\rho,n}(r,\varphi)=\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-(\rho^{2n+1}r)^2}{1+(\rho^{2n+1}r)^2-2\rho^{2n+1}r\cos(\alpha-\varphi)}\,d\alpha. \tag{30}$$

Then

$$V_{2,\rho,n}(\rho,\varphi)=\frac{1}{2\pi}\int\limits_0^{2\pi}V_{2,\rho,n}(1,\zeta)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}\,d\zeta=$$
$$=\frac{1}{2\pi}\int\limits_0^{2\pi}\left[\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+2}}{1+\rho^{4n+2}-2\rho^{2n+1}\cos(\alpha-\zeta)}\,d\alpha\right]\times$$
$$\times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}\,d\zeta. \tag{31}$$

From (30) we have

$$V_{2,\rho,n}(\rho,\varphi)=\frac{1}{2\pi}\int\limits_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+4}}{1+\rho^{4n+4}-2\rho^{2n+2}\cos(\alpha-\varphi)}\,d\alpha. \tag{32}$$

(31) and (32) now yield

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+2}}{1+\rho^{4n+2}-2\rho^{2n+1}\cos(\alpha-\zeta)}\,d\alpha\right]\times$$
$$\times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\zeta-\varphi)}\,d\zeta=$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+4}}{1+\rho^{4n+4}-2\rho^{2n+2}\cos(\alpha-\varphi)}\,d\alpha. \tag{33}$$

Multiplying equality (29) by $\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\varphi-\beta)}$ and integrating it with respect to φ, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}\mu(\zeta)\frac{1-\rho^2}{1+\rho^2-2\rho r\cos(\zeta-\varphi)}\,d\zeta\right]\frac{1-\rho^2}{1+\rho^2-2\rho r\cos(\varphi-\beta)}\,d\varphi-$$
$$-\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+2}}{1+\rho^{4n+2}-2\rho^{2n+1}\cos(\alpha-\varphi)}\,d\alpha\right]\times$$
$$\times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\varphi-\beta)}\,d\varphi=0. \tag{34}$$

in which the first summund on the left is, according to (16), equal to $\mu(\beta)$, and the second one, by virtue of (33), is equal to

$$-\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+4}}{1+\rho^{4n+4}-2\rho^{2n+2}\cos(\alpha-\beta)}\,d\alpha.$$

Therefore from (34) we finally get

$$\mu(\beta)-\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\frac{1-\rho^{4n+4}}{1+\rho^{4n+4}-2\rho^{2n+2}\cos(\alpha-\beta)}\,d\alpha=0. \tag{35}$$

Thus assuming that equality (24) is valid, we have stated that equality (35) is true. For $n=1$, the validity of (24) follows from (23). Thus equality (24) is proved for any $n\in\mathbb{N}$.

Passing to the limit as $n\to\infty$, from (24) we have

$$\mu(\beta)-\frac{1}{2\pi}\int_0^{2\pi}\mu(\alpha)\,d\alpha=0. \tag{36}$$

This implies that $\mu=\text{const}$. Consequently, only constant functions are the solutions of equation (22) and its conjugate for $\nu=0$ and $m=0$, respectively. Hence by the Fredholm property, equation (24) is solvable, if and only if

$$\int_0^{2\pi} C\,\nu(\zeta)\,d\zeta = 0, \quad C = \text{const},$$

i.e., if

$$\int_0^{2\pi} \left[f(\zeta) - \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\alpha-\zeta)}\,d\alpha \right] d\zeta = 0.$$

whence we have

$$\int_0^{2\pi} f(\zeta)\,d\zeta = \int_0^{2\pi} g(\zeta)\,d\zeta. \tag{37}$$

Thus equation (15) and, consequently, the system (14) is solvable, if and only if the given boundary functions f and g satisfy the condition (37). If this condition is fulfilled, then equation (22) has a solution μ depending on one arbitrary parameter C,

$$\mu = \widetilde{\mu} + C \tag{38}$$

where $\widetilde{\mu}$ is a particular solution of equation (22) of the class $L^p(I;\omega_1)$. From the second equation of the system (14) we find the function λ,

$$\lambda(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\mu}(\alpha) \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\zeta)}\,d\alpha + C - g(\zeta), \tag{39}$$

which belongs to $L^p(I;\omega_2)$. The condition (9) implies that

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \lambda(\zeta)\,d\zeta = \frac{1}{(2\pi)^2} \int_0^{2\pi} \widetilde{\mu}(\alpha)\,d\alpha + C - \frac{1}{2\pi} \int_0^{2\pi} g(\zeta)\,d\zeta,$$

i.e.,

$$C = C_0 = -\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{\widetilde{\mu}(\zeta)}{2\pi} - g(\zeta) \right] d\zeta. \tag{40}$$

Therefore if in equalities (38) and (39) we take $C = C_0$, then the functions μ and λ defined by these equalities, provide us with a solution of the system (14) which satisfies the condition (9).

Let us show that this solution for $f = g = 0$ is given by a pair $\mu = 0$, $\lambda = 0$. Indeed, integrating the second of equalities (14) and taking into account (9), we find that

$$\int_0^{2\pi} \mu(\zeta)\,d\zeta = 0.$$

Then by virtue of (36), we have $\widetilde{\mu}=0$, and it follows from equalities (40) and (39) that $C_0=0$, $\lambda(\zeta)=0$. This, according to (8), implies that the homogeneous problem (19) has a unique solution $u=0$.

Thus the following theorem is proved.

Theorem 2. *If* $\omega\in W^p_E$, $p>1$, *then the Dirichlet problem* (13) *is solvable in the class* $E^p(K;\omega)$, *if and only if the condition* (37) *is fulfilled. If this condition is fulfilled, the problem has a unique solution given by equality* (8), *where the functions* μ *and* λ *are defined by equalities* (38) *and* (39) *in which* $\widetilde{\mu}$ *is a particular solution of equation* (22), *and* C *is given by equality* (40).

2. On the Conformal Mapping of the Circular Ring onto a Doubly-Connected Domain Bounded by Piecewise Smooth Curves

2.1. **Smirnov Classes in Doubly-Connected Domains.** Let D be a doubly-connected domain bounded by the rectifiable Jordan curves Γ_1 and Γ_2; Γ_2 lies in a finite domain which is bounded by Γ_1, and let $\rho(z)$ be an analytic in D, everywhere other than zero function.

Definition 4. We say that the one-valued analytic in D function $\Phi(z)$ is in the class $E^p(D;\rho)$, if there exists an increasing sequence of doubly-connected domains $\{D_i\}$ with rectifiable boundaries $\mathcal{L}^i$, exhausting the domain D and such that

$$\sup_i\int\limits_{\mathcal{L}^i}\big|\rho(z)\,\Phi(z)\big||dz|<\infty.$$

Assume $E^p(D)\equiv E^p(D;1)$, $e^p(D;\rho)=\{u:u=\operatorname{Re}\Phi,\ \Phi\in E^p(D;\rho)\}$.

Statement 5 ([7], [13]). (i) *For any function* Φ *from* $E^p(D)$ *we can take in the capacity of* $\mathcal{L}^i$ *the images of circumferences with center* $u=0$ *under the conformal mapping of the ring* K *onto* D; (ii) $\Phi(z)$ *has almost everywhere on* $\Gamma=\Gamma_1\cup\Gamma_2$ *angular boundary values* $\Phi^+(t)$, *where* $\Phi^+(t)\in L^p(\Gamma)$; (iii) $\Phi\in E^p(D)$, *if and only if the function* $\Phi(z(\omega))\sqrt[p]{z'(w)}$ *belongs to* $E^p(K)$, *where* $z=z(w)$ *is the conformal mapping of* K *onto* D, *such that* $z(\gamma_i)=\Gamma_i$, $i=1,2$; (iv) *the class* $E^p(D)$ *coincides with the class of functions* Φ, *representable in the form* $\Phi=\Phi_1+\Phi_2$, $\Phi_i\in E^p(D_i)$, *where* D_i *is that of the domains bounded by* Γ_i *which contains* D.

Statement 6. *If* $z=z(w)$ *is the conformal mapping of the ring* K *onto the domain* D *bounded by rectifiable Jordan curves, then* $z'(w)\in E^1(K)$.

Proof. In the case under consideration, $z(w)$ is the function, analytic in K and continuous in $\overline{K}$, therefore

$$z(w)=\frac{1}{2\pi i}\int\limits_{\gamma_1}\frac{z(\tau)d\tau}{\tau-w}+\frac{1}{2\pi i}\int\limits_{\gamma_2}\frac{z(\tau)d\tau}{\tau-w}=z_1(w)+z_2(w).$$

The function z_1 is analytic in the circle K_1 and continuous in $\overline{K}_1$; its boundary values form on γ_1 the function $z_1(\tau) = z(\tau)$, $\tau \in \gamma_1$, which is, in fact, the equation of the rectifiable curve. Hence z_1 is the function with bounded variation. As is known, in this case the function z_1 is absolutely continuous, and $z_1' \in E^1(K_1)$ (see [9], p.395).

Considering the function $\zeta(w) = z_2(\frac{\rho}{w})$ in K_1, we can, according to the above result, conclude that $\zeta' \in E^1(K_1)$, and consequently, $z_2 \in E^1(K_2)$. Thus $z' = z_1' + z_2'$, $z_i' \in E^1(K_i)$. By item (iv) of Statement 5, we have $z' \in E^1(K)$. □

2.2. **The Properties of the Derivative of Conformal Mapping of the Ring onto the Domain Bounded by Piecewise Smooth Curves.** Let Γ_1 and Γ_2 be the Jordan piecewise smooth curves bounding the doubly-connected domain D, where Γ_2 lies in a finite domain bounded by the curve Γ_1.

Denote by $t_1, t_2, \dots, t_n$ angular points of the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. We assume that at those points the sizes of angle, interior with respect to the domain D, are equal to $\pi\nu_k$, $0 \le \nu_k \le 2$. A set of such curves we denote by $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$.

Since $z' \in E^1(K)$, almost everywhere on γ there exist angular boundary valuers $z'(t)$. Therefore there exist the points A, B and C, D on γ_1 and γ_2, respectively, such that: (a) the pairs A, D and B, C lie on the radii, outcoming from the point $z = 0$; (b) at those points there exist angular boundary values; (c) the images of the arcs $\breve{BA}$ and $\breve{DC}$ under the conformal mapping are free from the angular points Γ. We assume that the point A follows the point B when passing round on γ_1 in the positive direction (i.e., counterclockwise).

Note here that as soon as the points satisfying the conditions (a)–(c) are found, there exist in any small one-sided neighborhoods of those points another points with the same properties.

We draw through the points A and D an ellipse whose largest axis is the segment $\overline{AD}$. Denote by l_{DA} that part of the ellipse under the motion along which from D to A the interior portion bounded by the ellipse remains on the left, and let l_{AD} be the remaining part of the ellipse. Just as above, we draw through the points B and C the ellipse and choose the arcs l_{CB} and l_{BC}. Assume that the constructed by us ellipses do not intersect. Suppose

$$L_1 = l_{DA} \cup \breve{AB} \cup l_{BC} \cap \breve{CD}, \quad L_2 = l_{AD} \cup \breve{DC} \cup l_{CB} \cap \breve{BA}.$$

By the construction, L_1 and L_2 are the simple closed Ljapunov curves. By G_1 and G_2 we denote finite domains, bounded respectively by L_1 and L_2.

Let $z_i(w) = z(w)$, $w \in G_i$, $i = 1, 2$, and assume $Q_i = z(G_i)$. Then Q_i are the simply connected domains. Note that by the condition (c), Q_1 is bounded by the piecewise smooth curve, while Q_2 by the smooth curve.

Below we will need the following facts.

Statement 7 (see, for e.g., [6], [4], p. 153). *If $z = z(w)$ is the conformal mapping of the unit circle onto the simply connected domain G bounded by a simple curve from $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 \le \nu_k \le 2$, then*

$$z'(w) = \prod_{k=1}^{n} (w - a_k)^{\nu_k - 1} z_0(w), \quad z(a_k) = t_k, \tag{41}$$

$$z_0^{\pm\alpha} \in \bigcap_{\delta>1} E^\delta(K), \quad z_0^\alpha(e^{i\zeta}) \in \bigcap_{\delta>1} W^\delta, \quad \alpha \in \mathbb{R}. \tag{42}$$

Statement 8 ([14]). *If Γ is the simple closed curve with the chord condition (i.e., for which the ratio of the length of the smallest arc, connecting two arbitrary points, to that of the chord, connecting the same points, is the bounded function), in particular, if Γ is an arbitrary piecewise smooth curve without cusps, and if for the given on that curve almost everywhere nonnegative finite and other than zero function ω, we have*

$$\sup_l \left(\frac{1}{|l|} \int_l \omega^p(s) ds \right)^{1/p} \left(\frac{1}{|l|} \int_l \omega^{-p'}(s) ds \right)^{1/p'} < \infty,$$

where the upper bound is taken over the whole arcs l, $|l| < |\Gamma|$, then $\omega \in W^p(\Gamma)$.

Let now G be a simply connected domain, bounded by a simple rectifiable Ljapunov curve, while the domain D by a piecewise smooth curve. Next, let $z = \varphi(\zeta)$ be the conformal mapping of the unit circle onto Q, and $\zeta = \zeta(w)$ be the same mapping of G onto the unit circle. The function $\varphi(\zeta(w))$ maps conformally the domain G onto Q. Using Kellogg's theorem (see, for e.g., [9], p.411) and the fact that: if the schlicht function is different from zero and belongs to the Hölder class, then the inverse function possesses the same properties, and from Statements 7 and 8 we have

Statement 9. *If $z = z(w)$ is the conformal mapping of the simply connected domain G bounded by the Ljapunov curve onto the domain, bounded by the curve from $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k < 2$, then for z' the representation* (41) *in which*

$$z_0^{\pm\alpha} \in \bigcap_{\delta>1} E^\delta(G), \quad z_0^{\pm\alpha}(t) \in \bigcap_{\delta>1} W^\delta(\Gamma), \quad \alpha \in \mathbb{R}.$$

is valid.

Using this statement, we can conclude that

$$z_2'(w) \in \bigcap_{\delta>1} E^\delta(G_2), \;\; z_2'(w) \in \bigcap_{\delta>1} W^\delta(\breve{BA}), \;\; z_2'(w) \in \bigcap_{\delta>1} W^\delta(\breve{DC}), \tag{43}$$

$$\begin{aligned} &z_1'(w) = \prod_{k=1}^{n}(w-a_k)^{\nu_k-1} z_{1,0}(w), \quad w \in G_1, \quad z(a_k) = t_k, \\ &z_{1,0} \in \bigcap_{\delta>1} E^\delta(G_1), \quad z_{1,0}^\alpha(t) \in \bigcap_{\delta>1} W^\delta(\breve{AB}), \\ &z_{1,0}^\alpha(t) \in \bigcap_{\delta>1} W^\delta(\breve{CD}), \quad \alpha \in \mathbb{R}. \end{aligned} \tag{44}$$

On the basis of the above facts we prove that for the doubly connected domain D with the boundary from $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k < 2$ we have

$$z'(w) = \prod_{k=1}^{n}(z-a_k)^{\nu_k-1} z_0(w), \quad z_0^{\pm 1} \in \bigcap_{\delta>1} E^\delta(K). \tag{45}$$

Indeed, let $\delta > 1$ be an arbitrary number. Since $z_2 \in E^\delta(G_2)$, $z_{1,0} \in E^\delta(G_1)$, there exist a sequence of the curves $L_{2n} \subset G_2$ and $L_{1n} \subset G_1$ converging to L_2, and that of the curves L_2 and $L_{1,} \subset G_2$ converging to L_1, such that

$$\sup_n \int_{L_{2n}} |z_2'|^\delta |dw| < \infty, \quad \sup_n \int_{L_{1n}} |z_{1,0}'|^\delta |dw| < \infty. \tag{46}$$

Since in G_i we have $z_i(w) = z(w)$, and the distance from the set $\{t_1, \dots, t_n\}$ to L_2 is positive, this implies that

$$\sup_n \int_{L_{in}} \left| \frac{z'(w)}{\prod_{k=1}^{n}(w-a_k)^{\nu_k-1}} \right|^\delta |dw| < \infty, \quad i = 1, 2. \tag{47}$$

As far as L_{1n} converges to L_1, the curve L_{1n} for large n intersects the segment $\overline{AD}$ in small neighborhoods of the points A and D. The first points of intersection we denote by a_{1n} and d_{1n}. Just in the same way, the curves L_{2n} intersect $\overline{AD}$ in small neighborhoods of the points A and D, and let a_{2n} and d_{2n} be the first points of intersection. Analogously, we choose the points $b_{1n}, c_{1n}, b_{2n}, c_{2n}$ on the chord $\overline{BC}$. Consider the curves

$$\begin{aligned} \gamma_{1n} &= L_{a_{1n}b_{1n}} \cup \overline{b_{1n}b_{2n}} \cup L_{b_{2n}a_{2n}} \cup \overline{a_{2n}a_{1n}}, \\ \gamma_{2n} &= L_{d_{1n}c_{1n}} \cup \overline{c_{1n}c_{2n}} \cup L_{c_{2n}d_{2n}} \cup \overline{d_{2n}d_{1n}}, \end{aligned}$$

where $L_{a_{1n}b_{1n}}$, $L_{d_{1n}c_{1n}}$ denote the parts of the curve L_{1n} with the ends $a_{1n}, b_{1,n}$ and d_{1n}, c_{1n}, respectively. The arcs $L_{b_{2n}a_{2n}}$ and $L_{c_{2n}d_{2n}}$ are defined analogously; $\overline{b_{1n}b_{2n}}$, $\overline{a_{1n}a_{2n}}$, $\overline{c_{1n}c_{2n}}$ and $\overline{d_{1n}d_{2n}}$ are the segments on the chords $\overline{AB}$ and $\overline{CD}$. Obviously, γ_{in}, $i = 1, 2$, converge to the circumferences γ_i.

Since the function z' at the points A, B, C, D has angular boundary values and inequalities (47) are valid, it is not difficult to conclude that

$$\sup_n \int\limits_{\gamma_{in}} \left|\frac{z'(w)}{\prod_{k=1}^n(\nu-a_k)^{\nu_k-1}}\right|^\delta |dw| < \infty.$$

Thus we have proved the inclusion $z_0 \in E^\delta(K)$ and hence (45).

Establish now that $\sqrt[p]{z'(e^{i\zeta})}$ and $\sqrt[p]{z'(\rho e^{i\zeta})}$ belong to W^p.

We start from (43) and (44) and note first of all that as is said above, in the small neighborhoods of the points A, B, C, D there exist the points $A_1, B_1 \in \smile{BA}$, $A_2, B_2 \in \smile{AB}$, $C_1, D_1 \in \smile{DC}$, $C_2, D_2 \in \smile{CD}$ such that the conditions (a)–(c) are fulfilled, and hence the following inclusions analogous to (43) and (44) are valid:

$$\begin{aligned} &z_2' \in \bigcap_{\delta>1} W^\delta(B_2\smile{}A_2), \quad z_2'(w) \in \bigcap_{\delta>1} W^\delta(C_2\smile{}D_2),\\ &z_{1,0}^\alpha \in \bigcap_{\delta>1} W^\delta(A_1\smile{}B_1), \quad z_{1,0}^\alpha(t) \in \bigcap_{\delta>1} W^\delta(D_1\smile{}C_1), \quad \alpha \in \mathbb{R}. \end{aligned} \tag{48}$$

Let $f \in L^p(\gamma_1)$. We have

$$\begin{aligned} J &= \int\limits_{\gamma_1}\left|\sqrt[p]{z'(t)}\int\limits_{\gamma_1}\frac{f(\tau)}{\sqrt[p]{z'(\tau)}}\,\frac{d\tau}{\tau-t}\right|^p |dt| = \int\limits_{AB}\left|\sqrt[p]{z_1'(t)}\int\limits_{A_1B_1}\frac{f(\tau)}{\sqrt[p]{z_1'(\tau)}}\,\frac{d\tau}{\tau-t}+\right.\\ &+\left.\sqrt[p]{z_1'(t)}\int\limits_{B_1A_1}\frac{f(\tau)}{\sqrt[p]{z_2'(\tau)}}\,\frac{d\tau}{\tau-t}\right|^p |dt| + \int\limits_{BA}\left|\sqrt[p]{z_2'(t)}\int\limits_{B_2A_2}\frac{f(\tau)}{\sqrt[p]{z_2'(\tau)}}\,\frac{d\tau}{\tau-t}+\right.\\ &+\left.\sqrt[p]{z_2'(z)}\int\limits_{A_2B_2}\frac{f(\tau)}{\sqrt[p]{z_1'(\tau)}}\,\frac{d\tau}{\tau-t}\right|^p |dt| \le\\ &\le 2^p\Bigg(\int\limits_{AB}\left|\sqrt[p]{z_1'(t)}\int\limits_{A_1B_1}\frac{f(\tau)}{\sqrt[p]{z_1'(\tau)}}\,\frac{d\tau}{\tau-t}\right|^p |dt|+\\ &+\int\limits_{AB}\left|\sqrt[p]{z_1'(t)}\int\limits_{B_1A_1}\frac{f(\tau)}{\sqrt[p]{z_2'(\tau)}}\,\frac{d\tau}{\tau-t}\right|^p |dt|+\\ &+\int\limits_{BA}\left|\sqrt[p]{z_2'(t)}\int\limits_{B_2A_2}\frac{f(\tau)}{\sqrt[p]{z_2'(\tau)}}\,\frac{d\tau}{\tau-t}\right|^p |dt|+\\ &+\int\limits_{BA}\left|\sqrt[p]{z_2'(t)}\int\limits_{A_2B_2}\frac{f(\tau)}{\sqrt[p]{z_1'(\tau)}}\,\frac{d\tau}{\tau-t}\right|^p |dt|\Bigg) = 2^p(J_1+J_2+J_3+J_4). \end{aligned} \tag{49}$$

We assume that

$$-\frac{1}{p} < \frac{\nu_k-1}{p} < \frac{1}{p'}\,, \quad \text{i.e.} \quad 0<\nu_k<p.$$

Then $\prod\limits_{k=1}^{n}(w-a_k)^{\nu_k-1}$ belongs to W^p, and hence $W^{p+\varepsilon}$, as well. Moreover, according to (48), we have $z_{1,0}^{1/p} \in \bigcap\limits_{\delta>1} W^{\delta}(\overset{\smile}{A_1B_1})$. Using Stein's interpolation theorem on the weighted functions for the operator $f \to S_{A_1B_1}f$ ([15]), we obtain $\sqrt[p]{z_1'} \in W^p(\overset{\smile}{A_1B_1})$. Therefore $J_1 \leq M_{1p}\|f\|^p_{A_1B_1}$. For J_2, we have $|\tau - t| \geq m > 0$, and hence

$$\bigg|\int\limits_{B_1A_1} \frac{f(\tau)}{\sqrt[p]{z_2'(\tau)}}\,\frac{d\tau}{\tau - t}\bigg|^p |dt| \leq \frac{1}{m^p}\int\limits_{B_1A_1}|f|^p|dt|\bigg(\int\limits_{B_1A_1}\frac{d\tau}{|z_2'(\tau)|^{p-1}}\bigg)^{p-1}.$$

Moreover, $\sqrt[p]{z_1'(t)} \in L^p(\overset{\smile}{AB})$ and thus $J_2 \leq M_{2p}\|f\|^p_{B_1A_1}$.

Analogously we estimate J_3 and J_4, and from (49) we get $J \leq M_p\|f\|^p_{\gamma_1}$. Just in the same way we can estimate the value

$$\int\limits_{\gamma_2}\bigg|\sqrt[p]{z'(t)}\int\limits_{\gamma_2}\frac{f(\tau)}{\sqrt[p]{z'(\tau)}}\,\frac{d\tau}{\tau - t}\bigg|^p|dt|.$$

As a result of the above reasoning, we have

Theorem 3. *If the doubly connected domain D is bounded by the boundary from $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k < p$, $p > 1$, and $z = z(w)$ is the conformal mapping of the circle K onto D, then the relations* (45) *are valid, and, moreover, the functions $\sqrt[p]{z'(e^{i\zeta})}$ and $\sqrt[p]{z'(\rho e^{i\zeta})}$ belong to W^p.*

3. The Dirichlet Problem in the Class $e^p(D;\rho)$

Again, let D be the doubly-connected domain with the boundary Γ, $\Gamma \in C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$, $\tau_1, \tau_2, \dots, \tau_m$ be the points on Γ (some of them may coincide with the points t_k). Let

$$\rho(z) = \prod_{k=1}^{m}(z - \tau_k)^{\beta_k}. \tag{50}$$

Consider the Dirichlet problems which is formulated as follows: Find the function, satisfying the conditions

$$\begin{cases} u \in e^p(D;\rho), \\ u\big|_{\Gamma_1} = f, \quad u\big|_{\Gamma_2} = g, \quad f \in L^p(\Gamma_1;\rho), \quad g \in L^p(\Gamma_2;\rho) \end{cases} \tag{51}$$

If $u = \operatorname{Re}\Phi$ is a solution of the problem (51), then the function $\Phi(z(w))\rho(z(w))\sqrt[p]{z'(w)}$ belongs to $E^p(K)$, i.e., $\Phi(z(w)) \in E^p(K;\omega)$, $\omega = \rho(z(w))\sqrt[p]{z'(w)}$. If we assume that $U(w) = u(z(w)$, $f_1(\tau) = f(z(\tau))$, $g_1(\tau) = g(z(\tau))$, then with regard to U we obtain the problem

$$\begin{cases} U \in e^p(K;\omega), \quad \omega(w) = \rho(z(w))\sqrt[p]{z'(w)}, \\ U\big|_{\Gamma_1} = f_1, \quad U\big|_{\Gamma_2} = g_1, \quad f_1 \in L^p(I;\omega_1), \quad g_1 \in L^p(I;\omega_2). \end{cases} \tag{52}$$

In order to apply Theorem 2 to the problem (52), the condition $\omega(w) \in W^p_E$ should be fulfilled. This condition will be fulfilled under certain assumptions with respect to Γ and ρ.

We have $\rho(z(w)) = \prod\limits_{k=1}^{m} (z(w) - z(b_k))^{\beta_k}$, $z(b_k) = \tau_k$.

Here we shall use one result from [6]: If $z = \varphi(w)$ is the conformal mapping of the unit circle onto the domain with the boundary from $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$, then

$$\varphi(w) - \varphi(a_k) = \prod (w - a_k)^{\nu_k} \varphi_k(w), \quad \varphi(a_k) = t_k,$$

where

$$\varphi_k \in \bigcap_{\delta>1} H^\delta, \quad \varphi_k^{\pm\alpha}(e^{i\zeta}) \in \bigcap_{\delta>1} W^\delta, \quad H^\delta \quad \text{is Hardy's class.}$$

Relying on this statement, just in the same way as in proving Theorem 3, we can prove that under the above-adopted assumptions with regard to Γ, we have

$$\varphi(w) - \varphi(a_k) = (w - a_k)^{\nu} z_k(w),$$

where $z_k(w) \in \bigcap\limits_{\delta>1} E^\delta(K)$, $z_k^{\pm\alpha}(e^{i\zeta})$ and $z_k^{\pm\alpha}(\rho e^{i\zeta})$ belong to the set $\bigcap\limits_{\delta>1} W^\delta$ for any $\alpha \in \mathbb{R}$.

From this fact we immediately find that $\omega^{\pm 1}(w) \in E^\eta(K)$, $\eta > 0$. Since the functions $z_k(w)$, $k = \overline{1,m}$ and $\frac{z'(w)}{\prod\limits_{k=1}^{m} (w-a_k)^{\nu_k - 1}}$, $w \in \gamma_i$, belong to the set $\bigcap\limits_{\delta>1} W^\delta$, owing to Stein's theorem, the function $\omega(w)$, $w \in \gamma_i$, will belong to W^p, if and only if the functions

$$\prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{p}} \prod_{k=1}^{m} (w - b_k)^{\beta_k \mu_k},$$

$$\mu_k = \begin{cases} 1, & \text{if} \quad \tau_k \overline{\in} \{t_1, t_2, \dots, t_k\}, \quad z(a_k) = t_k, \\ & \qquad z(b_k) = \tau_k, \quad w \in \gamma_s, \quad s = 1, 2, \\ \nu_j, & \text{if} \quad \tau_k = t_j, \end{cases}$$

belong to W^p. This condition will be fulfilled, if

$$0 < \nu_k < p, \quad \text{when} \quad t_k \overline{\in} \{\tau_1, \tau_2, \dots, \tau_m\}, \tag{53}$$

$$-\frac{1}{p} < \beta_k < \frac{1}{p'}\,, \quad \text{when} \quad \tau_k \overline{\in} \{t_1, t_2, \dots, t_m\}, \tag{54}$$

$$-\frac{1}{p} < \frac{\nu_k - 1}{\rho} + \nu_j \beta_j < \frac{1}{p'}\,, \quad \text{when} \quad t_k = \tau_j,$$

i.e.

$$0 < \nu_k + \beta_j \nu_j p < p, \quad \text{when} \quad t_k = \tau_j. \tag{55}$$

Thus if $0 < \nu_k < 2$ and the conditions (53)–(55) are fulfilled, then $\omega(w) \in W_E^p$.

Assuming that the above assumptions are fulfilled, we are able to solve the problem (52) and hence the problem (51).

Theorem 4. *Let D be the doubly-connected domain with the boundary Γ from $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$, and let $\rho(z)$ be the weighted function given by equality* (50) *in which $\tau_k \in \Gamma$, $-\frac{1}{p} < \beta_k < \frac{1}{p'}$.*

If the conditions (53)–(55) *are fulfilled, then the Dirichlet problem* (51) *is solvable, if and only if the condition*

$$\int_{\Gamma_1} f(t)\,dt = \int_{\Gamma_2} g(t)\,dt \tag{56}$$

is fulfilled. If the above condition is fulfilled, then the problem is solvable uniquely.

In particular, for $\rho = 1$, the conditions (53)–(55) take the form

$$0 < \nu_k < \min(2; p). \tag{57}$$

Theorem 4 generalizes the result obtained in [16] dealing with the Dirichlet problem in the class $E^p(D)$ for domains with piecewise Ljapunov boundary to the case of domains with piecewise smooth boundary.

The condition (57) makes it impossible to consider the boundaries with cusps and the cases when $1 < p \le 2$ and at least for one j, $\nu_j \ge p$. Under these assumptions, a picture of solvability in the case of a simply connected domain differs from that considered in Theorem 4, i.e., the unique solvability violates. The same situation is expected for the doubly-connected domains. We did not succeeded in extending the method of investigation applied in [1]–[2] for the simply connected domains to the multiply connected domains. Unfortunately, the method applied in the present work does not cover the above-mentioned cases. Here the work [17] is worth mentioning in which the Riemann–Hilbert problem is solved in the class of harmonic in the circle K functions with the Hölder continuous boundary values. A particular case of that problem is the Dirichlet problem.

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