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NON-ABELIAN HOMOLOGY OF HOM-LIE ALGEBRAS AND APPLICATIONS

Introduction

A Hom-Lie algebra is a triple $(L, [-,-], \alpha)$, where α is a linear self-map, in which the skew-symmetric bracket satisfies an α-twisted version of the Jacobi identity, called the Hom-Jacobi identity. When α is the identity map, the Hom-Jacobi identity reduces to the usual Jacobi identity, and L is a Lie algebra. Hom-Lie algebras were introduced in [4] to construct deformations of the Witt algebra, which is the Lie algebra of derivations on the Laurent polynomial algebra $C[z^{\pm}]$. Since the introduction, there have been several works dealing generalizations of known theories from Lie to Hom-Lie algebras (see [1], [6]–[12]).

In this paper we introduce the zero and first non-abelian homology of Hom-Lie algebras generalizing the zero and first non-abelian homology of Lie algebras developed in [3, 5], as well as the low dimensional homology of Hom-Lie algebras given in [10, 12]. We use the non-abelian homology of Hom-Lie algebras in the description of a relationship between cyclic and Milnor cyclic homologies of Hom-associative algebras satisfying certain additional condition.

Throughout this paper we fix a ground field $\mathbb{K}$. Vector spaces are considered over $\mathbb{K}$ and linear maps are $\mathbb{K}$-linear maps. We write $\otimes$ (resp. $\wedge$) for the tensor product $\otimes_{\mathbb{K}}$ (resp. exterion product $\wedge_{\mathbb{K}}$).

1. Preliminaries on Hom-Lie Algebras

We start by reviewing some notions and terminology.

Definition 1.1. A Hom-Lie algebra (L, α_L) is a non-associative algebra L together with a linear map $\alpha_L : L \to L$ satisfying

$$[x, y] = -[y, x], \qquad \text{(skew-symmetry)}$$

$$[\alpha_L(x), [y, z]] + [\alpha_L(z), [x, y]] + [\alpha_L(y), [z, x]] = 0 \quad \text{(Hom-Jacobi identity)}$$

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for all $x, y, z \in L$, where $[-,-]$ denotes the product in L.

In this paper we deal only with (the so called *multiplicative*) Hom-Lie algebras (L, α_L) such that $\alpha_L[x,y] = [\alpha_L(x), \alpha_L(y)]$, $x, y \in L$.

It is clear that any Lie algebra L can be considered as a Hom-Lie algebra (L, id_L). Moreover, any Hom-associative algebra [7] becomes a Hom-Lie algebra (see Section 4 below).

A homorphism of Hom-Lie algebras $f : (L, \alpha_L) \to (L', \alpha_{L'})$ is an algebra homomorphism $f : L \to L'$ such that $f \circ \alpha_L = \alpha_{L'} \circ f$.

Definition 1.2. A Hom-Lie subalgebra (H, α_H) of (L, α_L) is a vector subspace H of L closed under the product, together with the endomorphism $\alpha_H : H \to H = \alpha_L|_H$. In such a case we write $\alpha_{L|}$ for α_H.

A Hom-Lie subalgebra $(H, \alpha_{L|})$ of (L, α_L) is said to be an ideal if $[x,y] \in H$ for any $x \in H$, $y \in L$.

Let $(H, \alpha_{L|})$ and $(K, \alpha_{L|})$ be ideals of a Hom-Lie algebra (L, α_L). The commutator of $(H, \alpha_{L|})$ and $(K, \alpha_{L|})$, denoted by $([H,K], \alpha_{L|})$, is the Hom-Lie subalgebra of (L, α_L) spanned by all $[h,k]$, $h \in H$, $k \in K$.

Definition 1.3. Let (L, α_L), (M, α_M) be Hom-Lie algebras. A Hom-action of (L, α_L) on (M, α_M) is a linear map $L \otimes M \to M$, $x \otimes m \mapsto {}^x m$ satisfying, for all $x, y \in L$ and $m, m' \in M$, the following equalities:

$$^{[x,y]}\alpha_M(m) = {}^{\alpha_L(x)}({}^y m) - {}^{\alpha_L(y)}({}^x m),$$
$$^{\alpha_L(x)}[m, m'] = [{}^x m, \alpha_M(m')] + [\alpha_M(m), {}^x m'],$$
$$\alpha_M({}^x m) = {}^{\alpha_L(x)}\alpha_M(m).$$

For example, if (L, α_L) is a Hom-subalgebra of a Hom-Lie algebra (K, α_K) and (H, α_H) is an ideal of (K, α_K), then there is a Hom-action of (L, α_L) on (H, α_H) given by the product in K.

Remark 1.4. If (M, α_M) is an abelian Hom-Lie algebra (i. e. [m,m']=0 for all $m, m' \in M$) enriched with a Hom-action of (L, α_L), then (M, α_M) is nothing else but a Hom-module over (L, α_L) (see [10]).

2. Non-Abelian Tensor Product of Hom-Lie Algebras

In this section we introduce a Hom-Lie algebra version of the non-abelian tensor product of Lie algebras [2], and study its properties.

Definition 2.1. Let (M, α_M) and (N, α_N) be Hom-Lie algebras with Hom-actions on each other. The Hom-actions are said to be compatible if, for all $m, m' \in M$ and $n, n' \in N$,

$$^{({}^m n)}m' = [m', {}^n m] \quad \text{and} \quad ^{({}^n m)}n' = [n', {}^m n].$$

Let (M, α_M) and (N, α_N) be Hom-Lie algebras acting on each other compatibly. Consider the Hom-vector space $(M \otimes N, \alpha_{M\otimes N})$, where $\alpha_{M\otimes N}(m \otimes$

$n) = \alpha_M(m) \otimes \alpha_N(n)$. Denote by $D(M,N)$ subspace of $M \otimes N$ generated by all elements of the form

$$[m,m'] \otimes \alpha_N(n) - \alpha_M(m) \otimes {}^{m'}n + \alpha_M(m') \otimes {}^{m}n,$$

$$\alpha_M(m) \otimes [n,n'] - {}^{n'}m \otimes \alpha_N(n) + {}^{n}m \otimes \alpha_N(n'),$$

$${}^{n}m \otimes {}^{m}n,$$

$${}^{n}m \otimes {}^{m'}n' + {}^{n'}m' \otimes {}^{m}n,$$

$$[{}^{n}m, {}^{n'}m'] \otimes \alpha_N({}^{m''}n'') + [{}^{n'}m', {}^{n''}m''] \otimes \alpha_N({}^{m}n) + [{}^{n''}m'', {}^{n}m] \otimes \alpha_N({}^{m'}n'),$$

for $m, m', m'' \in M$ and $n, n', n'' \in N$.

Proposition 2.2. *The quotient vector space* $(M \otimes N)/D(M,N)$ *with the product*

$$[m \otimes n, m' \otimes n'] = -{}^{n}m \otimes {}^{m'}n' \tag{1}$$

and the endomorphism $(M \otimes N)/D(M,N) \to (M \otimes N)/D(M,N)$ *induced by* $\alpha_{M\otimes N}$*, is a Hom-Lie algebra.*

Proof. It is clear that $\alpha_{M\otimes N}$ preserves the elements of $D(M,N)$ and the product given by (1). This product is compatible with the defining relations of $(M \otimes N)/D(M,N)$ and can be extended to any elements. Since the actions are compatible, direct calculations show that the skew-symmetry and Hom-Jacobi identity are satisfied. □

Definition 2.3. The above described Hom-Lie algebra structure on $(M \otimes N)/D(M,N)$ is called the non-abelian tensor product of Hom-Lie algebras (M, α_M) and (N, α_N). It will be denoted by $(M \boxtimes N, \alpha_{M\boxtimes N})$ and the equivalence class of $m \otimes n$ will be denoted by $m \boxtimes n$.

Remark 2.4. If $\alpha_M = \mathrm{id}_M$ and $\alpha_N = \mathrm{id}_N$ then $M \boxtimes N$ is the non-abelian tensor product of Lie algebras developed in [2] (see also [5]).

The Hom-Lie tensor product is symmetric in the sense of the following isomorphism of Hom-Lie algebras

$$(M \boxtimes N, \alpha_{M\boxtimes N}) \xrightarrow{\approx} (N \boxtimes M, \alpha_{N\boxtimes M}), \quad m \boxtimes n \mapsto n \boxtimes m.$$

Sometimes the non-abelian tensor product of Hom-Lie algebras can be described as the tensor product of vector spaces.

Proposition 2.5. *If the Hom-Lie algebras* (M, α_M) *and* (N, α_N) *act trivially on each other and both* α_M*,* α_N *are epimorphisms, then there is an isomorphism of abelian Hom-Lie algebras*

$$(M \boxtimes N, \alpha_{M\boxtimes N}) \approx (M^{ab} \otimes N^{ab}, \alpha_{M^{ab}\otimes N^{ab}}),$$

where $M^{ab} = M/[M,M]$*,* $N^{ab} = N/[N,N]$ *and* $\alpha_{M^{ab}\otimes N^{ab}}$ *is induced by* α_M *and* α_N*.*

Proof. Since the Hom-actions are trivial, (1) enables us to see that $(M \boxtimes N, \alpha_{M\boxtimes N})$ is abelian. Further, since α_M, α_N are epimorphisms, the vector space $M \boxtimes N$ is the quotient of $M \otimes N$ by the relations $[m, m'] \otimes n = 0 = m \otimes [n, n']$. The later is isomorphic to $M^{ab} \otimes N^{ab}$. □

The Hom-Lie tensor product is functorial in the following sense: if $f : (M, \alpha_M) \to (M', \alpha_{M'})$ and $g : (N, \alpha_N) \to (N', \alpha_{N'})$ are homomorphisms of Hom-Lie algebras together with compatible Hom-actions of (M, α_M) (resp. $(M', \alpha_{M'})$) and (N, α_N) (resp. $(N', \alpha_{N'})$) on each other such that f, g preserve these Hom-actions, i.e. $f({}^n m) = {}^{g(n)} f(m)$, $g({}^m n) = {}^{f(m)} g(n)$ for $m \in M$, $n \in N$, then there is a homomorphism

$$f \boxtimes g : (M \boxtimes N, \alpha_{M\boxtimes N}) \to (M' \boxtimes N', \alpha_{M'\boxtimes N'}), \quad (m \boxtimes n) \mapsto f(m) \boxtimes g(n).$$

Proposition 2.6. *Let* $0 \to (M_1, \alpha_{M_1}) \xrightarrow{f} (M_2, \alpha_{M_2}) \xrightarrow{g} (M_3, \alpha_{M_3}) \to 0$ *be a short exact sequence of Hom-Lie algebras. Let* (N, α_N) *be a Hom-Lie algebra together with compatible Hom-actions of* (N, α_N) *and* (M_i, α_{M_i}) $(i = 1, 2, 3)$ *on each other and* f, g *preserve these Hom-actions. Then there is an exact sequence of Hom-Lie algebras*

$$(M_1 \boxtimes N, \alpha_{M_1\boxtimes N}) \xrightarrow{f\boxtimes \mathrm{id}_N} (M_2 \boxtimes N, \alpha_{M_2\boxtimes N}) \xrightarrow{g\boxtimes \mathrm{id}_N} (M_3 \boxtimes N, \alpha_{M_3\boxtimes N}) \longrightarrow 0.$$

Proof. Clearly $g \boxtimes \mathrm{id}_N$ is an epimorphism and $\mathrm{Im}(f \boxtimes \mathrm{id}_N) \subseteq \mathrm{Ker}(g \boxtimes \mathrm{id}_N)$. Now $\mathrm{Im}(f \boxtimes \mathrm{id}_N)$ is generated by elements of the form $f(m_1) \boxtimes n_1$ with $m_1 \in M_1$, $n_1 \in N$. It is an ideal in $(M_2 \boxtimes N, \alpha_{M_2\boxtimes N})$ since

$$[f(m_1) \boxtimes n_1, m_2 \boxtimes n_2] = -f({}^{n_1} m_1) \boxtimes {}^{m_2} n_2 \in \mathrm{Im}(f \boxtimes \mathrm{id}_N)$$

for any generator $m_2 \boxtimes n_2 \in M_2 \boxtimes N$. Thus, $g \boxtimes \mathrm{id}_N$ yields a factorization

$$\xi : \big((M_2 \boxtimes N)/\,\mathrm{Im}(f \boxtimes \mathrm{id}_N), \overline{\alpha}_{M_2\boxtimes N}\big) \to (M_3 \boxtimes N, \alpha_{M_3\boxtimes N}).$$

In fact this is an isomorphism of Hom-Lie algebras with the inverse

$$\xi' : (M_3 \boxtimes N, \alpha_{M_3\boxtimes N}) \to \big((M_2 \boxtimes N)/\,\mathrm{Im}(f \boxtimes \mathrm{id}_N), \overline{\alpha}_{M_2\boxtimes N}\big)$$

given by $\xi'(m_3 \boxtimes n) = \overline{m_2 \boxtimes n}$, where $m_2 \in M_2$ such that $g(m_2) = m_3$. The remaining details are straightforward. □

3. Zero and First Non-Abelian Homologies.

In this section we extend the zero and first non-abelian homology of Lie algebras [5] to Hom-Lie algebras. The following lemma will be needed.

Lemma 3.1. *Let* (M, α_M) *and* (N, α_N) *be Hom-Lie algebras with compatible actions on each other.*

(a) *There is a Hom-action of* (M, α_M) *on* $(M \boxtimes N, \alpha_{M\boxtimes N})$ *given by*

$${}^{m'}(m \boxtimes n) = [m', m] \boxtimes \alpha_N(n) + \alpha_M(m) \boxtimes {}^{m'} n.$$

And the induced Hom-action of $\mathrm{Im}(\psi)$ *on* $\mathrm{Ker}(\psi)$ *is trivial.*

(b) *There is a homomorphisms of Hom-Lie algebras*

$$\psi : (M \boxtimes N, \alpha_{M\boxtimes N}) \to (M, \alpha_M), \quad \psi_M(m \boxtimes n) = -^n m$$

satisfying the following equalities

$$\psi(^{m'}(m \boxtimes n)) = [\alpha_M(m'), \psi(m \boxtimes n)],$$
$$^{\psi(m\boxtimes n)}(m' \boxtimes n') = [\alpha_{M\boxtimes N}(m \boxtimes n), m' \boxtimes n'].$$

Proof. Everything can be readily checked thanks to the compatibility conditions and the relation (1). □

Definition 3.2. Let (M, α_M) and (N, α_N) be Hom-Lie algebras with compatible actions on each other. We define the zero and first non-abelian homology of (M, α_M) with coefficients in (N, α_N) by setting

$$\mathcal{H}_0^\alpha(M, N) = \operatorname{Coker} \psi, \quad \mathcal{H}_1^\alpha(M, N) = \operatorname{Ker} \psi.$$

Remark 3.3. (a) If $\alpha_M = id_M$ and $\alpha_N = id_N$, then ψ is a Lie crossed module [2] and $\mathcal{H}_0^\alpha(M, N)$, $\mathcal{H}_1^\alpha(M, N)$ are zero and first non-abelian homologies of the Lie algebra M with coefficients in N [5], respectively.

(b) If (N, α_N) is a Hom-module over (M, α_M) together with the trivial Hom-action of (N, α_N) on (M, α_M), then $\mathcal{H}_0^\alpha(M, N)$ and $\mathcal{H}_1^\alpha(M, N)$ coincide with the zero and first Chevalley-Eilenberg homologies of Hom-Lie algebras (see [10, 12]), respectively.

Theorem 3.4. *Let* $0 \to (N_1, \alpha_{N_1}) \xrightarrow{f} (N_2, \alpha_{N_2}) \xrightarrow{g} (N_3, \alpha_{N_3}) \to 0$ *be a short exact sequence of Hom-Lie algebras. Let* (M, α_M) *be a Hom-Lie algebra together with compatible Hom-actions of* (M, α_M) *and* (N_i, α_{N_i}) $(i = 1, 2, 3)$ *on each other and* f, g *preserve these Hom-actions. Then there is a six-term exact non-abelian homology sequence*

$$\mathcal{H}_1^\alpha(M, N_1) \to \mathcal{H}_1^\alpha(M, N_2) \to \mathcal{H}_1^\alpha(M, N_3) \to$$
$$\to \mathcal{H}_0^\alpha(M, N_1) \to \mathcal{H}_0^\alpha(M, N_2) \to \mathcal{H}_0^\alpha(M, N_3) \to 0.$$

Proof. This is a consequence of Proposition 2.6 and Snake Lemma. □

4. Application in Cyclic Homology of Hom-Associative Algebras

In this section we assume that $\mathbb{K}$ is a field of characteristic 0.

Definition 4.1. A Hom-associative algebra (see e.g. [7]) is a pair (A, α_A) consisting of a vector space A and a linear map $\alpha_A : A \to A$, together with a linear map (multiplication) $A \otimes A \to A$, $a \otimes b \mapsto ab$, such that, for all $a, b, c \in A$,

$$\alpha_A(a)(bc) = (ab)\alpha_A(c), \quad \alpha_A(ab) = \alpha_A(a)\alpha_A(b).$$

The Hom version of the classical cyclic bicomplex is constructed in [10] and the cyclic homology of a Hom-associative algebra is defined as the homology of its total complex. A reformulation of this cyclic homology via Connes's complex for Hom-associative algebras is also given in [10, Proposition 4.7]. It follows that, given a Hom-associative algebra (A, α_A), the first cyclic homology $HC_1^\alpha(A)$ is the kernel of the homomorphism of vector spaces

$$\psi : A \otimes A/J(A, \alpha) \to [A, A], \quad a \otimes b \mapsto ab - ba,$$

where $[A, A]$ is the subspace of A generated by the elements $ab - ba$, and $J(A, \alpha)$ is the subspace of $A \otimes A$ generated by the elements

$$a \otimes b + b \otimes a \quad \text{and} \quad ab \otimes \alpha_A(c) - \alpha_A(a) \otimes bc + ca \otimes \alpha_A(b).$$

Any Hom-associative algebra (A, α_A) is endowed with a Hom-Lie algebra structure by the induced product $[a, b] = ab - ba$ and the endomorphism α_A. Moreover, there is a Hom-Lie algebra structure on $(L^\alpha(A), \overline{\alpha}_A) = A \otimes A/J(A, \alpha)$ given by the product

$$[a \otimes b, a' \otimes b'] = [a, b] \otimes [a', b']$$

and the endomorphism $\overline{\alpha}_A$ induced by α_A.

Definition 4.2. We say that a Hom-associative algebra (A, α_A) satisfies the α-identity condition if

$$[A, \mathrm{Im}(\alpha_A - \mathrm{id}_A)] = 0, \tag{2}$$

where $[A, \mathrm{Im}(\alpha_A - \mathrm{id}_A)]$ is the subspace of A spanned by all elements $ab - ba$ with $a \in A$ and $b \in \mathrm{Im}(\alpha_A - \mathrm{id}_A)$.

Example 4.3. (i) Any Hom-associative algebra (A, α_A) with $\alpha_A = \mathrm{id}_A$ (i.e. an associative algebra) satisfies α-identity condition.
(ii) Any commutative Hom-associative algebra (A, α_A) (i.e. $ab = ba$ for all $a, b \in A$) with $\alpha_A = 0$ satisfies α-identity condition.
(iii) Consider the Hom-associative algebra (A, α_A), where as vector space A is 2-dimensional with basis $\{e_1, e_2\}$, the multiplication is given by $e_1 e_1 = e_2$ and zero elsewhere, α_A is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then (A, α_A) satisfies α-identity condition.
(iv) Consider the Hom-associative algebra (A, α_A), where as vector space A is 3-dimensional with basis $\{e_1, e_2, e_3\}$, the multiplication is given by $e_1 e_1 = e_2$, $e_1 e_2 = e_3$, $e_2 e_1 = e_3$ and zero elsewhere, α_A is represented by $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then (A, α_A) satisfies α-identity condition.

Lemma 4.4. *Let (A, α_A) be a Hom-associative algebra.*

(a) *There are Hom-actions of Hom-Lie algebras* (A, α_A) *and* $(L^\alpha(A), \overline{\alpha}_A)$ *on each other. Moreover, these Hom-actions are compatible if* (A, α_A) *satisfies the* α*-identity condition* (2).
(b) *There is a short exact sequence of Hom-Lie algebras*

$$0 \longrightarrow (HC_1^\alpha(A), \alpha_{HC}) \stackrel{i}{\longrightarrow} (L^\alpha(A), \overline{\alpha}_A) \stackrel{\psi}{\longrightarrow} \big([A, A], \alpha_{A|}\big) \longrightarrow 0,$$

where $(HC_1^\alpha(A), \alpha_{HC})$ *is an abelian Hom-Lie algebra with* α_{HC} *induced by* α_A, $\alpha_{A|}$ *is the restriction of* α_A *and* $\psi(a \otimes b) = [a, b]$.
(c) *The induced Hom-action of* (A, α_A) *on* $(HC_1^\alpha(A), \alpha_{HC})$ *is trivial. Moreover, if* (A, α_A) *satisfies the* α*-identity condition* (2)*, then both* i *and* ψ *preserve the Hom-actions of the Hom-Lie algebra* (A, α_A).

Proof. (a) The Hom-action of (A, α_A) on $(L^\alpha(A), \overline{\alpha}_A)$ is given by

$${}^{a'}(a \otimes b) = [a', a] \otimes \alpha_A(b) + \alpha_A(a) \otimes [a', b],$$

while the Hom-action of $(L^\alpha(A), \overline{\alpha}_A)$ on (A, α_A) is defined by

$${}^{(a \otimes b)}a' = [[a, b], a']$$

for all $a', a, b \in A$. Straightforward calculations show that these are indeed Hom-actions of Hom-Lie algebras, which are compatible if (A, α_A) satisfies α-identity condition (2).

(b) and (c) are immediate consequences of the definitions above. □

Definition 4.5. Let (A, α_A) be a Hom-associative algebra. The first Milnor cyclic homology $HC_1^M(A, \alpha_A)$ is the quotient vector space of $A \otimes A$ by the relations

$$\begin{aligned} &a \otimes b + b \otimes a = 0, \\ &ab \otimes \alpha_A(c) - \alpha_A(a) \otimes bc + ca \otimes \alpha_A(b) = 0, \\ &\alpha_A(a) \otimes bc - \alpha_A(a) \otimes cb = 0. \end{aligned}$$

Of course, for $\alpha_A = \mathrm{id}_A$ this is the definition of the first Milnor cyclic homology of the associative algebra A (see e.x. [5]).

Theorem 4.6. *Let* (A, α_A) *be a Hom-associative* (*non-commutative*) *algebra satisfying the* α*-identity condition* (2). *Then there is an exact sequence of vector spaces*

$$\begin{aligned} A/[A, A] \otimes HC_1^\alpha(A) &\to \mathcal{H}_1^\alpha(A, L^\alpha(A)) \to \mathcal{H}_1^\alpha(A, [A, A]) \to \\ &\to HC_1^\alpha(A) \to HC_1^M(A, \alpha_A) \to [A, A]/[A, [A, A]] \to 0. \end{aligned}$$

Proof. This is an easy consequence of Theorem 3.4. □

Let us remark that if $\alpha_A = \mathrm{id}_A$, the exact sequence in Theorem 4.6 coincides with that of [3, Theorem 5.7].

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References

1. J. M. Casas, M. A. Insua and N. Pacheco, On universal central extensions of Hom-Leibniz algebras. *J. Algebra Appl.* **13** (2014), No. 8, 1450053, 22 pp.
2. G. Ellis, A nonabelian tensor product of Lie algebras. *Glasgow Math.* J. 33 (1991), No. 1, 101–120.
3. D. Guin, Cohomologie des algébres de Lie croisées et K-théorie de Milnor additive. (French) Ann. Inst. Fourier **45** (1995), No. 1, 93–118.
4. J. T. Hartwig, D. Larsson and S. D. Silvestrov, Deformations of Lie algebras using σ-derivations. *J. Algebra* **295** (2006), No. 2, 314–361.
5. N. Inassaridze, E. Khmaladze and M. Ladra, Non-abelian homology of Lie algebras. *Glasg. Math. J.* **46** (2004), No. 2, 417–429.
6. Q. Jin and X. Li, Hom-Lie algebra structures on semi-simple Lie algebras. *J. Algebra* **319** (2008), No. 4, 1398–1408.
7. A. Makhlouf and S. Silvestrov, Hom-algebra structures. *J. Gen. Lie Theory Appl.* **2** (2008), No. 2, 51–64.
8. A. Makhlouf and S. Silvestrov, Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras. *Forum Math.* **22** (2010), No. 4, 715–739.
9. Y. Sheng, Representations of hom-Lie algebras. *Algebr. Represent. Theory* **15** (2012), No. 6, 1081–1098.
10. D. Yau, Hom-algebras as deformations and homology, arXiv 0712.3515v1, 2007.
11. D. Yau, Enveloping algebras of Hom-Lie algebras. *J. Gen. Lie Theory Appl.* **2** (2008), No. 2, 95–108.
12. D. Yau, Hom-algebras and homology. *J. Lie Theory* **19** (2009), No. 2, 409–421.

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