Available online at www.sciencedirect.com

ScienceDirect

Journal of Pure and Applied Algebra 209 (2007) 411–437

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Homotopy classification of braided graded categorical groups

A.M. Cegarra[a,*], E. Khmaladze[b]

[a] *Departamento de Álgebra, Universidad de Granada, 18071 Granada, Spain*
[b] *Georgian Acad Sci, A Razmadze Math Inst, Tbilisi, GE-0193, Georgia*

Received 28 April 2005; received in revised form 28 April 2006
Available online 28 July 2006
Communicated by I. Moerdijk

Abstract

For any group G, a certain cohomology theory of G-modules is developed. This cohomology arises from the homotopy theory of G-spaces and it is called the "abelian cohomology of G-modules". Then, as the main results of this paper, natural one-to-one correspondences between elements of the 3rd cohomology groups of G-modules, G-equivariant pointed simply-connected homotopy 3-types and equivalence classes of braided G-graded categorical groups are established. The relationship among all these objects with equivariant quadratic functions between G-modules is also discussed.

MSC: 18D10; 18D30; 55N25; 55P91

1. Introduction and summary

For any two abelian groups M and N, the abelian groups

$$H^n_{ab}(M,N) = \mathrm{Hom}_{\mathrm{Ho}\mathbf{S}_*}(K(M,2), K(N,n+1)),$$

where $\mathrm{Ho}\mathbf{S}_*$ is the homotopy category of pointed spaces (simplicial sets), define the (first level) cohomology theory of the abelian group M with coefficients in the abelian group N [10,11,19]. It is a well-known fact that for each $k \in H^n_{ab}(M,N)$, $n \geq 3$, there exists a pointed space $X = (X, *)$, unique up to weak homotopy equivalence, such that $\pi_2 X = M$, $\pi_n X = N$, $\pi_i X = 0$ for all $i \neq 2, n$ and k is the (unique non-trivial) Postnikov invariant of X. On the other hand, Joyal and Street proved in [18, Theorem 3.3] that every $k \in H^3_{ab}(M,N)$ also determines a braided categorical group $\mathbb{G}$, unique up to braided monoidal equivalence, such that M is the abelian group of isomorphism classes of its objects, N is the abelian group of automorphisms of its identity object, and k is the cohomology class of those abelian 3-cocycles of M with coefficients in N canonically deduced from the coherence pentagons and hexagons in $\mathbb{G}$. Hence, braided categorical groups arise as algebraic models for homotopy 3-types of simply connected pointed spaces [17].

The main purpose of this paper is to show how all the above facts are actually instances of what happens in a more general equivariant context. Indeed, our first motivation for carrying out this work was to state and prove a precise

* Corresponding author.
E-mail addresses: acegarra@ugr.es (A.M. Cegarra), khmal@rmi.acnet.ge (E. Khmaladze).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.06.011

classification theorem for *braided graded categorical groups*; that is, for groupoids $\mathbb{G}$ equipped with a grading functor to a group G, $gr : \mathbb{G} \to G$, a graded monoidal structure by graded functors $\otimes : \mathbb{G} \times_G \mathbb{G} \to \mathbb{G}$ and $I : G \to \mathbb{G}$, corresponding coherent 1-graded associativity and unit constraints $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, $X \otimes I \to X \leftarrow I \otimes X$ and natural 1-graded braiding morphisms $X \otimes Y \to Y \otimes X$ compatible with the graded monoidal structure, such that for any object X there is an object X' with an arrow $X \otimes X' \to I$ of grade 1 (the non-braided case was dealt with in [4]). These graded categorical groups were originally introduced by Fröhlich and Wall in [12], where they presented a suitable abstract setting to study Brauer groups in equivariant situations (cf. also [13,4] or [5] for more instances). For their classification, two braided G-graded categorical groups that are connected by a braided graded monoidal equivalence are considered the same. Therefore, the problem arises of giving a complete invariant of this equivalence relation, which we solve by means of triples (M, N, k), consisting of G-modules M, N and a cohomology class $k \in H^3_{G,ab}(M, N)$. Here, the $H^n_{G,ab}(M, -)$ are Quillen cohomology groups [21, II, Section 5] of the pointed G-space $K(M, 2)$ in the homotopy category of pointed G-spaces, $\mathrm{Ho}\mathbf{S}^G_*$, with respect to the closed model structure where weak equivalences are those G-equivariant pointed maps that are weak equivalences on the underlying spaces, that is,

$$H^n_{G,ab}(M, N) = \mathrm{Hom}_{\mathrm{Ho}\mathbf{S}^G_*}(K(M, 2), K(N, n + 1)).$$

Since equivariant weak homotopy types of pointed G-spaces X with only two non-trivial homotopy groups at dimensions 2 and n are classified by the elements of the cohomology group $H^n_{G,ab}(\pi_2 X, \pi_n X)$ (see Theorem 4 in Section 1), we get our main result, namely: *the homotopy category of G-equivariant pointed simply-connected 3-types is equivalent to the homotopy category of braided G-graded categorical groups.*

We should stress that the equivariant weak equivalences we are using should not be confused with the stronger notion of *weak equivariant-homotopy equivalences*, which are G-equivariant maps inducing weak equivalences on the fixed point subspaces of all subgroups of G. Thus, whereas the Postnikov invariants of weak equivariant-homotopy types live in Bredon–Moerdijk–Svensson's cohomology groups [1,20], the cohomology groups $H^n_{G,ab}(M, N)$ we use are isomorphic to ordinary reduced equivariant cohomology groups (see Proposition 3 in Section 1).

The plan of this paper, briefly, is as follows. The first section includes the definition of the cohomology groups of G-modules $H^n_{G,ab}$ and it is mainly dedicated to stating some concepts, results and notations concerning the homotopy theory of G-spaces we are going to use. The material in Section 1 is quite standard, so an expert reader may skip most of the proofs here. Both for theoretical and computational interests, it is appropriate to have an explicit description of a manageable cochain complex $C^\bullet_{G,ab}(M, N)$ to compute the cohomology groups $H^n_{G,ab}(M, N)$, and this is the goal of the second section. Next, in Section 3, we describe a particular subcomplex $\mathcal{E}^\bullet_G(M, N) \subseteq C^\bullet_{G,ab}(M, N)$, whose cohomology groups are precisely the abelian groups $\mathrm{Ext}^n_{\mathbb{Z}G}(M, N)$, allowing us to explicitly show the relationship between the groups $\mathrm{Ext}^n_{\mathbb{Z}G}$ and $H^n_{G,ab}$ without any argument based on the universal coefficient or the Borel spectral sequences (see [2, Section 3]). Namely, we describe isomorphisms $\mathrm{Ext}^n_{\mathbb{Z}G}(M, N) \cong H^{n+1}_{G,ab}(M, N)$ for $n = 0, 1$, and, in Section 4, a natural exact sequence

$$0 \to \mathrm{Ext}^2_{\mathbb{Z}G}(M, N) \to H^3_{G,ab}(M, N) \to \mathrm{Quad}_G(M, N) \to \mathrm{Ext}^3_{\mathbb{Z}G}(M, N),$$

where $\mathrm{Quad}_G(M, N)$ is the abelian group of all G-equivariant quadratic functions from M to N (cf. [19, Theorem 3], where it was stated that $H^3_{ab}(M, N)$ is isomorphic to the group of all quadratic functions from M to N, for any two abelian groups M and N). In the fifth and final section we include our theorems on the homotopy classification of braided G-graded categorical groups and their homomorphisms by means of the cohomology groups $H^3_{G,ab}(M, N)$ and $H^2_{G,ab}(M, N)$.

2. Cohomology of G-modules

The material of this section is fairly standard concerning basic facts of the homotopy theory of G-spaces. Our main goal is the introduction of cohomology groups of G-modules $H^n_{G,ab}$. We refer the reader to the book by Goerss and Jardine [14] for background.

Throughout G is a fixed group, $\mathbf{S}$ denotes the category of simplicial sets, and $\mathbf{S}^G_*$ is the category of all pointed simplicial sets $X = (X, *)$ with a (left) G-action by pointed automorphisms, hereafter referred to as pointed G-spaces.

There is a Quillen closed model category structure on $\mathbf{S}^G_*$ such that a pointed G-map $f : X \to Y$ is

- a *weak equivalence* if and only if f is a weak equivalence in $\mathbf{S}$, that is, $\pi_i(f) : \pi_i X \to \pi_i Y$ is an isomorphism for all $i \geq 0$;

– a *fibration* if and only if f is a (Kan) fibration in $\mathbf{S}$;
– a *cofibration* if and only if it is injective and $Y \setminus f(X)$ is a free G-set.

Thus, in this homotopy theory a pointed G-space $(X,*)$ is fibrant whenever X is a Kan simplicial set, while $(X,*)$ is cofibrant if no nonidentity element of G fixes a simplex different from the base point.

For any G-module N, the Eilenberg–MacLane minimal complexes $K(N,n)$ have an evident structure of pointed G-spaces and the G-equivariant cohomology groups of a pointed G-space X with coefficients in N, $H^n_G(X,N)$, are defined by

$$H^n_G(X,N) = \mathrm{Hom}_{\mathrm{Ho}\mathbf{S}^G_*}(X, K(N,n)), \quad n \geq 0,$$

where $\mathrm{Ho}\mathbf{S}^G_*$ is the homotopy category associated to the closed model category $\mathbf{S}^G_*$ described above.

For any G-group Π, the equivariant cohomology groups

$$H^n_G(\Pi,N) = H^n_G(K(\Pi,1),N),$$

even for twisted local coefficients N, are treated in [3] and several algebraic applications of this cohomology theory of groups with operators are shown in [4,5] or [6]. In this paper we deal with the case when Π is abelian, that is, with cohomology groups of a G-module M, $H^n_{G,ab}(M,N)$, which are defined as follows.

Definition 1. The cohomology groups of a G-module M with coefficients in a G-module N are defined by

$$H^n_{G,ab}(M,N) = H^{n+1}_G(K(M,2),N), \quad n \geq 1. \tag{2.1}$$

Note that when $G = 1$ is the trivial group, then the cohomology groups (2.1) are just those of Eilenberg–MacLane cohomology theory of abelian groups $H^n_{ab}(M,N)$ [10,11,19] (see the next section).

Both for theoretical and computational interests, it is appropriate to have an interpretation of the cohomology groups $H^n_{G,ab}(M,N)$ in terms of an ordinary singular cohomology with local coefficients. This is the aim of the definitions below.

Any G-space X can be regarded as a functor $X : G \to \mathbf{S}$, that is, as a diagram of spaces with the shape of G. Let

$$\int_G X = \operatorname*{hocolim}_G X.$$

Thus, $(\int_G X)_n = \{(\sigma_1,\ldots,\sigma_n,x) | \sigma_i \in G, x \in X_n\}$ and the face and degeneracy operators are given by

$$\begin{aligned}
&d_0(\sigma_1,\ldots,\sigma_n,x) = (\sigma_2,\ldots,\sigma_n, {}^{\sigma_1^{-1}}d_0 x),\\
&d_i(\sigma_1,\ldots,\sigma_n,x) = (\sigma_1,\ldots,\sigma_i\sigma_{i+1},\ldots,\sigma_n,d_i x), \quad 0 < i < n,\\
&d_n(\sigma_1,\ldots,\sigma_n,x) = (\sigma_1,\ldots,\sigma_{n-1},d_n x),\\
&s_i(\sigma_1,\ldots,\sigma_n,x) = (\sigma_1,\ldots,\sigma_i,1,\sigma_{i+1},\ldots,\sigma_n,s_i x), \quad 0 \leq i \leq n.
\end{aligned}$$

In particular, for $X = *$ we have

$$\int_G * = K(G,1) = BG \text{ (the classifying space of } G),$$

and therefore, a pointed G-space $X = (X,*)$ gives rise to a retraction diagram

$$\textstyle\int_G X : \quad \int_G X \underset{\rho}{\overset{\epsilon}{\rightleftarrows}} \int_G * = BG, \qquad \rho\epsilon = id, \tag{2.2}$$

where $\rho(\sigma_1,\ldots,\sigma_n,x) = (\sigma_1,\ldots,\sigma_n)$ and $\epsilon(\sigma_1,\ldots,\sigma_n) = (\sigma_1,\ldots,\sigma_n,*)$. Note that the retraction ρ is a fibration if and only if X is fibrant and also that the fibre space of ρ at the unique vertex of BG is isomorphic to X as simplicial set.

This construction defines a functor

$$\textstyle\int_G : \mathbf{S}^G_* \longrightarrow \mathbf{S}^{BG}_{BG}, \qquad X \longmapsto \int_G X,$$

from the category of pointed G-spaces to the double comma category of spaces over and under BG, that is, the category $\mathbf{S}_{BG}^{BG}$ whose objects R are retraction diagrams

$$R: \quad R \underset{r}{\overset{s}{\rightleftarrows}} BG, \qquad rs = id, \tag{2.3}$$

and whose morphisms are simplicial maps $f : R \to R'$ such that $r'f = r$ and $fs = s'$.

The category $\mathbf{S}_{BG}^{BG}$ has a closed model structure induced by the usual one of simplicial sets; that is, a map f in $\mathbf{S}_{BG}^{BG}$ is a weak equivalence, cofibration or fibration if and only if f is a weak equivalence, cofibration or fibration of simplicial sets respectively. The following proposition states that the homotopy theory of pointed G-spaces is equivalent to the homotopy theory of spaces over and under BG.

Proposition 2. *The functor $\int_G : \mathbf{S}_*^G \to \mathbf{S}_{BG}^{BG}$ is a right Quillen equivalence. Then it induces an equivalence*

$$\operatorname{Ho}\mathbf{S}_*^G \simeq \operatorname{Ho}\mathbf{S}_{BG}^{BG}$$

between the associated homotopy categories.

Proof. The proposition is known to a large extent so we omit its proof in full detail. To describe the left adjoint functor $\mathcal{L}_G : \mathbf{S}_{BG}^{BG} \to \mathbf{S}_*^G$ to $\int_G$, let EG be the universal cover of G, that is, EG is the G-space with $(EG)_n = G^{n+1}$ and face maps given by $d_i(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_n)$, $i < n$, $d_n(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \ldots, \sigma_{n-1})$, and with G-action ${}^{\sigma}(\sigma_0, \ldots, \sigma_n) = (\sigma\sigma_0, \sigma_1, \ldots, \sigma_n)$. The space EG is contractile and $EG/G \cong BG$ via the canonical projection $EG \to BG$, $(\sigma_0, \ldots, \sigma_n) \mapsto (\sigma_1, \ldots, \sigma_n)$. Then, for any R in $\mathbf{S}_{BG}^{BG}$ as in (2.3), the pointed G-space $\mathcal{L}_G R$ is defined by the cofibre product

$$\begin{tikzcd} EG \arrow[r, "\bar{s}"] \arrow[d] & EG \times_{BG} R \arrow[d, "\pi"] \\ * \arrow[r] & \mathcal{L}_G R, \end{tikzcd} \tag{2.4}$$

where $\bar{s}(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \ldots, \sigma_n, s(\sigma_1, \ldots, \sigma_n))$ and the G-action on it is induced by the one on EG. The unit of the adjunction $R \to \int_G \mathcal{L}_G R$ is given by $x \mapsto (r(x), [1, r(x), x])$ and, for any pointed G-space X, the counit $\int_G \mathcal{L}_G X \to X$ is given by

$$[\sigma_0, \ldots, \sigma_n, (\sigma_0, \ldots, \sigma_n, x)] \mapsto {}^{\sigma_0}x.$$

By [14, VI, Lemma 4.2 and IV, Proposition 1.7] the functor $\int_G$ preserves both fibrations and weak equivalences. Then, by [16, Lemma 1.3.4] $(\mathcal{L}_G, \int_G)$ is a Quillen adjunction. For any R in $\mathbf{S}_{BG}^{BG}$, by [8, Proposition 2.3] the map $R \to \int_G EG \times_{BG} R$, $x \mapsto (r(x), (1, r(x), x))$ is a weak equivalence. Since the map π in (2.4) is a weak equivalence, the induced one $\int_G EG \times_{BG} R \to \int_G \mathcal{L}_G R$ is also a weak equivalence. The unit of the adjunction is just the composition $R \to \int_G EG \times_{BG} R \to \int_G \mathcal{L}_G R$ and therefore it is also a weak equivalence. An analogous argument, using once again [8, Proposition 2.3], proves that the counit is a weak equivalence and then the proposition follows from [16, Proposition 1.3.13].

Since every object in $\mathbf{S}_{BG}^{BG}$ is cofibrant and, for any G-module N, the retractions $\int_G K(N,n) \rightleftarrows BG$ are fibrant (indeed, they are split minimal fibrations), it is a consequence of the above Proposition 2 that for any pointed G-space X there are natural isomorphisms

$$\begin{aligned} H_G^n(X, N) &\cong \operatorname{Hom}_{\operatorname{Ho}\mathbf{S}_{BG}^{BG}}\left(\int_G X, \int_G K(N,n)\right) \\ &\cong \left[\int_G X, \int_G K(N,n)\right]_{\mathbf{S}_{BG}^{BG}} \\ &\cong H^n\left(\int_G X, BG; N\right), \end{aligned}$$

where $\left[\int_G X, \int_G K(N,n)\right]_{\mathbf{S}_{BG}^{BG}}$ is the abelian group of homotopy classes of maps and $H^n(\int_G X, BG; N)$ is the ordinary cohomology of the simplicial set $\int_G X$ relative to $BG = \int_G *$ with local coefficients in the G-module N. The last isomorphism sends a homotopy class represented by a simplicial map $\int_G X \xrightarrow{f} \int_G K(N,n)$ to the cohomology class represented by the cocycle $G^n \times X_n \xrightarrow{f_n} G^n \times N \xrightarrow{pr} N$ (see [14, VI, Proposition 4.13]). In particular, we have

Proposition 3. *For any G-modules M and N there are natural isomorphisms*

$$H^n_{G,ab}(M,N) \cong H^{n+1}\left(\int_G K(M,2), BG; N\right).$$

We shall finish this section by proving the following result.

Theorem 4. *Every pointed G-space X with $\pi_i X = 0$ for all $i \neq 2, n$, where $n \geq 3$, determines a cohomology class $k(X) \in H^n_{G,ab}(\pi_2 X, \pi_n X)$. Two such G-spaces X, Y are in the same equivariant weak homotopy class (i.e., they are isomorphic in $\mathrm{Ho}\mathbf{S}_*^G$) if and only if there are G-module isomorphisms $p : \pi_2 X \cong \pi_2 Y$ and $q : \pi_n X \cong \pi_n Y$ such that $p^* q_* k(X) = k(Y)$.*

We refer to $k(X)$ as the *equivariant Postnikov invariant of X*.

Proof. Let M and N be any two G-modules. From Proposition 2, the classification of pointed G-spaces X with $\pi_2 X = M$, $\pi_n X = N$ and $\pi_i X = 0$ for all $i \neq 2, n$, by its equivariant weak homotopy type, is equivalent to the classification of minimal fibre sequences

$$\begin{tikzcd} F \arrow[r, hook] & R \arrow[r, shift right, "r"'] & BG \arrow[l, shift right, "s"'] \end{tikzcd} \tag{2.5}$$

such that $\pi_2 F = M$, $\pi_n F = N$ and $\pi_i F = 0$ for all $i \neq 2, n$, two such split minimal fibrations being equivalent if there exists an isomorphism $f : R \cong R'$ such that $r' f = r$ and $f s' = s$.

In the case when $N = 0$, the split minimal fibre sequence (2.5) is necessarily isomorphic to the split minimal fibre sequence

$$\begin{tikzcd} K(M,2) \arrow[r, hook] & \int_G K(M,2) \arrow[r, shift right, "\rho"'] & BG \arrow[l, shift right, "\epsilon"'] \end{tikzcd}$$

and, therefore, for N arbitrary, the natural Postnikov system of (2.5) is of the form

$$\begin{tikzcd} K(N,n) \arrow[r, equal] \arrow[d, hook] & K(N,n) \arrow[d, hook] & \\ F \arrow[r, hook] \arrow[d] & R \arrow[r, shift right, "r"'] \arrow[d, "p"'] & BG \arrow[l, shift right, "s"'] \arrow[d, equal] \\ K(M,2) \arrow[r, hook] & \int_G K(M,2) \arrow[r, shift right, "\rho"'] & BG \arrow[l, shift right, "\epsilon"'] \end{tikzcd}$$

where $K(N,n) \hookrightarrow R \xrightarrow{p} \int_G K(M,2)$ is a minimal fibre sequence. It follows that the classification of the split minimal fibre sequence (2.5) is equivalent to the classification of the diagrams

$$\begin{tikzcd} & & BG \arrow[dl, "s"'] \arrow[d, "\epsilon"] \\ K(N,n) \arrow[r, hook] & R \arrow[r, "p"] & \int_G K(M,2), \end{tikzcd} \qquad ps = \epsilon,$$

that is, of all minimal fibre sequences $K(N,n) \hookrightarrow R \xrightarrow{p} \int_G K(M,2)$ with a crossed section from BG. Since these fibre sequences are classified precisely by the relative cohomology group $H^{n+1}(\int_G K(M,2), BG; N)$, the assertion follows from Proposition 3. $\square$

3. An equivariant Bar reduction. The complex $C^{\bullet}_{G,ab}(M,N)$

In [10,11,19] Eilenberg and MacLane defined a chain complex, $\mathcal{A}(M,2)$, associated to any abelian group M, to compute the (co)homology groups of a space $K(M,2)$. In fact, they proved the existence of a cochain equivalence

$$c_M : C^{\bullet}(K(M,2),N) \to \mathrm{Hom}(\mathcal{A}(M,2),N) \tag{3.1}$$

for any abelian group N [11, I, Theorem 20.3]. Hence, the cohomology groups of an abelian group M with coefficients in an abelian group N can be computed as

$$H^n_{ab}(M,N) = H^{n+1}(\mathrm{Hom}(\mathcal{A}(M,2),N)).$$

The main result of this section is to show an explicit description of a cochain complex, $C^{\bullet}_{G,ab}(M,N)$, to compute the cohomology $H^{\bullet}_{G,ab}(M,N)$, of a G-module M with coefficients in a G-module N, given in Definition 1.

We shall use the following notations for X, Y any two groups:

- $X^p|Y^q = \{(\mathbf{x}|\mathbf{y}) = (x_1,\ldots,x_p|y_1,\ldots,y_q),\ x_i \in X, y_j \in Y\},\ p,q \geq 0.$
- $X^0 = \{()\}$ is the trivial group. Then, $X^0|Y^q = Y^q$ and $X^p|Y^0 = X^p$.
- **Shuf**(p,q) is the set of all (p,q)-shuffles. Any $\pi \in \mathbf{Shuf}(p,q)$ defines a map

$$\pi : X^p|X^q \longrightarrow X^{p+q},$$

given by $\pi(x_1,\ldots,x_p|x_{p+1},\ldots,x_{p+q}) = (x_{\pi(1)},\ldots,x_{\pi(p+q)})$.

Definition 5. Let M, N be two G-modules. The complex $C^{\bullet}_{G,ab}(M,N)$ is defined to be trivial in dimension zero, that is, $C^0_{G,ab}(M,N) = 0$ and for $n \geq 1$ the elements of $C^n_{G,ab}(M,N)$, related as *abelian n-cochains of the G-module M with coefficients in the G-module N*, are the maps

$$f : \bigcup M^{p_1}|M^{p_2}|\cdots|M^{p_r}|G^q \to N,$$

where the union is taken over all $p_1,\ldots,p_r \geq 1$, $q \geq 0$ with $r + \sum_{i=1}^r p_i + q = n+1$, which are normalized in the sense that

$$f(x^1_1,\ldots,x^1_{p_1}|\cdots|x^r_1,\ldots,x^r_{p_r}|\sigma_1,\ldots,\sigma_q) = 0,$$

whenever some $x^i_j = 0$ or some $\sigma_k = 1$.

Addition in $C^n_{G,ab}(M,N)$ is given by adding pointwise in the abelian group N. The coboundary homomorphism $\partial : C^n_{G,ab}(M,N) \to C^{n+1}_{G,ab}(M,N)$ is defined by the formula

$$\begin{aligned}(\partial f)(\mathbf{x}^1|\mathbf{x}^2|\cdots|\mathbf{x}^r|\overline{\sigma}) = {}& {}^{\sigma_1}f(\mathbf{x}^1|\cdots|\mathbf{x}^r|d_0\overline{\sigma}) + \sum_{1\leq i\leq q-1}(-1)^i f(\mathbf{x}^1|\cdots|\mathbf{x}^r|d_i\overline{\sigma})\\ &+ (-1)^q f({}^{\sigma_q}\mathbf{x}^1|\cdots|{}^{\sigma_q}\mathbf{x}^r|d_q\overline{\sigma})\\ &+ \sum_{\substack{1\leq i\leq r\\ 0\leq j\leq p_i}}(-1)^{q+\epsilon_{i-1}+j} f(\mathbf{x}^1|\cdots|d_j\mathbf{x}^i|\cdots|\mathbf{x}^r|\overline{\sigma})\\ &+ \sum_{\substack{1\leq i\leq r-1\\ \pi\,\in\,\mathbf{Shuf}(p_i,p_{i+1})}}(-1)^{q+\epsilon_i+\epsilon(\pi)} f(\mathbf{x}^1|\cdots|\pi(\mathbf{x}^i|\mathbf{x}^{i+1})|\cdots|\mathbf{x}^r|\overline{\sigma}),\end{aligned}$$

where $(\mathbf{x}^1|\mathbf{x}^2|\cdots|\mathbf{x}^r|\overline{\sigma}) \in M^{p_1}|\cdots|M^{p_r}|G^q$, $\overline{\sigma} = (\sigma_1,\ldots,\sigma_q)$; $d_i : G^q \to G^{q-1}$, are the face operators of $BG = K(G,1)$; $d_j : M^{p_i} \to M^{p_i-1}$ are the face operators of $BM = K(M,1)$; $\epsilon_i = p_1 + \cdots + p_i + i$ and $\epsilon(\pi)$ is the parity of the shuffle π.

Thus,
$C^1_{G,ab}(M,N)$ consists of all normalized maps

$$M \xrightarrow{f} N,$$

$C^2_{G,ab}(M,N)$ consists of all normalized maps

$$M^2 \cup M|G \xrightarrow{g} N,$$

$C^3_{G,ab}(M,N)$ consists of normalized maps

$$M^3 \cup M|M \cup M^2|G \cup M|G^2 \xrightarrow{h} N,$$

$C^4_{G,ab}(M,N)$ consists of normalized maps

$$M^4 \cup M^2|M \cup M|M^2 \cup M^3|G \cup M|M|G \cup M^2|G^2 \cup M|G^3 \longrightarrow N,$$

... ...

with the coboundary maps

$$(\partial f)(x,y) = f(x) - f(x+y) + f(y), \tag{3.2}$$
$$(\partial f)(x|\sigma) = {}^{\sigma}f(x) - f({}^{\sigma}x); \tag{3.3}$$
$$(\partial g)(x,y,z) = g(y,z) - g(x+y,z) + g(x,y+z) - g(x,y), \tag{3.4}$$
$$(\partial g)(x|y) = g(x,y) - g(y,x), \tag{3.5}$$
$$(\partial g)(x,y|\sigma) = {}^{\sigma}g(x,y) - g({}^{\sigma}x,{}^{\sigma}y) - g(y|\sigma) + g(x+y|\sigma) - g(x|\sigma), \tag{3.6}$$
$$(\partial g)(x|\sigma,\tau) = {}^{\sigma}g(x|\tau) - g(x|\sigma\tau) + g({}^{\tau}x|\sigma); \tag{3.7}$$
$$(\partial h)(x,y,z,t) = h(y,z,t) - h(x+y,z,t) + h(x,y+z,t) - h(x,y,z+t) + h(x,y,z), \tag{3.8}$$
$$(\partial h)(x|y,z) = h(x|z) - h(x|y+z) + h(x|y) + h(x,y,z) - h(y,x,z) + h(y,z,x), \tag{3.9}$$
$$(\partial h)(x,y|z) = h(y|z) - h(x+y|z) + h(x|z) - h(x,y,z) + h(x,z,y) - h(z,x,y), \tag{3.10}$$
$$\begin{aligned}(\partial h)(x,y,z|\sigma) \;=\; & {}^{\sigma}h(x,y,z) - h({}^{\sigma}x,{}^{\sigma}y,{}^{\sigma}z) - h(y,z|\sigma)\\ & + h(x+y,z|\sigma) - h(x,y+z|\sigma) + h(x,y|\sigma),\end{aligned} \tag{3.11}$$
$$(\partial h)(x|y|\sigma) = {}^{\sigma}h(x|y) - h({}^{\sigma}x|{}^{\sigma}y) - h(x,y|\sigma) + h(y,x|\sigma), \tag{3.12}$$
$$(\partial h)(x,y|\sigma,\tau) = {}^{\sigma}h(x,y|\tau) - h(x,y|\sigma\tau) + h({}^{\tau}x,{}^{\tau}y|\sigma) + h(y|\sigma,\tau) - h(x+y|\sigma,\tau) + h(x|\sigma,\tau), \tag{3.13}$$
$$(\partial h)(x|\sigma,\tau,\gamma) = {}^{\sigma}h(x|\tau,\gamma) - h(x|\sigma\tau,\gamma) + h(x|\sigma,\tau\gamma) - h({}^{\gamma}x|\sigma,\tau). \tag{3.14}$$

... ...

Below is our main result in this section.

Theorem 6. *For any G-modules M and N, $C^{\bullet}_{G,ab}(M,N)$ is actually a cochain complex and there are natural isomorphisms*

$$H^n_{G,ab}(M,N) \cong H^n(C^{\bullet}_{G,ab}(M,N)), \quad n \geq 0.$$

Proof. Let $X_G(M)$ be the bisimplicial set whose (p,q)-simplices are the elements of the cartesian product $G^p \times K(M,2)_q$. The vertical face and degeneracy maps are defined by those of the Eilenberg–MacLane simplicial set $K(M,2)$, and the horizontal face and degeneracy maps are those of $BG = K(G,1)$ except d^h_0, which is defined by $d^h_0(\sigma_1,\ldots,\sigma_p,\mathbf{x}) = (\sigma_2,\ldots,\sigma_p,{}^{\sigma_1^{-1}}\mathbf{x})$. Hence

$$\mathrm{diag}X_G(M) = \int_G K(M,2). \tag{3.15}$$

Then we obtain a double cosimplicial abelian group $C^{\bullet\bullet}(X_G(M), N)$ in which the group $C^{p,q}(X_G(M), N)$ consists of all maps $f : X_G(M)_{p,q} \to N$, the horizontal cofaces are defined by

$$(d_i^h f)(\sigma_1, \ldots, \sigma_p, \mathbf{x}) = \begin{cases} f(\sigma_2, \ldots, \sigma_p, {}^{\sigma_1^{-1}}\mathbf{x}), & \text{if } i = 0, \\ f(\sigma_1, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_p, \mathbf{x}), & \text{if } 0 < i < p, \\ {}^{\sigma_p} f(\sigma_1, \ldots, \sigma_{p-1}, \mathbf{x}), & \text{if } i = p, \end{cases}$$

and the vertical cofaces are defined by

$$(d_j^v f)(\overline{\sigma}, \mathbf{x}) = f(\overline{\sigma}, d_j\mathbf{x}), \quad 0 \le j \le q.$$

We denote again by $C^{\bullet\bullet}(X_G(M), N)$ the associated double complex of normalized cochains, where differentials are obtained from the faces by taking alternating sums, and by $\operatorname{Tot} C^{\bullet\bullet}(X_G(M), N)$ the associated total complex. Observe that, by equality (3.15), we have $\operatorname{diag} C^{\bullet\bullet}(X_G(M), N) \cong C^\bullet(\int_G K(M, 2), N)$. Then, as a result of Dold and Puppe [7], there is a natural equivalence of cochain complexes

$$\varphi_M : \operatorname{Tot} C^{\bullet\bullet}(X_G(M), N) \to C^\bullet\left(\int_G K(M, 2), N\right).$$

Now, let us note that the map c_M in (3.1) is an equivalence of cochain complexes of G-modules, if we consider the diagonal G-action on both complexes $C^\bullet(K(M, 2), N)$ and $\operatorname{Hom}(\mathcal{A}(M, 2), N)$. Then, c_M induces a homomorphism of bicomplexes

$$c_M : C^\bullet(G, C^\bullet(K(M, 2), N)) \to C^\bullet(G, \operatorname{Hom}(\mathcal{A}(M, 2), N)),$$

where the cochain map $c_M : C^q(G, C^\bullet(K(M, 2), N)) \to C^q(G, \operatorname{Hom}(\mathcal{A}(M, 2), N))$ is an equivalence for every q. Therefore, the induced cochain map on the total complexes

$$c_M : \operatorname{Tot} C^\bullet(G, C^\bullet(K(M, 2), N)) \to \operatorname{Tot} C^\bullet(G, \operatorname{Hom}(\mathcal{A}(M, 2), N))$$

is also an equivalence. Since there is quite an obvious isomorphism of bicomplexes

$$C^\bullet(G, C^\bullet(K(M, 2), N)) \cong C^{\bullet\bullet}(X_G(M), N),$$

we deduce a natural equivalence

$$c_M : \operatorname{Tot} C^{\bullet\bullet}(X_G(M), N) \to \operatorname{Tot} C^\bullet(G, \operatorname{Hom}(\mathcal{A}(M, 2), N)).$$

By combining the quasi-isomorphisms $\varphi_M, c_M, \varphi_0$ and c_0 (the corresponding ones for $M = 0$), we get the following commutative diagram of cochain complexes induced by the retraction $M \leftrightarrows 0$:

$$\begin{tikzcd} C^\bullet(\int_G K(M,2), N) \arrow[r, shift left, "r_1"] & C^\bullet(\int_G K(0,2), N) \arrow[l, shift left, "s_1"] \\ \operatorname{Tot} C^{\bullet\bullet}(X_G(M), N) \arrow[u, "\varphi_M"] \arrow[d, "c_M"'] \arrow[r, shift left, "r_2"] & \operatorname{Tot} C^{\bullet\bullet}(X_G(0), N) \arrow[l, shift left, "s_2"] \arrow[u, "\varphi_0"'] \arrow[d, "c_0"] \\ \operatorname{Tot} C^\bullet(G, \operatorname{Hom}(\mathcal{A}(M,2), N)) \arrow[r, shift left, "r_3"] & \operatorname{Tot} C^\bullet(G, \operatorname{Hom}(\mathcal{A}(0,2), N)), \arrow[l, shift left, "s_3"] \end{tikzcd} \tag{3.16}$$

where $r_i s_i = id$, $i = 1, 2, 3$.

Let us observe now that $K(0, 2) = *$ is the one point simplicial set, so that

$$C^\bullet\left(\int_G K(0, 2), N\right) = C^\bullet(BG, N)$$

and the top retraction in (3.16) is precisely the one induced by the split fibration $\int_G K(M, 2) \leftrightarrows BG$ (see (2.2)). Therefore, the kernel of r_1 is the relative cochain complex $C^\bullet(\int_G K(M, 2), BG; N)$ whose homology groups are $H^n_{G,ab}(M, N)$ up to a shift dimension (see Proposition 3).

The bicomplexes $C^{\bullet\bullet}(X_G(0), M)$ and $C^\bullet(G, \mathrm{Hom}(\mathcal{A}(0,2), N))$ are both isomorphic to the double cochain complex which is the complex $C^\bullet(BG, N)$ constant in the vertical direction, and a straightforward identification shows that the complex $C^\bullet_{G,ab}(M, N)$, with its dimension raised by 1, occurs in the diagram (3.16) as the kernel of r_3. It follows that $C^\bullet_{G,ab}(M, N)$ is actually a cochain complex and the induced maps by φ_M and c_M

$$C^\bullet\left(\int_G K(M,2), BG; N\right) \xleftarrow{\varphi_M} \mathrm{Ker}(r_2) \xrightarrow{c_M} C^\bullet_{G,ab}(M, N)$$

are quasi-isomorphisms, whence $H^n_{G,ab}(M, N) \cong H^n(C^\bullet_{G,ab}(M, N))$ for all $n \geq 0$. □

4. Relationship between $\mathrm{Ext}^n_{\mathbb{Z}G}(M, N)$ and $H^n_{G,ab}(M, N)$

In this section, for any group G and G-modules M, N we present a new description by cocycles of the abelian groups $\mathrm{Ext}^n_{\mathbb{Z}G}(M, N)$. This description gives us the possibility to establish explicitly their relationship with $H^n_{G,ab}(M, N)$ groups in an elemental way, that is, without any argument based on the universal coefficient or the Borel spectral sequences, as was done by Breen in [2].

Let M be any fixed G-module. We begin by inductively constructing an exact sequence of G-modules

$$0 \to R_n(M) \xrightarrow{i_n} P_{n-1}(M) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1(M) \xrightarrow{\partial_1} P_0(M) \xrightarrow{\partial_0} M \to 0, \tag{4.1}$$

in which all $P_i(M)$ are free G-modules, as follows. Denote by $\mathbb{Z}[M^*]$ the free abelian group on the set of elements $[x]$, $x \in M$, with the only relation $[0] = 0$ (so that $\mathbb{Z}[M^*]$ is the free abelian group on $M^* = M \setminus \{0\}$) and let $\mathbb{Z}[G]$ be the free abelian group on the elements $[\sigma]$, $\sigma \in G$ (so that $\mathbb{Z}[G]$ is the free G-module on the unitary set $\{[1]\}$). Then we define

$$P_0(M) = \mathbb{Z}[M^*] \otimes \mathbb{Z}[G]$$

with G-action determined by ${}^\sigma(\omega \otimes [\tau]) = \omega \otimes [\sigma\tau]$, $\omega \in \mathbb{Z}[M^*]$, $\sigma, \tau \in G$. Hence, $P_0(M)$ is the free G-module on the set of elements $[x] \otimes [1]$, $x \in M$, $x \neq 0$. The G-module epimorphism ∂_0 is given on the free generators by $\partial_0([x] \otimes [1]) = x$ and we define $R_1(M)$ as the kernel of ∂_0, so that we have the short exact sequence

$$0 \to R_1(M) \xrightarrow{i_1} P_0(M) \xrightarrow{\partial_0} M \to 0.$$

Proceeding by induction, suppose we have defined the sequence (4.1) for $n \geq 1$. Then we define

$$P_n(M) = R_n(M) \otimes \mathbb{Z}[G]$$

with G-action such that ${}^\sigma(\omega \otimes [\tau]) = \omega \otimes [\sigma\tau]$, $\omega \in R_n(M)$, $\sigma, \tau \in G$. Note that $P_n(M)$ is a free G-module with basis $\{e_i \otimes [1]\}$ where $\{e_i\}$ is a basis of the free abelian group $R_n(M)$. The G-module epimorphism $\partial_n : P_n(M) \to R_n(M)$ is given by $\partial(\omega \otimes [\sigma]) = {}^\sigma\omega$ and we define $R_{n+1}(M)$ as the kernel of ∂_n, so that we have the short exact sequence

$$0 \to R_{n+1}(M) \xrightarrow{i_{n+1}} P_n(M) \xrightarrow{\partial_n} R_n(M) \to 0. \tag{4.2}$$

Next we show a description of the G-module $R_n(M)$ in (4.1) by a set of generators and relations.

Proposition 7. *For any $n \geq 1$, the G-module $R_n(M)$ is defined by a set of generators*

$$\{[x, y, \sigma_1, \ldots, \sigma_{n-1}],\ [x, \sigma_1, \ldots, \sigma_n];\ x, y \in M,\ \sigma_1, \ldots, \sigma_n \in G\}$$

with the relations

$$[x, y, \sigma_1, \ldots, \sigma_{n-1}] = 0 = [x, \sigma_1, \ldots, \sigma_n], \quad \textit{if } x = 0 \textit{ or some } \sigma_i = 1, \tag{4.3}$$

$$[x, y, \sigma_1, \ldots, \sigma_{n-1}] = [y, x, \sigma_1, \ldots, \sigma_{n-1}], \tag{4.4}$$

$$[y, z, \sigma_1, \ldots, \sigma_{n-1}] + [x, y + z, \sigma_1, \ldots, \sigma_{n-1}] = [x + y, z, \sigma_1, \ldots, \sigma_{n-1}] + [x, y, \sigma_1, \ldots, \sigma_{n-1}], \tag{4.5}$$

$$\begin{aligned}{}^{\sigma_1}[x, y, \sigma_2, \cdots, \sigma_n] = &\sum_{i=1}^{n-1}(-1)^{i+1}[x, y, \sigma_1, \ldots, \sigma_i\sigma_{i+1}, \cdots, \sigma_n] \\ &+(-1)^{n+1}\left([{}^{\sigma_n}x, {}^{\sigma_n}y, \sigma_1, \ldots, \sigma_{n-1}] + [x, \sigma_1, \ldots, \sigma_n]\right. \\ &\left.+[y, \sigma_1, \ldots, \sigma_n] - [x+y, \sigma_1, \ldots, \sigma_n]\right),\end{aligned} \tag{4.6}$$

$${}^{\sigma_1}[x, \sigma_2, \cdots, \sigma_{n+1}] = \sum_{i=1}^{n}(-1)^{i+1}[x, \sigma_1, \ldots, \sigma_i\sigma_{i+1}, \cdots, \sigma_{n+1}] + (-1)^n[{}^{\sigma_{n+1}}x, \sigma_1, \ldots, \sigma_n]. \tag{4.7}$$

Furthermore, as an abelian group, $R_n(M)$ is defined by the same set of generators and with the only relations (4.3), (4.4) and (4.5).

Proof. We proceed by induction. For $n = 1$, let $[x, y], [x, \sigma] \in R_1(M)$ be the elements defined by the equalities

$$[x, y] = [x] \otimes [1] + [y] \otimes [1] - [x+y] \otimes [1], \quad x, y \in M, \tag{4.8}$$

$$[x, \sigma] = [x] \otimes [\sigma] - [{}^{\sigma}x] \otimes [1], \quad x \in M, \sigma \in G. \tag{4.9}$$

To see that these elements generate $R_1(M)$, let us denote by $R_1'(M) \subseteq R_1(M)$ the subgroup generated by them. We claim that $\partial_0 : P_0(M) \to M$ is the cokernel of the inclusion of $R_1'(M)$ into $P_0(M)$. In effect, for any abelian group homomorphism $f : P_0(M) \to A$, which verifies $f(R_1'(M)) = 0$, the map $\tilde{f} : M \to A$ given by $\tilde{f}(x) = f([x] \otimes [1])$ is a group homomorphism satisfying $\tilde{f}\partial_0 = f$ since

$$\tilde{f}(x+y) = f([x+y] \otimes [1]) = f([x] \otimes [1]) + f([y] \otimes [1]) - f[x, y] = \tilde{f}(x) + \tilde{f}(y),$$
$$\tilde{f}\partial_0([x] \otimes [\sigma]) = \tilde{f}({}^{\sigma}x) = f([{}^{\sigma}x] \otimes [1]) = f([x] \otimes [\sigma]) + f[x, \sigma] = f([x] \otimes [\sigma]).$$

Therefore, $R_1'(M) = \mathrm{Ker}(\partial_0) = R_1(M)$, and thus, $R_1(M)$ is generated (even as an abelian group) by the elements $[x, y], [x, \sigma], x, y \in M, \sigma \in G$. The elements (4.8) and (4.9), clearly verify the relations (4.3) and (4.4). Next we observe that the relations (4.5)–(4.7) hold. In effect,

$$\begin{aligned}
[y, z] + [x, y+z] &= ([y] + [z] - [y+z]) \otimes [1] + ([x] + [y+z] - [x+y+z]) \otimes [1] \\
&= ([x] + [y] + [z] - [x+y+z]) \otimes [1] \\
&= ([x] + [y] - [x+y]) \otimes [1] - ([x+y] + [z] - [x+y+z]) \otimes [1] \\
&= [x, y] + [x+y, z], \\
{}^{\sigma}[x, y] + [x+y, \sigma] &= [x] \otimes [\sigma] + [y] \otimes [\sigma] - [x+y] \otimes [\sigma] + [x+y] \otimes [\sigma] - [{}^{\sigma}x + {}^{\sigma}y] \otimes [1] \\
&= ([x] \otimes [\sigma] - [{}^{\sigma}x] \otimes [1]) + ([y] \otimes [\sigma] - [{}^{\sigma}y] \otimes [1]) + ([{}^{\sigma}x] + [{}^{\sigma}y] - [{}^{\sigma}x + {}^{\sigma}y]) \otimes [1] \\
&= [x, \sigma] + [y, \sigma] + [{}^{\sigma}x, {}^{\sigma}y], \\
{}^{\sigma}[x, \tau] &= [x] \otimes [\sigma\tau] - [{}^{\tau}x] \otimes [\sigma] = [x] \otimes [\sigma\tau] - [{}^{\sigma\tau}x] \otimes [1] + [{}^{\sigma\tau}x] \otimes [1] - [{}^{\tau}x] \otimes [\sigma] \\
&= [x, \sigma\tau] - [{}^{\tau}x, \sigma].
\end{aligned}$$

Let us suppose now that A is any abelian group and $a_{[x,y]}, a_{[x,\sigma]} \in A$ are elements satisfying (4.3)–(4.5) (with respect to the indexes). We shall prove the existence of a (necessarily unique) group homomorphism $f : R_1(M) \to A$ such that $f([x, y]) = a_{[x,y]}$ and $f([x, \sigma]) = a_{[x,\sigma]}$. For, we first build the abelian group extension

$$0 \to A \xrightarrow{j} A \star M \xrightarrow{p} M \to 0,$$

where $A \star M$ is the cartesian product set $A \times M$ with the addition

$$(a, x) + (a', y) = (a + a' + a_{[x,y]}, x + y),$$

j and p are given by $j(a) = (a, 0)$ and $p(a, x) = x$. Then, we define a group homomorphism $\varphi : P_0(M) \to A \star M$ by putting $\varphi([x] \otimes [\sigma]) = (a_{[x,\sigma]}, {}^{\sigma}x)$. Since $p\varphi = \partial_0 : P_0(M) \to M$, there is a unique homomorphism of abelian groups $f : R_1(M) \to A$ such that $(f(\omega), 0) = \varphi(\omega)$, for all $\omega \in R_1(M)$. In particular, we have

$$\begin{aligned}
(f([x, y]), 0) &= \varphi([x, y]) = \varphi([x] \otimes [1]) + \varphi([y] \otimes [1]) - \varphi([x+y] \otimes [1]) \\
&= (0, x) + (0, y) - (0, x+y) = (a_{[x,y]}, 0),
\end{aligned}$$

$$\begin{aligned}(f([x,\sigma]),0) &= \varphi([x,\sigma]) = \varphi([x]\otimes[\sigma]) - \varphi([{}^{\sigma}x]\otimes[1])\\ &= (a_{[x,\sigma]}, {}^{\sigma}x) - (0, {}^{\sigma}x) = (a_{[x,\sigma]}, 0).\end{aligned}$$

Thus, $f([x,y]) = a_{[x,y]}$ and $f([x,\sigma]) = a_{[x,\sigma]}$, as required.

Here we observe, simply by checking on the generators, that if A is a G-module and the elements $a_{[x,y]}$, $a_{[x,\sigma]}$ also satisfy (4.6) and (4.7), then the above group homomorphism $f : R_1(M) \to A$ is actually of G-modules. This completes the proof for $n = 1$.

Next, we suppose that the proposition is true for $n \geq 1$. The short exact sequence (4.2) is split by the group homomorphism $s : R_n(M) \to P_n(M)$ given by $s(\omega) = \omega \otimes [1]$. Therefore, we have an isomorphism of abelian groups $P_n(M) \cong R_{n+1}(M) \times R_n(M)$, with projection $q : P_n(M) \to R_{n+1}(M)$ defined by

$$q(\omega\otimes[\sigma]) = \omega\otimes[\sigma] - s\partial_n(\omega\otimes[\sigma]) = \omega\otimes[\sigma] - {}^{\sigma}\omega\otimes[1].$$

Therefore, as an abelian group and so also as a G-module, $R_{n+1}(M)$ is generated by the elements

$$\begin{aligned}[][x,y,\sigma_1,\ldots,\sigma_n] &= q([x,y,\sigma_2,\ldots,\sigma_n]\otimes[\sigma_1])\\ &= [x,y,\sigma_2,\ldots,\sigma_n]\otimes[\sigma_1] - {}^{\sigma_1}[x,y,\sigma_2,\ldots,\sigma_n]\otimes[1],\\ [x,\sigma_1,\ldots,\sigma_{n+1}] &= q([x,\sigma_2,\ldots,\sigma_{n+1}]\otimes[\sigma_1])\\ &= [x,\sigma_2,\ldots,\sigma_{n+1}]\otimes[\sigma_1] - {}^{\sigma_1}[x,\sigma_2,\ldots,\sigma_{n+1}]\otimes[1].\end{aligned}$$

It is routine and we leave it to the reader to check that these elements verify all the equations (4.3)–(4.7), since such equations are satisfied by the corresponding generators of $R_n(M)$ due to the hypothesis of induction.

Let us suppose now that A is any abelian group and

$$a_{[x,y,\sigma_1,\ldots,\sigma_n]}, a_{[x,\sigma_1,\ldots,\sigma_{n+1}]} \in A$$

are elements satisfying (4.3), (4.4), (4.6) and (4.7). We shall prove the existence of a (necessarily unique) group homomorphism $\varphi : R_{n+1}(M) \to A$ such that $\varphi([x,y,\sigma_1,\ldots,\sigma_n]) = a_{[x,y,\sigma_1,\ldots,\sigma_n]}$ and $\varphi([x,\sigma_1,\ldots,\sigma_{n+1}]) = a_{[x,\sigma_1,\ldots,\sigma_{n+1}]}$. For, we first observe, by using inductive hypothesis on $R_n(M)$, that for each fixed $\sigma_1 \in G$, the mapping $[x,y,\sigma_2,\ldots,\sigma_n] \mapsto a_{[x,y,\sigma_1,\ldots,\sigma_n]}$, $[x,\sigma_2,\ldots,\sigma_{n+1}] \mapsto a_{[x,\sigma_1,\ldots,\sigma_{n+1}]}$ extends to an abelian group homomorphism from $R_n(M)$ to A. Then, we can define a homomorphism of abelian groups $\varphi : P_n(M) \to A$ by equalities

$$\begin{aligned}&\varphi([x,y,\sigma_2,\ldots,\sigma_n]\otimes[\sigma_1]) = a_{[x,y,\sigma_1,\ldots,\sigma_n]},\\ &\varphi([x,\sigma_2,\ldots,\sigma_{n+1}]\otimes[\sigma_1]) = a_{[x,\sigma_1,\ldots,\sigma_{n+1}]}.\end{aligned}$$

Clearly $\varphi s = 0 : R_n(M) \to A$, thanks to condition (4.3). Therefore, $\varphi = \varphi\, id = \varphi(i_{n+1}q + s\partial_n) = \varphi i_{n+1}q$, that is, $\varphi(\alpha) = \varphi(q(\alpha))$ for all $\alpha \in P_n(M)$. In particular, the restriction of φ to $R_{n+1}(M)$ satisfies

$$\begin{aligned}\varphi([x,y,\sigma_1,\ldots,\sigma_n]) &= \varphi\, q([x,y,\sigma_2,\ldots,\sigma_n]\otimes[\sigma_1]) = \varphi([x,y,\sigma_2,\ldots,\sigma_n]\otimes[\sigma_1]) = a_{[x,y,\sigma_1,\ldots,\sigma_n]},\\ \varphi([x,\sigma_1,\ldots,\sigma_{n+1}]) &= \varphi\, q([x,\sigma_2,\ldots,\sigma_{n+1}]\otimes[\sigma_1]) = \varphi([x,\sigma_2,\ldots,\sigma_{n+1}]\otimes[\sigma_1])\\ &= a_{[x,\sigma_1,\ldots,\sigma_{n+1}]},\end{aligned}$$

as required.

Finally observe, simply by checking on the generators, that if A is in addition a G-module and the elements $a_{[x,y,\sigma_1,\ldots,\sigma_n]}, a_{[x,\sigma_1,\ldots,\sigma_{n+1}]}$ also satisfy (4.6) and (4.7), then the above $\varphi : R_{n+1}(M) \to A$ is actually a homomorphism of G-modules. Therefore, as an abelian group, $R_{n+1}(M)$ is indeed defined by the generators $[x,y,\sigma_1,\ldots,\sigma_n]$, $[x,\sigma_1,\ldots,\sigma_{n+1}]$ and the relations (4.3)–(4.5), while as a G-module it is generated by the same elements but with the relations (4.3)–(4.7). □

Now we are ready to describe the abelian groups $\mathrm{Ext}^n_{\mathbb{Z}G}(M,N)$ as the cohomology groups of the cochain complex introduced below.

Definition 8. Let M, N be two G-modules. The cochain complex

$$\mathcal{E}^{\bullet}_G(M,N) : 0 \to \mathcal{E}^0_G(M,N) \xrightarrow{\partial^1} \mathcal{E}^1_G(M,N) \xrightarrow{\partial^2} \cdots,$$

is defined as follows. Each $\mathcal{E}^n_G(M,N)$ is the abelian group of all normalized maps

$$\widetilde{f} : M^2|G^{n-1} \cup M|G^n \to N,$$

satisfying

$$\widetilde{f}(x,y|\sigma_1,\ldots,\sigma_{n-1}) = \widetilde{f}(y,x|\sigma_1,\ldots,\sigma_{n-1}), \tag{4.10}$$
$$\widetilde{f}(y,z|\sigma_1,\ldots,\sigma_{n-1})+\widetilde{f}(x,y+z|\sigma_1,\ldots,\sigma_{n-1}) = \widetilde{f}(x+y,z\mid\sigma_1,\ldots,\sigma_{n-1})+\widetilde{f}(x,y\mid\sigma_1,\ldots,\sigma_{n-1}). \tag{4.11}$$

The coboundary homomorphism $\partial : \mathcal{E}^n_G(M,N) \to \mathcal{E}^{n+1}_G(M,N)$ is defined by the formulas

$$\begin{aligned}(\partial\widetilde{f})(x,y\mid\sigma_1,\ldots,\sigma_n) &= {}^{\sigma_1}\widetilde{f}(x,y\mid\sigma_2,\ldots,\sigma_n)+\sum_{i=1}^{n-1}(-1)^i\widetilde{f}(x,y\mid\sigma_1,\ldots,\sigma_i\sigma_{i+1},f\ldots,\sigma_n)\\ &\quad+(-1)^n\left[\widetilde{f}({}^{\sigma_n}x,{}^{\sigma_n}y\mid\sigma_1,\ldots,\sigma_{n-1})+\widetilde{f}(x\mid\sigma_1,\ldots,\sigma_n)\right.\\ &\quad\left.-\ \widetilde{f}(x+y\mid\sigma_1,\ldots,\sigma_n)+\widetilde{f}(y\mid\sigma_1,\ldots,\sigma_n)\right],\\ (\partial\widetilde{f})(x\mid\sigma_1,\ldots,\sigma_{n+1}) &= {}^{\sigma_1}\widetilde{f}(x\mid\sigma_2,\ldots,\sigma_{n+1})+\sum_{i=1}^{n}(-1)^i\widetilde{f}(x\mid\sigma_1,\ldots,\sigma_i\sigma_{i+1},\ldots,\sigma_{n+1})\\ &\quad+(-1)^{n+1}\widetilde{f}({}^{\sigma_{n+1}}x\mid\sigma_1,\ldots,\sigma_n).\end{aligned}$$

Thus,
$\mathcal{E}^0_G(M,N)$ consists of all normalized maps $\widetilde{f} : M \to N$,
$\mathcal{E}^1_G(M,N)$ consists of normalized maps $\widetilde{g} : M^2 \cup M \mid G \to N$ such that

$$\widetilde{g}(x,y) = \widetilde{g}(y,x),$$
$$\widetilde{g}(y,z)+\widetilde{g}(x,y+z) = \widetilde{g}(x+y,z)+\widetilde{g}(x,y),$$

$\mathcal{E}^2_G(M,N)$ consists of normalized maps $\widetilde{h} : M^2 \mid G \cup M \mid G^2 \to N$ satisfying

$$\widetilde{h}(x,y|\sigma) = \widetilde{h}(y,x|\sigma),$$
$$\widetilde{h}(y,z|\sigma)+\widetilde{h}(x,y+z|\sigma) = \widetilde{h}(x+y,z|\sigma)+\widetilde{h}(x,y|\sigma),$$

$\mathcal{E}^3_G(M,N)$ consists of normalized maps $\widetilde{\kappa} : M^2|G^2 \cup M|G^3 \to N$ such that

$$\widetilde{\kappa}(x,y|\sigma,\tau) = \widetilde{\kappa}(y,x|\sigma,\tau),$$
$$\widetilde{\kappa}(y,z|\sigma,\tau)+\widetilde{\kappa}(x,y+z|\sigma,\tau) = \widetilde{\kappa}(x+y,z|\sigma,\tau)+\widetilde{\kappa}(x,y|\sigma,\tau),$$

… …
with the coboundary maps

$$\begin{aligned}&(\partial\widetilde{f})(x,y) = \widetilde{f}(y)-\widetilde{f}(x+y)+\widetilde{f}(x),\\ &(\partial\widetilde{f})(x\mid\sigma) = {}^{\sigma}\widetilde{f}(x)-\widetilde{f}({}^{\sigma}x);\\ &(\partial\widetilde{g})(x,y\mid\sigma) = {}^{\sigma}\widetilde{g}(x,y)-\widetilde{g}({}^{\sigma}x,{}^{\sigma}y)-\widetilde{g}(y\mid\sigma)+\widetilde{g}(x+y\mid\sigma)-\widetilde{g}(x\mid\sigma),\\ &(\partial\widetilde{g})(x\mid\sigma,\tau) = {}^{\sigma}\widetilde{g}(x\mid\tau)-\widetilde{g}(x\mid\sigma\tau)+\widetilde{g}({}^{\tau}x\mid\sigma);\\ &(\partial\widetilde{h})(x,y\mid\sigma,\tau) = {}^{\sigma}\widetilde{h}(x,y\mid\tau)-\widetilde{h}(x,y\mid\sigma\tau)+\widetilde{h}({}^{\tau}x,{}^{\tau}y\mid\sigma)+\widetilde{h}(y\mid\sigma,\tau)-\widetilde{h}(x+y\mid\sigma,\tau)+\widetilde{h}(x|\sigma,\tau),\\ &(\partial\widetilde{h})(x\mid\sigma,\tau,\gamma) = {}^{\sigma}\widetilde{h}(x\mid\tau,\gamma)-\widetilde{h}(x\mid\sigma\tau,\gamma)+\widetilde{h}(x\mid\sigma,\tau\gamma)-\widetilde{h}({}^{\gamma}x\mid\sigma,\tau);\\ &\ldots\ \ldots\end{aligned}$$

Theorem 9. *For any two G-modules M and N there are natural isomorphisms*

$$\mathrm{Ext}^n_{\mathbb{Z}G}(M,N) \cong H^n\mathcal{E}^\bullet_G(M,N).$$

Proof. Let $P_\bullet(M) = \cdots \to P_1(M) \to P_0(M) \to M \to 0$ be the free resolution of the G-module M built using the sequences (4.1). Since

$$\mathrm{Hom}_{\mathbb{Z}G}(P_n(M), N) = \mathrm{Hom}_{\mathbb{Z}G}(R_n(M) \otimes \mathbb{Z}(G), N) \cong \mathrm{Hom}_{\mathbb{Z}}(R_n(M), N),$$

Proposition 7 determines isomorphisms $\mathrm{Hom}_{\mathbb{Z}G}(P_n(M), N) \cong \mathcal{E}^n_G(M, N)$, $f \mapsto \widetilde{f}$, where

$$\widetilde{f}(x, y \mid \sigma_1, \ldots, \sigma_{n-1}) = f([x, y, \sigma_1, \ldots, \sigma_{n-1}] \otimes [1])$$
$$\widetilde{f}(x \mid \sigma_1, \ldots, \sigma_n) = f([x, \sigma_1, \ldots, \sigma_n] \otimes [1]).$$

These isomorphisms define a cochain complex isomorphism $\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet(M), N) \cong \mathcal{E}^\bullet_G(M, N)$, whence the theorem follows. $\square$

There is a natural injective cochain map of degree $+1$,

$$i : \mathcal{E}^\bullet_G(M, N) \to C^{\bullet+1}_{G,ab}(M, N), \tag{4.12}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^0_G(M,N) \arrow[r] \arrow[d, equal] & \mathcal{E}^1_G(M,N) \arrow[r] \arrow[d, equal] & \mathcal{E}^2_G(M,N) \arrow[r] \arrow[d, hook, "i_2"'] & \cdots & \arrow[r] & \mathcal{E}^{n-1}_G(M,N) \arrow[r] \arrow[d, hook, "i_3"'] & \cdots \\
0 \arrow[r] & C^1_{G,ab}(M,N) \arrow[r] & C^2_{G,ab}(M,N) \arrow[r] & C^3_{G,ab}(M,N) \arrow[r] & \cdots & \arrow[r] & C^n_{G,ab}(M,N) \arrow[r] & \cdots
\end{tikzcd}$$

which sends an $(n-1)$-cochain $\widetilde{f} : M^2|G^{n-2} \cup M|G^{n-1} \to N$ of $\mathcal{E}^\bullet_G(M, N)$ to the n-cochain $f : \bigcup M^{p_1}|\cdots|M^{p_r}|G^q \to N$ of $C^\bullet_{G,ab}(M, N)$ such that

$$f|_{M^2|G^{n-2} \cup M|G^{n-1}} = \widetilde{f},$$
$$f|_{M^{p_1}|\cdots|M^{p_r}|G^q} = 0 \quad \text{whenever } q < n-2.$$

Then, as a consequence, we have the following:

Theorem 10. *For any G-modules M and N there is a natural homomorphism*

$$\mathrm{Ext}^n_{\mathbb{Z}G}(M, N) \xrightarrow{i} H^{n+1}_{G,ab}(M, N)$$

for every $n \geq 0$, induced by the above cochain map (4.12). This homomorphism is actually an isomorphism for $n = 0$ and $n = 1$, so that

$$H^1_{G,ab}(M, N) \cong \mathrm{Hom}_G(M, N),$$
$$H^2_{G,ab}(M, N) \cong \mathrm{Ext}^1_{\mathbb{Z}G}(M, N),$$

and for $n = 2$ it is a monomorphism

$$\mathrm{Ext}^2_{\mathbb{Z}G}(M, N) \hookrightarrow H^3_{G,ab}(M, N).$$

5. $H^3_{G,ab}(M, N)$ and quadratic maps

In [10,11,19], Eilenberg and MacLane proved that, for any two abelian groups M, N, the cohomology group $H^3_{ab}(M, N)$ is isomorphic to the abelian group of quadratic functions from M to N. This fact was an algebraic proof of a classic result by Whitehead [22], which states that the quadratic map $\eta^* : \pi_2 X \to \pi_3 X$, induced by the Hopf map $\eta : S^3 \to S^2$, completely determines the homotopy type of a path-connected CW-complex X with only non-trivial homotopy groups $\pi_2 X$ and $\pi_3 X$ (recall that $H^3_{ab}(M, N) \cong H^4(K(M, 2), N)$). When X is a pointed G-space, the quadratic map $\eta^* : \pi_2 X \to \pi_3 X$ is a G-equivariant quadratic map and it is natural to enquire about the relationship between the G-equivariant homotopy type of X, which is determined by its equivariant Postnikov invariant $k(X) \in H^3_{G,ab}(\pi_2 X, \pi_3 X)$ (see Theorem 4), and the induced G-equivariant quadratic map η^*. And also, to enquire whether any G-equivariant quadratic map $q : M \to N$ is induced from an equivariant homotopy 3-type of a simply connected pointed G-space X.

Let M, N be two G-modules. Then the third Eilenberg–MacLane cohomology group of the abelian group M with coefficients in the abelian group N, $H^3_{ab}(M,N) = H^n_{1,ab}(M,N)$ is a G-module, where the G-action on 3-cocycles is given diagonally. Let $H^3_{ab}(M,N)^G$ be the subgroup of $H^3_{ab}(M,N)$ of all G-invariant elements. We state below a main result in this section.

Theorem 11. *For any G-modules M and N, there is a natural exact sequence of abelian groups*

$$0 \to \mathrm{Ext}^2_{\mathbb{Z}G}(M,N) \xrightarrow{i} H^3_{G,ab}(M,N) \xrightarrow{j} H^3_{ab}(M,N)^G \xrightarrow{\kappa} \mathrm{Ext}^3_{\mathbb{Z}G}(M,N). \tag{5.1}$$

Proof. We begin by describing the homomorphisms i, j and κ. The monomorphism i is exactly the one given in Theorem 10.

The "forgetting G" homomorphism j is the cohomological expression of the obvious map $\mathrm{Hom}_{\mathbf{HoS}^G_*}(K(M,2), K(N,3)) \to \mathrm{Hom}_{\mathbf{HoS}_*}(K(M,2), K(N,3))$. That is, for any abelian 3-cocycle $h : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N$

$$j[h] = [h|_{M^3\cup M|M}].$$

To describe the homomorphism κ, take $[h] \in H^3_{ab}(M,N)^G$. Then

$$h : M^3 \cup M|M \to N$$

is a normalized map satisfying $\partial h = 0$ in Eqs. (3.8)–(3.10) and ${}^\sigma[h] = [h]$ for all $\sigma \in G$. This last condition means that there are normalized maps $g_\sigma : M^2 \to N$ such that

$${}^\sigma h({}^{\sigma^{-1}}x, {}^{\sigma^{-1}}y, {}^{\sigma^{-1}}z) = h(x,y,z) + g_\sigma(y,z) - g_\sigma(x+y,z) + g_\sigma(x,y+z) - g_\sigma(x,y), \tag{5.2}$$

$${}^\sigma h({}^{\sigma^{-1}}x \mid {}^{\sigma^{-1}}y) = h(x|y) + g_\sigma(x,y) - g_\sigma(y,x). \tag{5.3}$$

Inserting ${}^\sigma x$, ${}^\sigma y$ and ${}^\sigma z$ instead of x, y and z respectively in the above expressions and denoting again by h the map

$$h : M^2|G \to N$$

defined by $h(x,y \mid \sigma) = g_\sigma({}^\sigma x, {}^\sigma y)$, the equalities (5.2) and (5.3) become

$${}^\sigma h(x,y,z) - h({}^\sigma x, {}^\sigma y, {}^\sigma z) - h(y,z \mid \sigma) + h(x+y,z \mid \sigma) - h(x,y+z \mid \sigma) + h(x,y \mid \sigma) = 0, \tag{5.4}$$

$${}^\sigma h(x|y) - h({}^\sigma x| {}^\sigma y) - h(x,y|\sigma) + h(y,x|\sigma) = 0. \tag{5.5}$$

Then, let

$$\widetilde{\kappa}_h : M^2|G^2 \cup M|G^3 \to N$$

be the map defined by

$$\widetilde{\kappa}_h(x,y \mid \sigma,\tau) = {}^\sigma h(x,y \mid \tau) - h(x,y \mid \sigma\tau) + h({}^\tau x, {}^\tau y \mid \sigma), \tag{5.6}$$

$$\widetilde{\kappa}_h(x \mid \sigma,\tau,\gamma) = 0. \tag{5.7}$$

By using (5.4) and (5.5), it is routine and requires only care over the definitions to show that $\widetilde{\kappa}_h$ is a 3-cocycle of the complex $\mathcal{E}^\bullet_G(M,N)$. Thanks to the isomorphism in Theorem 9 for $n = 3$, we define the homomorphism κ by

$$\kappa[h] = [\widetilde{\kappa}_h] \in \mathrm{Ext}^3_{\mathbb{Z}G}(M,N).$$

It is easy to verify that $\kappa : H^3_{ab}(M,N)^G \to \mathrm{Ext}^3_{\mathbb{Z}G}(M,N)$ is a well-defined homomorphism of abelian groups.

We next prove the exactness of the sequence

$$\mathrm{Ext}^2_{\mathbb{Z}G}(M,N) \xrightarrow{i} H^3_{G,ab}(M,N) \xrightarrow{j} H^3_{ab}(M,N)^G.$$

Obviously, $j\,i = 0$. To prove that $\mathrm{Ker}(j) \subseteq \mathrm{Im}(i)$, let $j[h] = [h|_{M^3\cup M|M}] = 0$. Then, there exists an abelian 2-cochain $g : M^2 \to N$ such that

$$h(x,y,z) = g(y,z) - g(x+y,z) + g(x,y+z) - g(x,y),$$

$$h(x|y) = g(x,y) - g(y,x),$$

and we define a map $\widetilde{h}: M^2|G \cup M|G^2 \to N$ by taking

$$\widetilde{h}(x, y \mid \sigma) = h(x, y \mid \sigma) - {}^{\sigma}g(x, y) + g({}^{\sigma}x, {}^{\sigma}y),$$
$$\widetilde{h}(x|\sigma, \tau) = h(x \mid \sigma, \tau).$$

An easy direct calculation shows that $\widetilde{h} \in \mathcal{E}^2_G(M, N)$, $\partial\widetilde{h} = 0$ and $i[\widetilde{h}] = [h]$. Thus, $\mathrm{Im}(i) = \mathrm{Ker}(j)$.

And finally, we prove the exactness of the sequence

$$H^3_{G,ab}(M, N) \xrightarrow{j} H^3_{ab}(M, N)^G \xrightarrow{\kappa} \mathrm{Ext}^3_{\mathbb{Z}G}(M, N).$$

For any $[h] \in H^3_{G,ab}(M, N)$ we have $\kappa j[h] = \kappa[h|_{M^3 \cup M|M}] = [\kappa_h]$, where

$$\kappa_h(x \mid \sigma, \tau, \gamma) = 0,$$
$$\kappa_h(x, y \mid \sigma, \tau) \;=\; {}^{\sigma}h(x, y \mid \tau) - h(x, y \mid \sigma\tau) + h({}^{\tau}x, {}^{\tau}y \mid \sigma)$$
$$\stackrel{(3.13)}{=} h(y \mid \sigma, \tau) + h(x|\sigma, \tau) - h(x + y \mid \sigma, \tau).$$

Then the map $\widetilde{h}: M^2|G \cup M|G^2 \to N$ given by the equalities

$$\widetilde{h}(x|\sigma, \tau) = -h(x|\sigma, \tau),$$
$$\widetilde{h}(x, y|\sigma) = 0$$

is a 2-cochain of $\mathcal{E}^{\bullet}_G(M, N)$ and clearly $\partial\widetilde{h} = \kappa_h$. Thus, $\kappa j = 0$.

Let us now consider any $[h] \in H^3_{ab}(M, N)^G$ into the kernel of κ, that is, $\kappa[h] = [\widetilde{\kappa}_h] = 0$. Recall that $\widetilde{\kappa}_h$ is defined by equalities (5.6) and (5.7), via a map $h : M^2 \mid G \to N$ such that equalities (5.4) and (5.5) hold. Since $[\widetilde{\kappa}_h] = 0$, there exists $\widetilde{h} \in \mathcal{E}^2_G(M, N)$ such that $\partial\widetilde{h} = \widetilde{\kappa}_h$. Then, it is straightforward to verify that the map

$$h' : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N$$

defined by

$$h'(x, y, z) = h(x, y, z),$$
$$h'(x|y) = h(x|y),$$
$$h'(x, y|\sigma) = h(x, y|\sigma) - \widetilde{h}(x, y|\sigma),$$
$$h'(x|\sigma, \tau) = \widetilde{h}(x|\sigma, \tau)$$

is actually an abelian 3-cocycle and clearly $j[h'] = [h]$. This complete the proof. □

Let us recall that a map $q : M \to N$ between abelian groups is called *quadratic* if the function $M \times M \to N$, $(x, y) \mapsto q(x) + q(y) - q(x + y)$, is bilinear and $q(-x) = q(x)$. The Whitehead quadratic functor Γ [23] is characterized by the universal property:

$$\mathrm{Hom}(\Gamma(M), N) \cong \mathrm{Quad}(M, N),$$

where $\mathrm{Quad}(M, N)$ is the abelian group of quadratic functions from M to N.

The *trace* of an abelian 3-cocycle $h : M^3 \cup M|M \to N$ is the map

$$t_h : M \to N, \quad t_h(x) = h(x|x).$$

An easy calculation shows that traces are quadratic maps, and Eilenberg and MacLane [10,11,19] proved that trace determines an isomorphism

$$H^3_{ab}(M, N) \cong \mathrm{Quad}(M, N), \quad [h] \mapsto t_h. \tag{5.8}$$

Suppose now that M and N are two G-modules. Then, both $H^3_{ab}(M, N)$ and $\mathrm{Quad}(M, N)$ are G-modules by the diagonal G-action, and (5.8) is actually a G-module isomorphism. Therefore, it restricts to the corresponding subgroups of all G-invariant elements

$$H^3_{ab}(M, N)^G \cong \mathrm{Quad}(M, N)^G = \mathrm{Quad}_G(M, N),$$

where $\mathrm{Quad}_G(M, N)$ is the abelian group of all quadratic G-maps from M to N. Taking into account this isomorphism, the exact sequence (5.1) yields

Theorem 12. *For any G-modules M and N there is an exact sequence*

$$0 \to \mathrm{Ext}^2_{\mathbb{Z}G}(M, N) \xrightarrow{i} H^3_{G,ab}(M, N) \xrightarrow{j} \mathrm{Quad}_G(M, N) \xrightarrow{\kappa} \mathrm{Ext}^3_{\mathbb{Z}G}(M, N). \tag{5.9}$$

Theorem 11 has an interesting interpretation in the equivariant homotopy theory of pointed spaces:

For M and N, two given G-modules, let

$$\mathbf{S}^G_*[M, N]$$

denote the set of equivalence classes of triples (X, α, β), where X is a pointed G-space with $\pi_i X = 0$ for all $i \neq 2, 3$ and α, β are isomorphisms of G-modules $\alpha : \pi_2 X \cong M$, $\beta : \pi_3 X \cong N$. We say that (X, α, β) is congruent with (X', α', β') if there is G-equivariant pointed simplicial map $f : X \to X'$ such that $\alpha'\pi_2(f) = \alpha$ and $\beta'\pi_3(f) = \beta$ and then, that (X, α, β) is equivalent to (X', α', β') if there is a zig-zag chain of congruences

$$(X, \alpha, \beta) \leftarrow (X_1, \alpha_1, \beta_1) \to \cdots \leftarrow (X_n, \alpha_n, \beta_n) \to (X', \alpha', \beta').$$

By Theorem 4, elements of $\mathbf{S}^G_*[M, N]$ are in one-to-one correspondence with the elements of the group $H^3_{G,ab}(M, N)$; more precisely, we have:

Proposition 13. *For any G-modules M and N, there is a bijection*

$$\mathbf{S}^G_*[M, N] \cong H^3_{G,ab}(M, N), [X, \alpha, \beta] \mapsto \beta_*\alpha^*k(X),$$

where $k(X)$ is the equivariant Postnikov invariant of X.

It follows from Theorem 12 that any $[X, \alpha, \beta] \in \mathbf{S}_G[M, N]$ determines the G-equivariant quadratic map

$$q_{[X,\alpha,\beta]} = j\beta_*\alpha^*k(X) : M \to N.$$

Therefore, we have a partition

$$\mathbf{S}^G_*[M, N] = \coprod_q \mathbf{S}^G_*[M, N, q],$$

where, for each G-equivariant quadratic map $q : M \to N$, $\mathbf{S}^G_*[M, N, q]$ is the set of classes $[X, \alpha, \beta] \in \mathbf{S}^G_*[M, N]$ that fulfill q in the sense that $q_{[X,\alpha,\beta]} = q$.

On the other hand, each equivariant quadratic map $q : M \to N$ determines a cohomology class

$$\mathrm{Obs}(q) = \kappa(q) \in \mathrm{Ext}^3_{\mathbb{Z}G}(M, N),$$

which we refer to as the *obstruction of* q. Then, the exactness of (5.9) in $\mathrm{Quad}_G(M, N)$ implies the following:

Theorem 14. *A G-equivariant quadratic function $q : M \to N$ is realizable, that is, $\mathbf{S}^G_*[M, N, q] \neq \emptyset$, if and only if its obstruction vanishes.*

And the exactness of (5.9) in $H^3_{G,ab}(M, N)$ means that:

Theorem 15. *If the obstruction of a quadratic G-map $q : M \to N$ vanishes, then there is a bijection*

$$\mathbf{S}^G_*[M, N, q] \cong \mathrm{Ext}^2_{\mathbb{Z}G}(M, N).$$

At the end of this section we consider two particular cases of Theorem 11.

Proposition 16. *Let G act trivially on M and N, then there is an isomorphism*

$$H^3_{G,ab}(M, N) \cong \mathrm{Ext}^2_{\mathbb{Z}G}(M, N) \oplus \mathrm{Quad}(M, N).$$

Proof. In the hypothesis of the theorem, we have $H^3_{ab}(M,N)^G = H^3_{ab}(M,N) \cong \mathrm{Quad}(M,N)$. Moreover, any abelian 3-cocycle of the abelian group M in the abelian group N, say

$$h : M^3 \cup M|M \to N,$$

defines an abelian 3-cocycle of the G-module M in the G-module N

$$h' : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N,$$

simply by putting $h'|_{M^3 \cup M|M} = h$ and $h'|_{M^2|G \cup M|G^2} = 0$. Then, the map

$$H^3_{ab}(M,N) \to H^3_{G,ab}(M,N), [h] \mapsto [h'],$$

is a group homomorphism that splits the homomorphism j of the sequence (5.1), and the proposition follows. □

Proposition 17. *Let M, N be G-modules such that $H_2(M) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(H_3(M), N) = 0$, where the $H_i(M) = H_i(M,\mathbb{Z})$ are the integral homology groups of the group M. Then the sequence*

$$0 \to \mathrm{Ext}^2_{\mathbb{Z}G}(M,N) \xrightarrow{i} H^3_{G,ab}(M,N) \xrightarrow{j} \mathrm{Quad}_G(M,N) \to 0.$$

is short exact.

Proof. Let us fix any quadratic G-map $q : M \to N$. By (5.8), there is an abelian 3-cocycle of the abelian group M with coefficients in N, say $h : M^3 \cup M|M \to N$, with trace q, that is, such that $h(x|x) = q(x)$ for all $x \in M$. Since $h|_{M^3} : M^3 \to N$ is an ordinary 3-cocycle of the group M with coefficients in the (trivial) M-module N, and by the Universal Coefficient Theorem $H^3(M,N) = 0$, there is a normalized map $g : M^2 \to N$ such that

$$h(x,y,z) = g(y,z) - g(x+y,z) + g(x,y+z) - g(x,y).$$

Then, h is cohomologous to the abelian 3-cocycle $h' : M^3 \cup M|M \to N$ given by

$$h'(x,y,z) = 0,$$
$$h'(x|y) = h(x \mid y) - g(x,y) + g(y,x).$$

Let us now define, for each $\sigma \in G$, the map $h_\sigma : M^3 \cup M|M \to N$ by

$$h_\sigma(x,y,z) = 0,$$
$$h_\sigma(x|y) = {}^\sigma h'(x|y) - h'({}^\sigma x \mid {}^\sigma y).$$

It is easy to check that so defined, each h_σ is an abelian 3-cocycle of the abelian group M with coefficients in N and, moreover, with zero trace:

$$t_{h_\sigma}(x) = h_\sigma(x \mid x) = {}^\sigma h'(x \mid x) - h'({}^\sigma x \mid {}^\sigma x) = {}^\sigma q(x) - q({}^\sigma x) = 0,$$

since q is G-equivariant. It follows that $0 = [h_\sigma] \in H^3_{ab}(M,N)$, that is, there is a map $g_\sigma : M^2 \to N$ such that

$$h_\sigma(x,y,z) = 0 = g_\sigma(y,z) - g_\sigma(x+y,z) + g_\sigma(x,y+z) - g_\sigma(x,y),$$
$$h_\sigma(x|y) = g_\sigma(x,y) - g_\sigma(y,x).$$

Since $H_2(M) = 0$, the Universal Coefficient Theorem implies that the canonical injection $\mathrm{Ext}^1_{\mathbb{Z}}(M,N) \to H^2(M,N)$ is an isomorphism and, therefore, every group extension of M by N is abelian. This is equivalent to saying that every ordinary 2-cocycle of the group M with coefficients in the (trivial) M-module N, say $f : M^2 \to N$, satisfies the symmetric condition $f(x,y) = f(y,x)$. In particular, we have $g_\sigma(x,y) = g_\sigma(y,x)$, whence $h_\sigma(x \mid y) = 0$, for all $\sigma \in G$ and $x, y \in M$. Therefore, we have the equality ${}^\sigma h'(x|y) = h'({}^\sigma x \mid {}^\sigma y)$. Then the map

$$h' : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N,$$

defined by $h'(x,y,z) = 0$, $h'(x|y) = h(x \mid y) - g(x,y) + g(y,x)$, as above, and $h'(x,y|\sigma) = 0$ and $h'(x|\sigma,\tau) = 0$, is an abelian 3-cocycle of the G-module M with coefficients in the G-module N. Clearly its trace is

$$t_{h'}(x) = h'(x|x) = h(x|x) = q(x),$$

that is $j[h'] = q$. Hence the homomorphism j in the sequence (5.9) is surjective, whence the proposition follows. □

6. Braided graded categorical groups and classification results

As mentioned in the introduction, this last section is dedicated to showing precise theorems on the homotopy classification of braided G-graded categorical groups and their homomorphisms. The results are stated and proved by means of the cohomology theory of G-modules $H^n_{G,ab}(M,N)$ studied throughout the previous sections.

We shall begin by recalling some needed terminology about graded monoidal categories and the definition of braided G-graded categorical groups, for G a given group.

We regard the group G as a category with one object, where the morphisms are elements of G and the composition is the group operation. A *grading on a category* $\mathbb{G}$ is then a functor, say $gr : \mathbb{G} \to G$. For any morphism f in $\mathbb{G}$, we refer to $gr(f) = \sigma$ as *the grade of* f and say that f is a σ*-morphism*. The grading is said to be *stable* if for any object X of $\mathbb{G}$ and any $\sigma \in G$ there exists an isomorphism $X \cong Y$ with domain X and grade σ; in other words, the grading is a cofibration in the sense of Grothendieck [15]. A functor $F : \mathbb{G} \to \mathbb{H}$ between graded categories is called a *graded functor* if it preserves grades of morphisms. From [15, Corollary 6.12], every graded functor between stably G-graded categories is cocartesian. Suppose $F' : \mathbb{G} \to \mathbb{H}$ is another graded functor. Then, a *graded natural equivalence* $\theta : F \to F'$ is a natural equivalence of functors such that all isomorphisms $\theta_X : FX \to F'X$ are of grade 1. If $\mathbb{G}$ is a graded category, the category $\operatorname{Ker}\mathbb{G}$ is the subcategory consisting of all morphisms of grade 1. A graded functor $F : \mathbb{G} \to \mathbb{H}$ between stable graded categories is an equivalence if and only if the induced functor $F : \operatorname{Ker}\mathbb{G} \to \operatorname{Ker}\mathbb{H}$ is an equivalence of categories [15, Proposition 6.5].

For a G-graded category $\mathbb{G}$, we denote by $\mathbb{G} \times_G \mathbb{G}$ the subcategory of the product category $\mathbb{G}\times\mathbb{G}$ whose morphisms are all pairs of morphisms of $\mathbb{G}$ with the same grade. This category $\mathbb{G} \times_G \mathbb{G}$ has an obvious grading, which is stable if and only if the grading of $\mathbb{G}$ is as well.

A *braided G-graded monoidal category* $\mathbb{G} := (\mathbb{G}, gr, \otimes, \mathrm{I}, A, L, R, C)$ consists of a category $\mathbb{G}$, a stable grading $gr : \mathbb{G} \to G$, graded functors $\otimes : \mathbb{G} \times_G \mathbb{G} \to \mathbb{G}$ and $\mathrm{I} : G \to \mathbb{G}$, and graded natural equivalences defined by isomorphisms of grade 1 $A_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$, $L_X : \mathrm{I} \otimes X \xrightarrow{\sim} X$, $R_X : X \otimes \mathrm{I} \xrightarrow{\sim} X$ and $C_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, such that for any objects X, Y, Z, T of $\mathbb{G}$ the following four coherence conditions hold:

$$A_{X,Y,Z\otimes T}\ A_{X\otimes Y,Z,T} = (id_X \otimes A_{Y,Z,T})\ A_{X,Y\otimes Z,T}\ (A_{X,Y,Z} \otimes id_T), \tag{6.1}$$

$$(id_X \otimes L_Y)\ A_{X,I,Y} = R_X \otimes id_Y, \tag{6.2}$$

$$(id_Y \otimes C_{X,Z})\ A_{Y,X,Z}\ (C_{X,Y} \otimes id_Z) = A_{Y,Z,X}\ C_{X,Y\otimes Z}\ A_{X,Y,Z}, \tag{6.3}$$

$$(C_{X,Z} \otimes id_Y)\ A^{-1}_{X,Z,Y}\ (id_X \otimes C_{Y,Z}) = A^{-1}_{Z,X,Y}\ C_{X\otimes Y,Z}\ A^{-1}_{X,Y,Z}. \tag{6.4}$$

If $\mathbb{G}$, $\mathbb{H}$ are braided G-graded monoidal categories, then a *braided graded monoidal functor* $F := (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}$ consists of a graded functor $F : \mathbb{G} \to \mathbb{H}$, and natural graded equivalences defined by 1-graded isomorphisms $\Phi_{X,Y} : FX \otimes FY \xrightarrow{\sim} F(X \otimes Y)$ and $\Phi_* : \mathrm{I} \to F\mathrm{I}$, such that, for all $X, Y, Z \in \mathbb{G}$, the following coherence conditions hold:

$$\Phi_{X,Y\otimes Z}\ (id_{FX} \otimes \Phi_{Y,Z})\ A_{FX,FY,FZ} = F(A_{X,Y,Z})\ \Phi_{X\otimes Y,Z}\ (\Phi_{X,Y} \otimes id_{FZ}), \tag{6.5}$$

$$F(R_X)\ \Phi_{X,\mathrm{I}}\ (id_{FX} \otimes \Phi_*) = R_{FX}, \quad F(L_X)\ \Phi_{\mathrm{I},X}\ (\Phi_* \otimes id_{FX}) = L_{FX}, \tag{6.6}$$

$$\Phi_{Y,X}\ C_{FX,FY} = F(C_{X,Y})\ \Phi_{X,Y}. \tag{6.7}$$

Suppose $F' : \mathbb{G} \to \mathbb{H}$ is another braided graded monoidal functor. A *homotopy* (or *braided graded monoidal natural equivalence*) $\theta : F \xrightarrow{\sim} F'$ is a graded natural equivalence such that, for all objects $X, Y \in \mathbb{G}$, the following coherence conditions hold:

$$\Phi'_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X\otimes Y}\,\Phi_{X,Y}, \qquad \theta_I\,\Phi_* = \Phi'_*. \tag{6.8}$$

For later use we state here the lemma below, whose proof is parallel to Lemma 1.1 in [4].

Lemma 18. *Every braided graded monoidal functor $F = (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}$ is homotopic to a braided graded monoidal functor $F' = (F', \Phi', \Phi'_*)$ with $F'\mathrm{I} = \mathrm{I}$ and $\Phi'_* = id$.*

A *braided categorical group* [18] is a braided monoidal category in which every morphism is invertible and, for each object X, there is an object X' with a morphism $X \otimes X' \to \mathrm{I}$. If $\mathbb{G}$ is a braided G-graded monoidal category, then the subcategory Ker $\mathbb{G}$ inherits a braided monoidal structure. When Ker $\mathbb{G}$ is a braided categorical group, then $\mathbb{G}$ is said to be a *braided G-graded categorical group*. Since Ker $\mathbb{G}$ is a groupoid if and only if $\mathbb{G}$ is, we make the following:

Definition 19. A braided G-graded categorical group is a braided G-graded monoidal groupoid such that, for any object X, there is an object X' with an arrow $X \otimes X' \to \mathrm{I}$ of grade 1.

We denote by $\mathcal{BCG}_G$ the category of braided G-graded categorical groups, whose morphisms are the braided graded monoidal functors between them. Homotopy is an equivalence relation among morphisms of $\mathcal{BCG}_G$ and it is compatible with compositions in $\mathcal{BCG}_G$. We can therefore define the *homotopy category of braided graded categorical groups* to be the quotient category with the same objects, but morphisms are homotopy classes of braided graded monoidal functors. A braided graded monoidal functor inducing an isomorphism in the homotopy category is said to be a *braided graded monoidal equivalence* and two braided graded categorical groups are *equivalent* if they are isomorphic in the homotopy category.

The homotopy classification of braided G-graded categorical groups is our major objective. For, we will associate to each braided G-graded categorical group $\mathbb{G}$ the algebraic data $\pi_0\mathbb{G}$, $\pi_1\mathbb{G}$ and $k(\mathbb{G})$, which are invariants under braided graded monoidal equivalence. We next introduce the first two.

– $\pi_0\mathbb{G}$ is the abelian group of 1-isomorphism classes of the objects in $\mathbb{G}$ where multiplication is induced by the tensor product.
– $\pi_1\mathbb{G}$ is the abelian group of 1-automorphisms of the unit object I where the operation is composition.

Thus, $\pi_i\mathbb{G} = \pi_i\mathrm{Ker}\,\mathbb{G}$, $i = 0, 1$, the first invariants of the braided categorical group Ker $\mathbb{G}$ considered by Joyal and Street in [18]. Note that $\pi_1\mathbb{G}$ is abelian since the multiplication $\pi_1\mathbb{G} \times \pi_1\mathbb{G} \to \pi_1\mathbb{G}$, $(a, b) \mapsto L_\mathrm{I}(a \otimes b)L_\mathrm{I}^{-1}$, is a group homomorphism [9]. The group $\pi_0\mathbb{G}$ is also abelian because of the braiding. Next we observe that both $\pi_0\mathbb{G}$ *and* $\pi_1\mathbb{G}$ *are G-modules*.

If $\sigma \in G$ and $[X] \in \pi_0\mathbb{G}$, then we write

$$^{\sigma}[X] = [X'], \tag{6.9}$$

whenever there exists a morphism of grade σ, $X \to X'$. Since the grading on $\mathbb{G}$ is stable, $^{\sigma}[X]$ is defined for all $\sigma \in G$ and $[X] \in \pi_0\mathbb{G}$. The map $(\sigma, [X]) \mapsto {}^{\sigma}[X]$, is well defined since every morphism in $\mathbb{G}$ is invertible: if $g : X \to X'$ and $g' : Y \to Y'$ are both σ-morphisms and $h : X \to Y$ is an 1-morphism, then $h' = g'hg^{-1} : X' \to Y'$ is an 1-morphism. Therefore, $[X'] = [Y'] \in \pi_0\mathbb{G}$. If g and g' are any two σ-morphisms as before, then $g \otimes g' : X \otimes Y \to X' \otimes Y'$ is also a σ-morphism; whence, $^{\sigma}[X \otimes Y] = [X' \otimes Y'] = [X'][Y'] = {}^{\sigma}[X]\,{}^{\sigma}[Y]$. Furthermore, if $\sigma, \tau \in G$, then for any $[X] \in \pi_0\mathbb{G}$, any τ-morphism $h : X \to Y$ and any σ-morphism $g : Y \to Z$, since the composition fg is a $\sigma\tau$-morphism, we have $^{\sigma\tau}[X] = [Z] = {}^{\sigma}[Y] = {}^{\sigma}({}^{\tau}[X])$. Hence, $\pi_0\mathbb{G}$ is a G-module.

The G-module structure on $\pi_1\mathbb{G}$ is easier to explain. For any $\sigma \in G$ and any arrow $a : \mathrm{I} \to \mathrm{I}$ in $\mathbb{G}$ of grade 1, it is

$$^{\sigma}a = \mathrm{I}(\sigma)\, a\, \mathrm{I}(\sigma)^{-1}. \tag{6.10}$$

The mappings $\mathbb{G} \mapsto \pi_i\mathbb{G}$, $i = 0, 1$, are functorial from the category of braided G-graded categorical groups to the category of G-modules. Moreover, since any braided graded monoidal functor between braided graded categorical groups is a braided graded equivalence if and only if its restriction to kernels is a braided equivalence, then, by [18] and by a little modification of the proof of [4, Proposition 1.3] (see also [12]), the proposition below easily follows:

Proposition 20. (i) *Every braided graded monoidal functor $F : \mathbb{G} \to \mathbb{G}'$ between braided G-graded categorical groups induces homomorphisms of G-modules*

$$\pi_i F : \pi_i(\mathbb{G}) \to \pi_i(\mathbb{G}'), \quad i = 0, 1,$$

given by $\pi_0 F : [X] \mapsto [FX]$, $\pi_1 F : a \mapsto \Phi_*^{-1} F(a)\Phi_*$.

(ii) *Two homotopic braided graded monoidal functors induce the same homomorphisms of G-modules.*

(iii) *A braided graded monoidal functor is a braided graded monoidal equivalence if and only if the induced homomorphisms of G-modules, $\pi_0 F$ and $\pi_1 F$, are isomorphisms.*

We shall now establish the following terminology.

Definition 21. A *braided G-graded categorical group of type* (M, N), where M and N are G-modules, is a triple $(\mathbb{G}, \alpha, \beta)$ in which $\mathbb{G}$ is a braided G-graded categorical group, and $\alpha : \pi_0\mathbb{G} \cong M$ and $\beta : \pi_1\mathbb{G} \cong N$ are G-module isomorphisms.

If $(\mathbb{G}', \alpha', \beta')$ is a braided G-graded categorical group of type (M', N'), then a braided graded monoidal functor $F : \mathbb{G} \to \mathbb{G}'$ is said to be *of type* (p, q), where $p : M \to M'$ and $q : N \to N'$ are G-module homomorphisms, whenever the two diagrams below commute:

$$\begin{tikzcd} \pi_0\mathbb{G} \arrow[r, "\alpha"] \arrow[d, "\pi_0 F"'] & M \arrow[d, "p"] \\ \pi_0\mathbb{G}' \arrow[r, "\alpha'"] & M', \end{tikzcd} \qquad \begin{tikzcd} \pi_1\mathbb{G} \arrow[r, "\beta"] \arrow[d, "\pi_1 F"'] & N \arrow[d, "p"] \\ \pi_1\mathbb{G}' \arrow[r, "\beta'"] & N'. \end{tikzcd}$$

Two braided G-graded categorical groups of the same type (M, N), say $(\mathbb{G}, \alpha, \beta)$ and $(\mathbb{G}', \alpha', \beta')$, are *equivalent* if there exists a braided graded monoidal functor (necessarily an equivalence by Proposition 20) $F : \mathbb{G} \to \mathbb{G}'$ of type (id_M, id_N), that is, such that $\alpha'\pi_0 F = \alpha$ and $\beta'\pi_1 F = \beta$.

The set of equivalence classes of braided G-graded categorical groups of type (M, N) is denoted by

$$\mathcal{BCG}_G[M, N].$$

Next, we shall prove that 3-cocycles in $C^\bullet_{G,ab}(M, N)$ are the appropriate data to construct the manifold of all braided G-graded categorical groups of type (M, N) up to equivalence.

Every abelian 3-cocycle $h \in Z^3_{G,ab}(M, N)$,

$$h : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N,$$

gives rise to a braided G-graded categorical group of type (M, N),

$$\mathbb{G}(h) := (\mathbb{G}(h), \alpha_h, \beta_h), \tag{6.11}$$

which is defined as follows: the objects of $\mathbb{G}(h)$ are the elements of M; an arrow $x \to y$ in $\mathbb{G}(h)$ is a pair $(a, \sigma) : x \to y$ where $a \in N$ and $\sigma \in G$ verifying ${}^\sigma x = y$.

The composition of two morphisms $x \xrightarrow{(a,\sigma)} y \xrightarrow{(b,\tau)} z$ is defined by

$$(b, \tau)(a, \sigma) = (b + {}^\tau a + h(x \mid \tau, \sigma), \tau\sigma). \tag{6.12}$$

This composition is unitary thanks to the normalization condition of h and it is associative owing to the 3-cocycle condition $\partial h = 0$ in (3.14). Note that every morphism is invertible, indeed

$$(a, \sigma)^{-1} = (-{}^{\sigma^{-1}}a - h(x \mid \sigma^{-1}, \sigma), \sigma^{-1}).$$

Hence, $\mathbb{G}(h)$ is a groupoid.

The stable G-grading is given by $gr(a, \sigma) = \sigma$.

The graded tensor product $\otimes : \mathbb{G}(h) \times_G \mathbb{G}(h) \to \mathbb{G}(h)$ is defined by

$$\big(x \xrightarrow{(a,\sigma)} y\big) \otimes \big(x' \xrightarrow{(b,\sigma)} y'\big) = \big(x + x' \xrightarrow{(a+b+h(x,x'|\sigma),\sigma)} y + y'\big), \tag{6.13}$$

which is a functor thanks to the 3-cocycle condition $\partial h = 0$ in (3.13) and the normalization condition of h.

The associativity isomorphisms are

$$A_{x,y,z} = (h(x, y, z), 1) : (x + y) + z \longrightarrow x + (y + z), \tag{6.14}$$

which satisfy the coherence condition (6.1) because of the 3-cocycle condition $\partial h = 0$ in (3.8). The naturalness here follows since $\partial h = 0$ in (3.11).

The unit graded functor $\mathrm{I} : G \to \mathbb{G}(h)$ is defined by

$$\mathrm{I}(\sigma) = (0 \xrightarrow{(0,\sigma)} 0), \tag{6.15}$$

and the unit constraints are identities: $L_x = (0, 1) = R_x : x \to x$.

The braiding for $\mathbb{G}(h)$ is given by

$$C_{x,y} = (h(x|y), 1) : x + y \longrightarrow y + x. \tag{6.16}$$

The cocycle conditions $\partial h = 0$ in (3.9) and (3.10) amount precisely to the coherence conditions (6.3) and (6.4) respectively. The naturalness of the braiding follows since $\partial h = 0$ in (3.12).

Thus, $\mathbb{G}(h)$ is a braided G-graded monoidal groupoid, which is actually a braided G-graded categorical group since, for any object x of $\mathbb{G}(h)$, we have $x \otimes (-x) = x + (-x) = 0 = \mathrm{I}$.

Finally, we recognize that $\mathbb{G}(h)$ is of type (M, N) by means of the obvious G-module isomorphisms

$$\alpha_h : \pi_0\mathbb{G}(h) \cong M, \quad \beta_h : \pi_1\mathbb{G}(h) \cong N, \tag{6.17}$$

which are defined by

$$\alpha_h(x) = x, \quad \beta_h(a, 1) = a.$$

With the device $h \mapsto \mathbb{G}(h)$, $h \in Z^3_{G,ab}(M, N)$, we are ready to prove the main results in this section. The next theorem deals with the classification of braided graded monoidal functors between braided G-graded categorical groups of the form $\mathbb{G}(h)$, and the following one shows that every braided G-graded categorical group is equivalent to $\mathbb{G}(h)$ for some h.

Theorem 22. *Let $h \in Z^3_{G,ab}(M, N)$, $h' \in Z^3_{G,ab}(M', N')$ be abelian 3-cocycles, where M, N, M', N' are G-modules, and suppose that $p : M \to M'$ and $q : N \to N'$ are any given G-module homomorphisms.*

Then, there exists a braided graded monoidal functor $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (p, q) if and only if the abelian 3-cocycles $p^*h', q_*h \in Z^3_{G,ab}(M, N')$ represent the same cohomology class, that is, if and only if

$$0 = p^*[h'] - q_*[h] \in H^3_{G,ab}(M, N').$$

Furthermore, when $p^*[h'] = q_*[h]$, homotopy classes of braided monoidal functors $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (p, q) are in bijection with elements of the group $H^2_{G,ab}(M, N')$.

Proof. We first assume that $p^*[h'] = q_*[h] \in H^3_{G,ab}(M, N')$. Then, we have $q_*h = p^*h' + \partial g$, for some $g \in C^2_{G,ab}(M, N')$ which determines a braided graded monoidal functor of type (p, q)

$$F = F(g) : \mathbb{G}(h) \to \mathbb{G}(h'), \tag{6.18}$$

given by

$$F\big(x \xrightarrow{(a,\sigma)} y\big) = \big(p(x) \xrightarrow{(q(a)+g(x|\sigma),\sigma)} p(y)\big), \tag{6.19}$$

together with the isomorphisms Φ of grade 1

$$\Phi_{x,y} = (g(x, y), 1) : p(x) + p(y) \to p(x + y) \tag{6.20}$$

and

$$\Phi_* = id = (0, 1) : p(0) \to 0.$$

So defined, it is routine to check that F is actually a functor because of the equality (3.7) and the normalization condition of g. The isomorphisms $\Phi_{x,y}$ define a graded natural equivalence $F(-) \otimes F(-) \cong F(- \otimes -)$ owing to the coboundary condition (3.6). The coherence conditions (6.5) and (6.7) hold thanks to (3.4) and (3.5) respectively, whilst (6.6) is trivially verified. Since $\alpha_{h'}\, \pi_0 F(x) = p(x) = p\, \alpha_h(x)$ and $\beta_{h'}\, \pi_1 F(a, 1) = q(a) = q\, \beta_h(a, 1)$, we see that F is actually of type (p, q).

Conversely, suppose that $F = (F, \Phi, \Phi_*) : \mathbb{G}(h) \to \mathbb{G}(h')$ is any braided graded monoidal functor of type (p, q). By Lemma 18 there is no loss of generality in assuming that F satisfies that $\Phi_* = id_0 = (0, 1)$. Then, F acts as p on objects, since $\alpha_{h'}\, \pi_0 F = p\, \alpha_h$, and

$$F\big(\, 0 \xrightarrow{(a,1)} 0 \,\big) = \; 0 \xrightarrow{(q(a),1)} 0,$$

for any $a \in N$, since $\beta_{h'}\pi_1 F = q\,\beta_h$. Furthermore, by coherence condition (6.6), one has $\Phi_{x,0} = id_x = \Phi_{0,x}$ for all $x \in M$ and then, since every morphism of grade 1, say $x \stackrel{(a,1)}{\longrightarrow} x$, can be expressed in the form $x \stackrel{(a,1)}{\longrightarrow} x = (0 \stackrel{(a,1)}{\longrightarrow} 0) \otimes (x \stackrel{(0,1)}{\longrightarrow} x)$, we deduce by naturalness that

$$\begin{aligned} F\left(x \stackrel{(a,1)}{\longrightarrow} x\right) &= F\left(0 \stackrel{(a,1)}{\longrightarrow} 0\right) \otimes F\left(x \stackrel{(0,1)}{\longrightarrow} x\right) = \left(0 \stackrel{(q(a),1)}{\longrightarrow} 0\right) \otimes \left(p(x) \stackrel{(0,1)}{\longrightarrow} p(x)\right) \\ &= p(x) \stackrel{(q(a),1)}{\longrightarrow} p(x). \end{aligned}$$

If we write for each $\sigma \in G$ and $x \in M$

$$F\big(x \stackrel{(0,\sigma)}{\longrightarrow} {}^{\sigma}x\big) = \big(p(x) \xrightarrow{(g(x|\sigma),\sigma)} {}^{\sigma}p(x)\big), \qquad g(x|\sigma) \in N',$$

and

$$\Phi_{x,y} = \big(x + y \xrightarrow{(g(x,y),1)} x + y\big), \qquad g(x, y) \in N', \tag{6.21}$$

for each $x, y \in M$, we get an abelian 2-cochain $g : M^2 \cup M|G \to N' \in C^2_{G,ab}(M, N')$, which determines F completely. Indeed, for any morphism in $\mathbb{G}(h)$ say $x \stackrel{(a,\sigma)}{\longrightarrow} y$, we have

$$\begin{aligned} F\big(x \stackrel{(a,\sigma)}{\longrightarrow} y\big) &= F\big(y \stackrel{(a,1)}{\longrightarrow} y\big)\, F\big(x \stackrel{(0,\sigma)}{\longrightarrow} y\big) \\ &= \big(p(y) \stackrel{(a,1)}{\longrightarrow} p(y)\big)\,\big(\,p(x) \xrightarrow{(g(x|\sigma),\sigma)} p(y)\,\big) \\ &= p(x) \xrightarrow{(a+g(x|\sigma),\sigma)} p(y). \end{aligned} \tag{6.22}$$

It is now straightforward to see that the equality $q_*h = p^*h' + \partial g$ amounts to the conditions of F being a braided graded monoidal functor. More precisely, the equality $q_*h(x, y, z) = p^*h'(x, y, z) + \partial g(x, y, z)$ follows from the coherence condition (6.5); $q_*h(x|y) = p^*h'(x|y) + \partial g(x|y)$ is a consequence of (6.7); $q_*h(x, y|\sigma) = p^*h'(x, y|\sigma) + \partial g(x, y|\sigma)$ owing to the naturalness of the isomorphisms $\Phi_{x,y}$ and the equality $q_*h(x|\sigma, \tau) = p^*h'(x|\sigma, \tau) + \partial g(x|\sigma, \tau)$ is a direct consequence of F being a functor. Therefore, q_*h and p^*h' are cohomologous abelian 3-cocycles of the G-module M with coefficients in N', as claimed.

To prove that homotopy classes of braided graded monoidal functors $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (p, q) are in bijection with elements of the group $H^2_{G,ab}(M, N')$, we shall stress that we have actually proved before that the mapping $g \mapsto F(g)$, given by construction (6.18), induces a surjection from the set of those abelian 2-cochains $g \in C^2_{G,ab}(M, N')$ such that $q_*h = p^*h' + \partial g$ onto the set of homotopy classes of braided graded monoidal functors $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (p, q) (compare formulas (6.21) and (6.22) with (6.19) and (6.20) respectively).

We now note that if we fix any $g_0 \in C^2_{G,ab}(M, N')$ satisfying $q_*h = p^*h' + \partial g_0$, which exists under the hypothesis $p^*[h'] = q_*[h]$, then any other such abelian 2-cochain $g_1 \in C^2_{G,ab}(M, N')$ with $q_*h = p^*h' + \partial g_1$, is necessarily written in the form $g_1 = g_0 + g$ with $\partial g = 0$, that is, where $g \in Z^2_{G,ab}(M, N')$.

Then, to complete the proof of the theorem, it suffices to prove that two braided graded monoidal functors $F(g_0+g)$ and $F(g_0 + g')$, where $g, g' \in Z^2_{G,ab}(M, N')$, are homotopic if and only if the g and g' are cohomologous:

Let $g' - g = \partial f$ for some $f \in C^1_{G,ab}(M, N')$. Then the following family of isomorphisms of grade 1 in $\mathbb{G}(h')$,

$$\theta_x : p(x) \stackrel{(f(x),1)}{\longrightarrow} p(x), \quad x \in M,$$

defines a graded natural equivalence $\theta : F(g_0 + g') \to F(g_0 + g)$ thanks to the condition (3.3), which also verifies the condition (6.8) due to the equality (3.2). That is, θ is a homotopy of braided graded monoidal functors.

And conversely, if $\theta : F(g_0 + g') \to F(g_0 + g)$ is any homotopy of braided graded monoidal functors and we write $\theta_x = (f(x), 1) : p(x) \to p(x)$ for a map $f : M \to N$, then one can easily check that $f \in C^1_{G,ab}(M, N')$ and $g' - g = \partial f$. $\square$

Corollary 23. *Let $h, h' \in Z^3_{G,ab}(M, N)$ be two abelian 3-cocycles of a G-module M with coefficients in a G-module N. Then, $\mathbb{G}(h)$ and $\mathbb{G}(h')$ are equivalent, as braided G-graded categorical groups of type (M, N), if and only if h and h' are cohomologous.*

Proof. By definition, $\mathbb{G}(h)$ and $\mathbb{G}(h')$ are equivalent whenever there exists a braided graded monoidal functor $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (id_M, id_N), which, by Theorem 22, occurs if and only if $0 = [h'] - [h] \in H^3_{G,ab}(M, N)$. □

Theorem 24. *For any G-modules M and N, the map $h \mapsto \mathbb{G}(h)$, $h \in Z^3_{G,ab}(M, N)$, induces a bijection between elements of the cohomology group $H^3_{G,ab}(M, N)$ and equivalence classes of braided G-graded categorical groups of type (M, N). Hence there is a bijection*

$$\mathcal{BCG}_G[M, N] \cong H^3_{G,ab}(M, N), \quad [\mathbb{G}, \alpha, \beta] \mapsto \alpha^* \beta_* k(\mathbb{G}),$$

where we refer to $k(\mathbb{G}) \in H^3_{G,ab}(\pi_0\mathbb{G}, \pi_1\mathbb{G})$ as the Postnikov invariant of the braided G-graded categorical group $\mathbb{G}$.

Proof. By the previous Corollary 23, the correspondence $[h] \mapsto [\mathbb{G}(h)]$ is a correctly defined injective map $H^3_{G,ab}(M, N) \to \mathcal{BCG}_G[M, N]$. Therefore it only remains to prove that this map is onto, that is, that every braided G-graded categorical group of type (M, N), say $\mathbb{G} = (\mathbb{G}, \alpha, \beta)$, is equivalent to one $\mathbb{G}(h)$ for some $h \in Z^3_{G,ab}(M, N)$. To simplify the notation, we can assume, without loss of generality, that the G-module isomorphisms α and β are identities, that is, $\pi_0\mathbb{G} = M$ and $\pi_1\mathbb{G} = N$.

Recall now that a (co)fibred category is skeletal when all its fibre categories are skeletal. In particular, a graded category is skeletal when any two objects isomorphic by an isomorphism of grade 1 are equal. *The braided G-graded categorical group $\mathbb{G}$ is equivalent to a skeletal one*, say $\widehat{\mathbb{G}}$, which can be constructed as follows: for each $x \in M$, let us choose an object $O_x \in x$, with $O_0 = \mathrm{I}$, and for any other $O \in x$, we fix a 1-morphism $\Phi_O : O \to O_x$, with $\Phi_{O_x} = id_{O_x}$, $\Phi_{\mathrm{I}\otimes O_x} = L_{O_x}$ and $\Phi_{O_x \otimes I} = R_{O_x}$. Let $\widehat{\mathbb{G}}$ be the full subcategory of $\mathbb{G}$ whose objects are all O_x, $x \in M$. Then $\widehat{\mathbb{G}}$ is stably G-graded with grading $\widehat{gr} = gr|_{\widehat{\mathbb{G}}} : \widehat{\mathbb{G}} \to G$; the inclusion functor $\widehat{\mathbb{G}} \hookrightarrow \mathbb{G}$ is a graded equivalence and clearly $\widehat{\mathbb{G}}$ is skeletal. Now, the braided graded categorical group structure of $\mathbb{G}$ can be transported to $\widehat{\mathbb{G}}$, in a unique way such that the inclusion functor $\widehat{\mathbb{G}} \hookrightarrow \mathbb{G}$, together with the isomorphisms $\Phi_{O_x, O_y} = \Phi_{O_x \otimes O_y} : O_x \otimes O_y \to O_x \widehat{\otimes} O_y = O_{x+y}$ and $\Phi_* = id_{\mathrm{I}}$, turns out to be a braided graded monoidal equivalence (see (6.5) and (6.6)). Note that in the resulting skeletal braided G-graded categorical group $\widehat{\mathbb{G}} = (\widehat{\mathbb{G}}, \widehat{gr}, \widehat{\otimes}, \widehat{\mathrm{I}}, \widehat{A}, \widehat{L}, \widehat{R}, \widehat{C})$, the unit $\widehat{\mathrm{I}}$ is strict in the sense that $\widehat{L} = id = \widehat{R}$.

Hence it is no loss of generality if we suppose that $\mathbb{G}$ *is a skeletal braided G-graded categorical group in which the unit constraints are identities*, and also $\pi_0\mathbb{G} = M$, $\pi_1\mathbb{G} = N$.

Then, the following facts hold: $\mathrm{Ob}\mathbb{G} = M$ and $x \otimes y = x + y$ for all $x, y \in M$. The (strict) unit object is $\mathrm{I} = 0$, $\mathrm{Aut}_1(0) = N$ and $a \otimes b = a + b = a\,b$ for all $a, b \in N$. For any $x \in M$, $N \cong \mathrm{Aut}_1(x)$ by the isomorphism $a \mapsto a \otimes id_x$; furthermore,

$$a \otimes id_x = id_x \otimes a \tag{6.23}$$

for any $a \in N$ (by [18, Proposition 2.1] and the naturalness of the braiding). If $x, y \in M$ and $\sigma \in G$, then there exists a σ-morphism in $\mathbb{G}$, $u : x \to y$ if and only if ${}^\sigma x = y$ (according to (6.9)). For any such morphism $u : x \to y$, of grade σ, we have the composite bijection

$$N \cong \mathrm{Aut}_1(y) \overset{u^*}{\cong} \mathrm{Hom}_\sigma(x, y), \quad a \mapsto (a \otimes id_y)u, \tag{6.24}$$

between elements of N and arrows of grade σ in $\mathbb{G}$ from x to $y = {}^\sigma x$; further,

$$\mathrm{I}(\sigma) \otimes u = u = u \otimes \mathrm{I}(\sigma) \tag{6.25}$$

(due to the naturalness of the unit constraints) and, for all $a \in N$,

$$u\,(a \otimes id_x) = ({}^\sigma a \otimes id_y)\,u. \tag{6.26}$$

In effect,

$$\begin{aligned}({}^\sigma a \otimes id_y)\,u &\overset{(6.25)}{=} ({}^\sigma a \otimes id_y)\,(\mathrm{I}(\sigma) \otimes u) = ({}^\sigma a\,\mathrm{I}(\sigma) \otimes u)\\ &\overset{(6.10)}{=} (\mathrm{I}(\sigma)\,a \otimes u) = (\mathrm{I}(\sigma) \otimes u)\,(a \otimes id_x) \overset{(6.25)}{=} u\,(a \otimes id_x).\end{aligned}$$

We now choose, for each $\sigma \in G$ and $x \in M$, a morphism in $\mathbb{G}$ with domain x and grade σ, say

$$u_{x,\sigma} : x \to y = {}^{\sigma}x, \quad \text{with } u_{0,\sigma} = I(\sigma), u_{x,1} = id_x$$

and, for each $a \in N$, we shall write

$$x \xrightarrow{(a,\sigma)} y \;:=\; x \xrightarrow{(a\otimes id_y)\,u_{x,\sigma}} y\,. \tag{6.27}$$

That is, we are denoting by $(a, \sigma) : x \to y$ the σ-morphism from x to y that corresponds to a by the bijection (6.24) for $u = u_{x,\sigma}$. Thus, for example, we have the equalities

$$x \xrightarrow{(0,\sigma)} y \;=\; x \xrightarrow{u_{x,\sigma}} y\,, \quad x \xrightarrow{(a,1)} x \;=\; x \xrightarrow{a\otimes id_x} x\,,$$

$$x \xrightarrow{(0,1)} x \;=\; x \xrightarrow{id_x} x\,, \quad 0 \xrightarrow{(0,\sigma)} 0 \;=\; 0 \xrightarrow{\mathrm{I}(\sigma)} 0\,, \tag{6.28}$$

$$x \xrightarrow{(a,\sigma)} y \quad \overset{(6.27)}{=} \quad \big(\, y \xrightarrow{(a,1)} y \,\big)\big(\, x \xrightarrow{(0,\sigma)} y \,\big), \tag{6.29}$$

$$x \xrightarrow{({}^{\sigma}a,\sigma)} y \quad \overset{(6.26)}{=} \quad \big(\, x \xrightarrow{(0,\sigma)} y \,\big)\big(\, x \xrightarrow{(a,1)} y \,\big), \tag{6.30}$$

$$\big(\, x \xrightarrow{(a,1)} x \,\big)\big(\, x \xrightarrow{(b,1)} x \,\big) \;=\; \big(\, x \xrightarrow{(a+b,1)} x \,\big), \tag{6.31}$$

since $(a \otimes id_x)(b \otimes id_x) = (a + b) \otimes id_x$,

$$\big(\, x \xrightarrow{(a,1)} x \,\big) \otimes \big(\, y \xrightarrow{(b,1)} y \,\big) \;=\; \big(\, x + y \xrightarrow{(a+b,1)} x + y \,\big), \tag{6.32}$$

since $(a \otimes id_x) \otimes (b \otimes id_y) \overset{(6.23)}{=} (a + b) \otimes id_{x+y}$.

All in all, we are now ready to build a 3-cocycle $h = h^{\mathbb{G}} \in Z^3_{G,ab}(M, N)$ such that $\mathbb{G}(h) \simeq \mathbb{G}$. For we begin by determining a 3-cochain

$$h : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N,$$

by the four equations below.

$$\big(\, y \xrightarrow{(0,\sigma)} z \,\big)\big(\, x \xrightarrow{(0,\tau)} y \,\big) \;=\; \big(\, x \xrightarrow{(h(x|\sigma,\tau),\sigma\tau)} z \,\big), \tag{6.33}$$

for $\sigma, \tau \in G,\ x \in M,\ {}^{\tau}x = y,\ {}^{\sigma}y = z$;

$$\big(\, x \xrightarrow{(0,\sigma)} y \,\big) \otimes \big(\, x' \xrightarrow{(0,\sigma)} y' \,\big) \;=\; \big(\, x + x' \xrightarrow{(h(x,x'|\sigma),\sigma)} y + y' \,\big), \tag{6.34}$$

for $\sigma \in G,\ x, x' \in M,\ {}^{\sigma}x = y,\ {}^{\sigma}x' = y'$;

$$\big(\, x + y \xrightarrow{C_{x,y}} y + x \,\big) \;=\; \big(\, x + y \xrightarrow{(h(x|y),1)} y + x \,\big), \tag{6.35}$$

for $x, y \in M$;

$$\big(\, (x + y) + z \xrightarrow{A_{x,y,z}} x + (y + z) \,\big) \;=\; \big(\, (x + y) + z \xrightarrow{(h(x,y,z),1)} x + (y + z) \,\big), \tag{6.36}$$

for $x, y, z \in M$.

So defined, this 3-cochain h completely determines the braided graded strictly unitary monoidal category structure of $\mathbb{G}$, since the following two equalities hold:

$$\big(\, y \xrightarrow{(b,\tau)} z \,\big)\big(\, x \xrightarrow{(a,\sigma)} y \,\big) \;=\; \big(\, x \xrightarrow{(b+{}^{\tau}a+h(x|\tau,\sigma),\tau\sigma)} z \,\big), \tag{6.37}$$

$$\big(\, x \xrightarrow{(a,\sigma)} y \,\big) \otimes \big(\, x' \xrightarrow{(b,\sigma)} y' \,\big) \;=\; \big(\, x + x' \xrightarrow{(a+b+h(x,x'|\sigma),\sigma)} y + y' \,\big). \tag{6.38}$$

for all $\sigma, \tau \in G$, $x, x', z \in M$, ${}^{\sigma}x = y$, ${}^{\sigma}x' = y'$, ${}^{\tau}y = z$. In effect,

$$\begin{aligned}
(b,\tau)\,(a,\sigma) &\overset{(6.29)}{=} (b,1)\,(0,\tau)\,(a,1)\,(0,\sigma) \overset{(6.30)}{=} (b,1)\,({}^{\tau}a,\tau)\,(0,\sigma)\\
&\overset{(6.29)}{=} (b,1)\,({}^{\tau}a,1)\,(0,\tau)\,(0,\sigma) \overset{(6.31),(6.33)}{=} (b+{}^{\tau}a,1)\,(h(x|\tau,\sigma),\tau\sigma)\\
&\overset{(6.29)}{=} (b+{}^{\tau}a+h(x|\tau,\sigma),\tau\sigma),
\end{aligned}$$

$$\begin{aligned}
(a,\sigma)\otimes(b,\sigma) &\overset{(6.29)}{=} ((a,1)\,(0,\sigma))\otimes((b,1)\,(0,\sigma))\\
&= ((a,1)\otimes(b,1))\,((0,\sigma)\otimes(0,\sigma)) \overset{(6.32),(6.34)}{=} (a+b,1)\,(h(x,x'|\sigma),\sigma)\\
&\overset{(6.29)}{=} (a+b+h(x,x'|\sigma),\sigma).
\end{aligned}$$

It is now easy to conclude from equalities (6.28), (6.37) and (6.38) that h is actually an abelian 3-cocycle of the G-module M with coefficients in N. Since the composition and tensor in $\mathbb{G}$ are unitary and I is a functor, then the normalization of h follows from the equalities (6.25), (6.2) and [18, Propositions 1.1 and 1.2]. The cocycle condition $\partial h = 0$ in (3.14) follows from the associativity law for morphisms in $\mathbb{G}$. That $\partial h = 0$ in (3.13) is a consequence of the graded tensor product $\otimes : \mathbb{G}\times_G \mathbb{G} \to \mathbb{G}$ being functorial. The equality $\partial h = 0$ in (3.8) holds because of the coherence pentagons (6.1), and $\partial h = 0$ in (3.11) follows from the naturalness of the associativity constraints. The cocycle conditions $\partial h = 0$ in (3.9) and (3.10) are verified owing to the coherence conditions (6.3) and (6.4) respectively. And, finally, the naturalness of the braiding implies that $\partial h = 0$ in (3.12).

Hence $h^{\mathbb{G}} = h \in Z^3_{G,ab}(M,N)$ and, by comparison equalities (6.12) with (6.37), (6.13) with (6.38), (6.15) with (6.28), (6.14) with (6.36) and (6.16)with (6.35) respectively, it is obvious that $\mathbb{G}(h)$ and $\mathbb{G}$ are isomorphic braided G-graded categorical groups of type (M,N). This completes the proof of theorem. □

The classifying results stated in this section can be summarized as follows (cf. [18, Theorem 3.3]).

Let $\mathcal{H}^3_{G,ab}$ be the category whose objects (M,N,k) consist of G-modules M, N and cohomology classes $k \in H^3_{G,ab}(M,N)$. An arrow $(M,N,k) \overset{(p,q)}{\to} (M',N',k')$ is a pair of G-module homomorphisms $p : M \to M'$, $q : N \to N'$ such that $q_*k = p^*k'$,

$$\begin{array}{ccccc}
H^3_{G,ab}(M,N) & \xrightarrow{q_*} & H^3_{G,ab}(M,N') & \xleftarrow{p^*} & H^3_{G,ab}(M',N'),\\
k & \longmapsto & q_*k = p^*k' & \longleftarrow\!\shortmid & k'.
\end{array}$$

Then, we have the *classifying* functor

$$\begin{aligned}
&cl : \mathcal{BCG}_G \longrightarrow \mathcal{H}^3_{G,ab},\\
&\mathbb{G} \mapsto \big(\pi_0\mathbb{G},\ \pi_1\mathbb{G},\ k(\mathbb{G})\big),\\
&F \mapsto (\pi_0 F,\ \pi_1 F),
\end{aligned}$$

which has the following properties:

(i) For any object $(M,N,k) \in \mathcal{H}^3_{G,ab}$, there exists a braided G-graded categorical group $\mathbb{G}$ with an isomorphism $cl(\mathbb{G}) \cong (M,N,k)$.

(ii) For any morphism $(p,q) : cl(\mathbb{G}) \to cl(\mathbb{H})$, there is a braided graded monoidal functor $F : \mathbb{G} \to \mathbb{H}$ such that $cl(F) = (p,q)$.

(iii) $cl(F)$ is an isomorphism if and only if F is a braided graded monoidal equivalence.

(iv) For any arrow $(p,q) : cl(\mathbb{G}) \to cl(\mathbb{H})$, homotopy classes of braided graded monoidal functors $F : \mathbb{G} \to \mathbb{H}$ such that $cl(F) = (p,q)$ are in bijection with elements of the group $H^2_{G,ab}(\pi_0\mathbb{G}, \pi_1\mathbb{H})$.

As a final comment, we stress that, as a bonus from Proposition 13 and Theorem 24, for any two G-modules M, N we have a bijection

$$\mathbf{S}^G_*[M,N] \cong \mathcal{BCG}_G[M,N],$$

between the set of equivariant weak homotopy classes of pointed G-spaces X with $\pi_i X = 0$ for all $i \neq 2, 3$, $\pi_2 X \cong M$ and $\pi_3 X \cong N$ and the set of homotopy classes of braided G-graded categorical groups $\mathbb{G}$ with $\pi_0\mathbb{G} \cong M$ and

$\pi_1\mathbb{G} \cong N$. This bijection can be illustrated by the construction below of a braided G-graded categorical group $\mathbb{G}(X)$ associated to a pointed G-space X, which represents its equivariant 3-type when X is 1-connected.

Let $X = (X, *)$ be a (topological) space on which the (discrete) group G acts by pointed homeomorphisms. Then, the objects of $\mathbb{G}(X)$ are the double loops in X based on $*$; that is, all the maps from the square $I \times I$ into X which are constant along the edges, say $\omega : (I^2, \partial I^2) \to (X, *)$. A morphism $\omega \to \omega'$ of grade $\sigma \in G$ is a pair $([h], \sigma)$, where $[h]$ is the homotopy class (relative to ∂I) of a path between 2-loops $h : {}^{\sigma}\omega \to \omega'$. That is, a σ-morphism in $\mathbb{G}$, $([h], \sigma) : \omega \to \omega'$, is represented by a relative map $h : (I^3, \partial I^2 \times I) \to (X, *)$ with $h(s, t, 0) = {}^{\sigma}\omega(s, t)$ and $h(s, t, 1) = \omega'(s, t)$; two such h, h' are equivalent whenever there exists a map $H : (I^4, \partial I^2 \times I^2) \to (X, *)$ such that $H(s, t, u, 0) = h(s, t, u)$, $H(s, t, u, 1) = h'(s, t, u)$, $H(s, t, 0, v) = {}^{\sigma}\omega(s, t)$ and $H(s, t, 1, v) = \omega'(s, t)$.

The composition is induced by the usual vertical composition of homotopies, according to the formula $([h'], \tau)([h], \sigma) = ([h' \circ {}^{\tau}h], \tau\sigma)$, where

$$(h' \circ {}^{\tau}h)(s, t, u) = \begin{cases} {}^{\tau}h(s, t, 2u) & 2u \leq 1 \\ h'(s, t, 2u - 1) & 2u \geq 1. \end{cases}$$

The graded monoidal structure is induced by the H-group structure of the loop space $\Omega^2(X, *)$; thus, the graded tensor product is given on objects by concatenation of 2-loops

$$\begin{array}{ccc} * & = & * \\ \| & \omega(s,t) & \| \\ * & = & * \end{array} \otimes \begin{array}{ccc} * & = & * \\ \| & \omega'(s,t) & \| \\ * & = & * \end{array} = \begin{array}{ccccc} * & = & * & = & * \\ \| & \omega(2s,t) & \| & \omega'(2s-1,t) & \| \\ * & = & * & = & * \end{array}$$

and on morphisms with the same grade by the horizontal composition of homotopies. The 1-graded associativity and unit constraints are defined to be the equivalence classes of the respective standard homotopies proving the associativity and unit of the loop composition, and the 1-graded braiding isomorphisms are the equivalence classes of the ordinary homotopies showing the commutativity of the second homotopy groups of spaces, namely

$$\begin{array}{ccccc} * & = & * & = & * \\ \| & & \| & & \| \\ \| & \omega & \| & \omega' & \| \\ \| & & \| & & \| \\ * & = & * & = & * \end{array} \simeq \begin{array}{ccccc} * & = & * & = & * \\ \| & \omega & \| & * & \| \\ * & = & * & = & * \\ \| & * & \| & \omega' & \| \\ * & = & * & = & * \end{array} \simeq \begin{array}{ccccc} * & = & * & = & * \\ \| & * & \| & \omega & \| \\ * & = & * & = & * \\ \| & \omega' & \| & * & \| \\ * & = & * & = & * \end{array} \simeq \begin{array}{ccccc} * & = & * & = & * \\ \| & & \| & & \| \\ \| & \omega' & \| & \omega & \| \\ \| & & \| & & \| \\ * & = & * & = & * \end{array}.$$

Acknowledgements

A.M. Cegarra was partially supported by DGI of Spain (Project: MTM2004-01060) and FEDER.

E. Khmaladze is very grateful to the University of Granada for hospitality. He was supported by INTAS YSF grant No 03-55-684 and INTAS grant No 03-51-3251.

References

[1] G.E. Bredon, Equivariant Cohomology Theories, in: Lecture Notes in Math., vol. 34, Springer, Berlin, 1967.
[2] L. Breen, Braided n-structures and Σ-structures, Contemp. Math. 230 (1988) 59–81.
[3] A.M. Cegarra, J.M. García-Calcines, J.A. Ortega, Cohomology of groups with operators, Homology Homotopy Appl. 4 (1) (2002) 1–23.
[4] A.M. Cegarra, J.M. García-Calcines, J.A. Ortega, On graded categorical groups and equivariant group extensions, Canadian J. Math. 54 (5) (2002) 970–997.
[5] A.M. Cegarra, A.R. Garzón, Equivariant group cohomology and Brauer group, Bull. Belg. Math. Soc. Sim. 10 (3) (2003) 451–459.
[6] A.M. Cegarra, A.M. Garzón, Some algebraic applications of graded categorical group theory, Theory Appl. Categ. 11 (10) (2003) 215–251.
[7] A. Dold, D. Puppe, Homology nicht-additiver Functoren, Anwendungen, Ann. Inst. Fourier 11 (1961) 201–312.
[8] E. Dror, W.G. Dwayer, D.M. Kan, Equivariant maps which are self homotopy equivalences, Proc. Amer. Math. Soc. 80 (4) (1980) 670–672.
[9] B. Eckmann, P.J. Hilton, Group-like structures in general categories, I. Multiplications and comultiplications, Math. Ann. 145 (1962) 227–255.
[10] S. Eilenberg, S. MacLane, Cohomology theory of abelian groups and homotopy theory, I, II, III, Proc. Natl. Acad. Sci. USA 36 (1950) 443–447, 657–663; 37 (1951) 307–310.
[11] S. Eilenberg, S. MacLane, On the groups $H(\pi, n)$, I, II, Ann. of Math. 58 (1953) 55–106; 70 (1954) 49–137.
[12] A. Fröhlich, C.T.C. Wall, Graded monoidal categories, Compos. Math. 28 (1974) 229–285.

[13] A. Fröhlich, C.T.C. Wall, Equivariant Brauer groups, Contemp. Math. 272 (2000) 57–71.
[14] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, PM 174, Birkhäuser Verlag, 1999.
[15] A. Grothendieck, Catégories fibrées et déscente, (SGA I, exposé VI), in: Lecture Notes in Math., vol. 224, Springer, Berlin, 1971, pp. 145–194.
[16] M. Hovey, Model Categories, in: Math. Surveys and Monographs, vol. 63, Amer. Math. Soc., 1999.
[17] A. Joyal, Letter to Grothendieck, 1984.
[18] A. Joyal, R. Street, Braided tensor categories, Adv. Math. (1) 82 (1991) 20–78.
[19] S. MacLane, Cohomology theory of abelian groups, in: Proc. International Congress of Mathematicians, vol. II, 1950, pp. 8–14.
[20] I. Moerdijk, J.A. Svensson, The equivariant Serre spectral sequence, Proc. Amer. Math. Soc. 118 (1) (1993) 263–277.
[21] D. Quillen, Homotopical Algebra, in: Lecture Notes in Math., vol. 43, Springer, Berlin, 1967.
[22] J.H.C. Whitehead, A certain exact sequence, Ann. of Math. 52 (1) (1950) 51–110.
[23] J.H.C. Whitehead, On group extensions with operators, Q. J. Math. Oxford 2 (1950) 219–228.