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ON NON-ABELIAN LEIBNIZ COHOMOLOGY

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Abstract. In this note, by using a generalized notion of the Leibniz algebra of derivations, we present the constructions of the zero, first, and second non-Abelian Leibniz cohomologies with coefficients in crossed modules, which generalize the classical zero, first, and second Leibniz cohomology. For Lie algebras we compare the non-Abelian Leibniz and Lie cohomologies. We describe the second non-Abelian Leibniz cohomology via extensions of Leibniz algebras by crossed modules.

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1. Introduction

From the 1960s, many authors have attempted to answer the question of what we mean by non-Abelian cohomology of various algebraic structures. A convincing answer for groups and Lie algebras was given by Guin [2, 3] in the 1980s. More recently, in the case of groups, Inassaridze [4] has demonstrated how Guin's definition can be naturally extended to higher dimensions. Further, the second non-Abelian cohomology of Lie algebras, generalizing the classical Chevalley–Eilenberg cohomology and Guin's low dimensional non-Abelian Lie cohomology at the same time, has been constructed in [5].

It is clear that one should be able to develop an analogous theory of non-Abelian cohomology for other algebraic structures, such as Leibniz algebras, introduced by Loday [6, 7] as a noncommutative generalization of Lie algebras. By using his construction of the Leibniz algebra of biderivations, Gnedbaye [1] introduced the zero and the first non-Abelian cohomology of Leibniz algebras with coefficients in crossed modules. Even though, in some sense, biderivations play the same role in Leibniz algebras as derivations in Lie algebras, the so defined first (zero) non-Abelian cohomology does not coincide with the first (zero) Leibniz cohomology [6–8], when coefficients are representations viewed as trivial crossed modules.

In this note, by using a generalized notion of the Leibniz algebra of derivations, we present the constructions of the zero, first, and second non-Abelian Leibniz cohomologies with coefficients in crossed modules, which generalize the classical zero, first, and second Leibniz cohomology. For Lie algebras we compare the non-Abelian Leibniz and Lie cohomologies. We describe the second non-Abelian Leibniz cohomology via extensions of Leibniz algebras by crossed modules.

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2. Zero Non-Abelian Cohomology

It is known from [8] that the zero cohomology $HL^0(\mathcal{R},\mathcal{M})$ of a Leibniz algebra $\mathcal{R}$ over a commutative ring $\mathbb{K}$, with coefficients in a representation $\mathcal{M}$ is the submodule of left $\mathcal{R}$-invariant elements of $\mathcal{M}$. Similarly, we state the following definition.

Definition 1 (the zero non-Abelian cohomology). $\mathfrak{HL}^0(\mathcal{R},\mathcal{M})$ of a Leibniz algebra $\mathcal{R}$ with coefficients in a crossed $\mathcal{R}$-module $(\mathcal{M},\mu)$ is defined as the set of all left $\mathcal{R}$-invariant elements of $\mathcal{M}$, i.e.,

$$\mathfrak{HL}^0(\mathcal{R},\mathcal{M}) = \{m \in \mathcal{M} \mid {}^r m = 0 \text{ for any } r \in \mathcal{R}\}.$$

Therefore, $\mathfrak{HL}^0(\mathcal{R},\mathcal{M})$ is a two-sided ideal of $\mathcal{M}$, it is contained in the right center $\mathcal{Z}^r(\mathcal{M})$, and it inherits only a $\mathbb{K}$-module (Abelian Leibniz algebra) structure.

Note that if $\mathcal{M}$ is a representation of a Leibniz algebra $\mathcal{R}$ thought of as the trivial crossed $\mathcal{R}$-module $(\mathcal{M},0)$, then we have $\mathfrak{HL}^0(\mathcal{R},\mathcal{M}) = HL^0(\mathcal{R},\mathcal{M})$.

The zero non-Abelian cohomology of Gnedbaye [1] is defined as

$$\overline{\mathfrak{HL}}^0(\mathcal{R},\mathcal{M}) = \{m \in \mathcal{M} \mid {}^r m = m^r = 0 \text{ for any } r \in \mathcal{R}\}.$$

It is clear that $\overline{\mathfrak{HL}}^0(\mathcal{R},\mathcal{M}) \subseteq \mathfrak{HL}^0(\mathcal{R},\mathcal{M})$ and they coincide for Lie algebras. More concretely, if $\mathcal{R}$ is a Lie algebra and $(\mathcal{M},\mu)$ a crossed $\mathcal{R}$-module in Lie algebras, then $(\mathcal{M},\mu)$ can be considered as a crossed $\mathcal{R}$-module in Leibniz algebras, and it follows that

$$\mathfrak{HL}^0(\mathcal{R},\mathcal{M}) = \overline{\mathfrak{HL}}^0(\mathcal{R},\mathcal{M}) = \mathfrak{H}^0(\mathcal{R},\mathcal{M}),$$

where $\mathfrak{H}^0(\mathcal{R},\mathcal{M})$ is the zero non-Abelian Lie cohomology of $\mathcal{R}$ with coefficients in $(\mathcal{M},\mu)$ (see [3]).

3. Generalized Leibniz Algebras of Biderivations and Derivations

In [1], Gnedbaye constructed the first non-Abelian Leibniz cohomology by using the Leibniz algebra of biderivations. For our purposes we need the notion of the Leibniz algebras of biderivations and derivations in a more general setting.

Given a Leibniz action of $\mathcal{P}$ on $\mathcal{M}$, *a derivation* d (respectively, *an anti-derivation* δ) from $\mathcal{P}$ to $\mathcal{M}$ is a linear map $\mathcal{P} \to \mathcal{M}$ such that $d[p,p'] = d(p)^{p'} + {}^p d(p')$ (respectively, $\delta[p,p'] = \delta(p)^{p'} - \delta(p')^p$) for all $p,p' \in \mathcal{P}$.

Definition 2. Let $(\mathcal{M},\mu)$ be a $\mathcal{P}$-crossed $\mathcal{R}$-module. Denote by $\mathrm{Bider}(\mathcal{P},(\mathcal{M},\mu))$ (respectively, $\mathrm{Der}(\mathcal{P},(\mathcal{M},\mu))$) the set of triples (d,δ,r) (respectively, the set of pairs (d,r)), where $d : \mathcal{P} \to \mathcal{M}$ is a derivation, $\delta : \mathcal{P} \to \mathcal{M}$ is an anti-derivation and $r \in \mathcal{R}$ such that, for all $p \in \mathcal{P}$, $s \in \mathcal{R}$,

$$\mu d(p) = -{}^p r, \quad \mu\delta(p) = r^p, \quad {}^s d(p) = {}^s\delta(p), \quad \delta({}^s p) = -\delta(p^s)$$
$$\big(\text{respectively, } \mu d(p) = -{}^p r\big).$$

This set will be called the set of biderivations (respectively, derivations) from $\mathcal{P}$ to $(\mathcal{M},\mu)$.

Example-Definition 3. Let $(\mathcal{M},\mu)$ be a $\mathcal{P}$-crossed $\mathcal{R}$-module. Any $m \in \mathcal{M}$ determines a derivation $d_m : \mathcal{P} \to \mathcal{M}$, $d_m(p) = {}^p m$, and an anti-derivation $\delta_m : \mathcal{P} \to \mathcal{M}$, $\delta_m(p) = -m^p$, called principal derivation and anti-derivation, respectively. The triple $(d_m, \delta_m, -\mu(m) + z)$ is an element of $\mathrm{Bider}(\mathcal{P},(\mathcal{M},\mu))$, where z is any $\mathcal{P}$-invariant element of $\mathcal{R}$. Moreover, the pair $(d_m, -\mu(m) + c)$ is an element of $\mathrm{Der}(\mathcal{P},(\mathcal{M},\mu))$, where c is any left $\mathcal{P}$-invariant element of $\mathcal{R}$.

Proposition 4. *There is a Leibniz algebra structure on* $\mathrm{Bider}(\mathcal{P},(\mathcal{M},\mu))$ *(respectively,* $\mathrm{Der}(\mathcal{P},(\mathcal{M},\mu))$*) given by*

$$(d,\delta,r)+(d',\delta',r')=(d+d',\delta+\delta',r+r'),\quad \lambda(d,\delta,r)=(\lambda d,\lambda\delta,\lambda r),$$
$$[(d,\delta,r),(d',\delta',r')]=(d*d',\delta*\delta',[r,r']);$$

respectively,

$$(d,r)+(d',r')=(d+d',r+r'),\quad \lambda(d,r)=(\lambda d,\lambda r),$$
$$[(d,r),(d',r')]=(d*d',[r,r'])\big),$$

where $(d*d')(p)=d'(p^r)-d(p^{r'})$ *and* $(\delta*\delta')(p)=-\delta(p^{r'})-d'({}^r p)$ *for all* $p\in\mathcal{P}$.

Theorem 5. *Let* (M,μ) *be a* $\mathcal{P}$*-crossed* $\mathcal{R}$*-module in Leibniz algebras. Denote by* $\mathrm{Ant}(\mathcal{P},\mathrm{Ker}\,\mu\cap\mathfrak{H}\mathfrak{L}^0(\mathcal{R},\mathcal{M}))$ *the Abelian Leibniz algebra of all anti-derivations* $\delta:\mathcal{P}\to\mathrm{Ker}\,\mu\cap\mathfrak{H}\mathfrak{L}^0(\mathcal{R},\mathcal{M})$*. Then there is an exact sequence of Leibniz algebras*

$$0\longrightarrow \mathrm{Ant}(\mathcal{P},\mathrm{Ker}\,\mu\cap\mathfrak{H}\mathfrak{L}^0(\mathcal{R},\mathcal{M}))\longrightarrow \mathrm{Bider}(\mathcal{P},(\mathcal{M},\mu))\xrightarrow{\Delta}\mathrm{Der}(\mathcal{P},(\mathcal{M},\mu)).$$

In particular, if $(\mathcal{M},\mu)$ *is a crossed* $\mathcal{R}$*-module in Lie algebras, then there is a split short exact sequence of Leibniz algebras*

$$0\to \mathrm{Hom}_{\mathbb{K}}\big(\mathcal{R}/[\mathcal{R},\mathcal{R}],\mathrm{Ker}\,\mu\cap\mathfrak{H}^0(\mathcal{R},\mathcal{M})\big)\to \mathrm{Bider}(\mathcal{R},(\mathcal{M},\mu))\overset{\Delta}{\to}\mathrm{Der}(\mathcal{R},(\mathcal{M},\mu))\to 0.$$

4. First Non-Abelian Cohomology

Definition 6. Let $\mathcal{R}$ be a Leibniz algebra and $(\mathcal{M},\mu)$ a crossed $\mathcal{R}$-module. Consider the two-sided ideal

$$\mathcal{J}=\{(d_m,\ -\mu(m)+c)\mid m\in\mathcal{M},\ c\in\mathcal{Z}^r(\mathcal{R})\}$$

in $\mathrm{Der}(\mathcal{R},(\mathcal{M},\mu))$, where $d_m:\mathcal{R}\to\mathcal{M}$ is the principal derivation induced by $m\in\mathcal{M}$. The first non-Abelian cohomology $\mathfrak{H}\mathfrak{L}^1(\mathcal{R},\mathcal{M})$ of $\mathcal{R}$ with coefficients in $(\mathcal{M},\mu)$ is defined as the following quotient Leibniz algebra:

$$\mathfrak{H}\mathfrak{L}^1(\mathcal{R},\mathcal{M})=\mathrm{Der}(\mathcal{R},(\mathcal{M},\mu))/\mathcal{J}.$$

Proposition 7. (i) *Let* $(\mathcal{M},\mu)$ *be a representation of a Leibniz algebra* $\mathcal{R}$ *considered as the trivial crossed* $\mathcal{R}$*-module* $(\mathcal{M},0)$*. Then there is an isomorphism of* $\mathbb{K}$*-modules*

$$\mathfrak{H}\mathfrak{L}^1(\mathcal{R},\mathcal{M})\approx HL^1(\mathcal{R},\mathcal{M}).$$

(ii) *Let* $(\mathcal{M},\mu)$ *be a crossed* $\mathcal{R}$*-module in Lie algebras. Then the first non-Abelian Lie and Leibniz cohomologies of* $\mathcal{R}$ *with coefficients in* $(\mathcal{M},\mu)$ *are isomorphic.*

5. Second Non-Abelian Cohomology

For the construction here we proceed the same way as in [4] for groups and in [5] for Lie algebras. Let

$$\mathcal{P}\underset{l_1}{\overset{l_0}{\rightrightarrows}}\mathcal{F}\xrightarrow{\ \epsilon\ }\mathcal{R}, \qquad (*)$$

be a diagram of Leibniz algebras, where $\mathcal{F}$ is a free Leibniz algebra over a $\mathbb{K}$-module and $(\mathcal{P},l_0,l_1)$ is the simplicial kernel of ϵ. Let $(\mathcal{M},\mu)$ be a crossed $\mathcal{R}$-module, let $\mathcal{R}$ act on $\mathcal{F}$, and let ϵ preserve the actions and have a $\mathbb{K}$-linear section. Then there is an induced action of $\mathcal{R}$ on $\mathcal{P}$, and $(\mathcal{M},\mu)$ can be viewed as an $\mathcal{F}$-crossed ($\mathcal{P}$-crossed) $\mathcal{R}$-module. Denote by $Z(\mathcal{P},(\mathcal{M},\mu))$ the subset of $\mathrm{Der}(\mathcal{P},(\mathcal{M},\mu))$ of all elements $(d,0)$ satisfying $d(x,x)=0$. Let $B(\mathcal{P},(\mathcal{M},\mu))$ denote the submodule of $Z(\mathcal{P},(\mathcal{M},\mu))$

consisting of all elements $(d,0)$ for which there exists $(\alpha,h)\in \operatorname{Der}(\mathcal{F},(\mathcal{M},\mu))$ with $\alpha l_0-\alpha l_1=d$. Let $\widetilde{B}(\mathcal{P},(\mathcal{M},\mu))$ denote the submodule of $B(\mathcal{P},(\mathcal{M},\mu))$ consisting of all $(d,0)$ for which the existing (α,h) satisfies the additional condition: $\Im\alpha$ is contained in the center of $\mathcal{M}$.

Proposition-Definition 8. *The quotient module*

$$Z(\mathcal{P},(\mathcal{M},\mu))/B(\mathcal{P},(\mathcal{M},\mu))$$

(respectively, $Z(\mathcal{P},(\mathcal{M},\mu))/\widetilde{B}(\mathcal{P},(\mathcal{M},\mu))$) is independent (up to isomorphism) of the choice of the diagram $(*)$. It will be called the second non-Abelian cohomology (respectively, quasi-cohomology) of $\mathcal{R}$ with coefficients in $(\mathcal{M},\mu)$ and will be denoted by $\mathfrak{HL}^2(\mathcal{R},\mathcal{M})$ (respectively, $\widetilde{\mathfrak{HL}}^2(\mathcal{R},\mathcal{M})$).

Theorem 9. *Given a crossed $\mathcal{R}$-module $(\mathcal{M},\mu)$ in Leibniz algebras, there is an epimorphism of $\mathbb{K}$-modules*

$$HL^2(\mathcal{R},\operatorname{Ker}\mu)\twoheadrightarrow \mathfrak{HL}^2(\mathcal{R},\mathcal{M}),$$

where HL^2 denotes the second Leibniz cohomology [8]. Moreover, if any element $(\alpha,h)\in$ $\operatorname{Der}(\mathcal{F},(\mathcal{M},\mu))$ satisfies that $h\in \mathcal{Z}^r(\mathcal{R})$, then θ is an isomorphism. In particular, if $\mathcal{M}$ is a representation of $\mathcal{R}$ considered as the trivial crossed $\mathcal{R}$-module $(M,0)$, then $HL^2(\mathcal{R},\mathcal{M})\approx \mathfrak{HL}^2(\mathcal{R},\mathcal{M})$.

Proposition 10. *Let R be a Lie algebra and $(\mathcal{M},\mu)$ a crossed $\mathcal{R}$-module in Lie algebras. Denote by $\mathfrak{H}^2(\mathcal{R},\mathcal{M})$ the second non-Abelian Lie cohomology introduced in* [5]. *Then there is a monomorphism $\mathfrak{H}^2(\mathcal{R},\mathcal{M})\hookrightarrow \mathfrak{HL}^2(\mathcal{R},\mathcal{M})$.*

Proposition 11. (i) *Let $\mathcal{M}$ be a representation of a Leibniz algebra $\mathcal{R}$ considered as the trivial crossed $\mathcal{R}$-module. Then*

$$\mathfrak{HL}^2(\mathcal{R},\mathcal{M})=\widetilde{\mathfrak{HL}}^2(\mathcal{R},\mathcal{M})=HL^2(\mathcal{R},\mathcal{M}).$$

(ii) *Let $\mathcal{F}$ be a free Leibniz algebra. Then for any crossed $\mathcal{F}$-module $(\mathcal{M},\mu)$ we have*

$$\mathfrak{HL}^2(\mathcal{F},\mathcal{M})=\widetilde{\mathfrak{HL}}^2(\mathcal{F},\mathcal{M})=0.$$

6. Nine-Term Exact Sequence

In [3], Guin constructed a seven-term exact non-Abelian Lie cohomology sequence. Under an additional necessary condition on coefficients, this sequence was extended to nine terms in [5]. Now we generalize the exact sequence in [5] from Lie to Leibniz algebras.

Theorem 12. *Let $\mathcal{R}$ be a Leibniz algebra and*

$$0\longrightarrow(\mathcal{L},0)\xrightarrow{\xi}(\mathcal{M},\mu)\xrightarrow{\theta}(\mathcal{N},\nu)\longrightarrow 0$$

be a short exact sequence of crossed $\mathcal{R}$-modules, which has a $\mathbb{K}$-linear splitting. Then there is an exact sequence of $\mathbb{K}$-modules

$$\begin{aligned}0\longrightarrow \mathfrak{HL}^0(\mathcal{R},\mathcal{L})&\xrightarrow{\xi^0}\mathfrak{HL}^0(\mathcal{R},\mathcal{M})\xrightarrow{\theta^0}\mathfrak{HL}^0(\mathcal{R},\mathcal{N})\xrightarrow{\partial^0}\mathfrak{HL}^1(\mathcal{R},\mathcal{L})\\ &\xrightarrow{\xi^1}\mathfrak{HL}^1(\mathcal{R},\mathcal{M})\xrightarrow{\theta^1}\mathfrak{HL}^1(\mathcal{R},\mathcal{N})\xrightarrow{\partial^1}\mathfrak{HL}^2(\mathcal{R},\mathcal{L})\xrightarrow{\xi^2}\mathfrak{HL}^2(\mathcal{R},\mathcal{M})\xrightarrow{\theta^2}\mathfrak{HL}^2(\mathcal{R},\mathcal{N}).\end{aligned}$$

Moreover, θ^1 is a Leibniz homomorphism, the Leibniz algebra $\mathfrak{HL}^1(\mathcal{R},\mathcal{N})$ acts on the (Abelian) Leibniz algebra $\mathfrak{HL}^2(\mathcal{R},\mathcal{L})$, and ∂^1 is a derivation.

7. Extensions by Crossed Modules

Definition 13. (i) An extension of a Leibniz algebra $\mathcal{R}$ by a crossed $\mathcal{R}$-module $(\mathcal{M},\mu)$ is a pair $E=(0\to\mathcal{M}\overset{\sigma}{\to}\mathcal{X}\overset{\psi}{\to}\mathcal{R}\to 0,\ \varphi)$, where $0\to\mathcal{M}\overset{\sigma}{\to}\mathcal{X}\overset{\psi}{\to}\mathcal{R}\to 0$ is a short exact sequence of Leibniz algebras and φ is a $\mathbb{K}$-linear section of ψ such that, for all $r\in\mathcal{R}$, $m\in\mathcal{M}$, ${}^{r}m=\sigma^{-1}[\varphi(r),\sigma(m)]$, $m^{r}=\sigma^{-1}[\sigma(m),\varphi(r)]$, and $\operatorname{Ker}\psi\cap\overline{\varphi(\mathcal{R})}\subseteq\operatorname{Ker}\mu$, where $\overline{\varphi(\mathcal{R})}$ denotes the Leibniz subalgebra of $\mathcal{X}$ generated by $\varphi(\mathcal{R})$.

(ii) Let $E'=(0\to\mathcal{M}\overset{\sigma'}{\to}\mathcal{X}'\overset{\psi'}{\to}\mathcal{R}\to 0,\ \varphi')$ be another extension. E and E' will be called equivalent if there exists a Leibniz homomorphism $\vartheta:\mathcal{X}\to\mathcal{X}'$ and an element $h\in\mathcal{R}$ such that $\vartheta\sigma=\sigma'$, $\psi'\vartheta=\psi$ and $\mu(\vartheta\varphi(r)-\varphi'(r))=-[r,h]$ for any $r\in R$.

Let $E^1(\mathcal{R},\mathcal{M})$ denote the set of equivalence classes of extensions of $\mathcal{R}$ by $(\mathcal{M},\mu)$.

Theorem 14. *Let $\mathcal{R}$ be a Leibniz algebra and $(\mathcal{M},\mu)$ a crossed $\mathcal{R}$-module. Then there is a bijection*

$$\eta:\widetilde{\mathfrak{H}\mathfrak{L}}^2(\mathcal{R},\mathcal{M})\overset{\approx}{\longrightarrow}E^1(\mathcal{R},\mathcal{M}).$$

Note that η maps the zero element of $\widetilde{\mathfrak{H}\mathfrak{L}}^2(\mathcal{R},\mathcal{M})$ to the equivalence class of the trivial extension.

Let $S(\mathcal{R},\mathcal{M})$ denote the subset of $E^1(\mathcal{R},\mathcal{M})$ of those equivalence classes of extensions $E=(0\to\mathcal{M}\overset{\sigma}{\to}\mathcal{X}\overset{\psi}{\to}\mathcal{R}\to 0,\ \varphi)$ such that there exists a $\mathbb{K}$-homomorphism $f:\mathcal{R}\to\mathcal{M}$ with $(\sigma\alpha+\delta)(x)=(\sigma\alpha+\delta)(y)$ for $x,y\in\mathcal{F}(\mathcal{R})$, $\epsilon(x)=\epsilon(y)$, $\alpha:\mathcal{F}(\mathcal{R})\to\mathcal{M}$ is the derivation induced by f, and $\delta:\mathcal{F}(\mathcal{R})\to\mathcal{X}$ is the Leibniz homomorphism induced by φ. Then we have the following proposition.

Proposition 15. *Let $\mathcal{R}$ be a Leibniz algebra and $(\mathcal{M},\mu)$ a crossed $\mathcal{R}$-module. Then there is a short exact sequence of pointed sets*

$$0\longrightarrow S(\mathcal{R},\mathcal{M})\longrightarrow E^1(\mathcal{R},\mathcal{M})\longrightarrow\mathfrak{H}\mathfrak{L}^2(\mathcal{R},\mathcal{M})\longrightarrow 0.$$

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