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Homotopy classification of graded Picard categories ☆

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Abstract

Certain low-dimensional symmetric cohomology groups of G-modules, for any given group G, are computed as the cohomology of an explicit cochain complex. This result is used to establish natural one-to-one correspondences between elements of the 3rd symmetric cohomology groups of G-modules, G-equivariant pointed 2-connected homotopy 4-types, and equivalence classes of G-graded Picard categories. The simplicial nerve of a G-graded Picard category is also constructed and studied.

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1. Introduction and summary

Graded symmetric tensor (or monoidal) categories, and graded Picard categories in particular, provide a suitable setting for the treatment of an extensive list of subjects of equivariant nature with recognized mathematical interest. Let us briefly recall that, if G is a group, then a *G-graded*

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Picard category is a (small) groupoid $\mathbb{G}$ equipped with a degree functor $\mathrm{gr}:\mathbb{G}\to G$ and with a graded symmetric (or commutative) structure, by graded functors $\otimes:\mathbb{G}\times_G\mathbb{G}\to\mathbb{G}$, $\mathrm{I}:G\to\mathbb{G}$ and coherent graded natural equivalences $A:(X\otimes Y)\otimes Z\cong X\otimes(Y\otimes Z)$, $C:X\otimes Y\cong Y\otimes X$ and $R:X\otimes I\cong X$, such that every object has a dual (or quasi-inverse). Under the name of graded group-like categories, these graded structures were originally considered by Fröhlich and Wall in [17] (see also [18]), where the problem of its homotopy classification was also suggested, whose solution this paper is mainly dedicated to.

The appropriate framework for our discussion is suggested by the known classification theorem for (non-graded) Picard categories, which appeared originally in the unpublished thesis [29] of H. Sinh, in terms of the associated abelian groups $M=K_0\mathbb{G}$, the group of iso-classes of objects of $\mathbb{G}$, $N=K_1\mathbb{G}$, the group of automorphisms of the unit object of $\mathbb{G}$, and an element $q\in\mathrm{Hom}_{\mathbb{Z}}(M/2M,N)$, canonically deduced from the coherence diagrams in $\mathbb{G}$. Previously, Deligne [12] (see also [30]) had proved that the classification of restricted Picard categories (i.e., when $C_{X,X}=id_{X\otimes X}$) is trivial, in the sense that the pair of abelian groups (M,N) is a complete invariant for the equivalence type of such a restricted Picard category. As a bonus, Sinh's results pointed out the utility of Picard categories in algebraic topology: they are algebraic models for homotopy 4-types of pointed 2-connected spaces. This follows from the well-known Eilenberg–Mac Lane equality $H^5(K(M,3),N)=\mathrm{Hom}_{\mathbb{Z}}(M/2M,N)$ [16, II, Theorem 23.1], since one deduces that, for each triple (M,N,q), where $q\in\mathrm{Hom}_{\mathbb{Z}}(M/2M,N)$, there is a pointed CW-complex, say X, unique up to homotopy, such that $\pi_3X=M$, $\pi_4X=N$, $\pi_iX=0$ for all $i\neq 3,4$ and q is the unique non-trivial Postnikov invariant of X. Notice that, for any two abelian groups M and N, the groups $H^n_s(M,N)=H^{n+2}(K(M,3),N)$ define the (second level or *symmetric*) Eilenberg–Mac Lane cohomology theory of the abelian group M with coefficients in the abelian group N [15]. Therefore, Picard categories are just classified up to equivalence by an element of the 3rd symmetric cohomology groups $H^3_s(M,N)$. An extension of Sinh's results was proved by Joyal and Street in [23], where they stated a classification theorem for *braided categorical groups* (which are defined similarly as Picard categories, but where the usual symmetry condition $C^2=id$ is not assumed), in terms of elements of the 3rd *abelian* cohomology groups $H^3_{ab}(M,N)=H^4(K(M,2),N)$; in topological terms, this expresses that braided categorical groups are algebraic homotopy 3-types of 1-connected pointed spaces (cf. [8]). The classification of *categorical groups* (or non-commutative Picard categories, also called Gr-categories) goes back to Mac Lane and Whitehead in [26], since strict categorical groups are the same as crossed modules (cf. [4]), which are simply classified by elements of ordinary 3rd cohomology groups $H^3(M,N)=H^3(K(M,1),N))$ (with non-necessarily abelian M here) and, therefore, they represent homotopy 2-types of pointed spaces. These results on the classification of categorical groups have recently been extended to the graded case in [11].

As we mentioned above, the main objective of this paper is to state and prove precise classification theorems for graded Picard categories and their homomorphisms, where, in this classification, two G-graded Picard categories connected by a symmetric graded tensor equivalence are considered the same. When $\mathbb{G}$ is a G-graded Picard category, then both $K_0\mathbb{G}$ and $K_1\mathbb{G}$ inherit G-module structures. With this in mind, we were naturally led to search for an adequate cohomology theory of G-modules, such that the equivalence class of a G-graded Picard category $\mathbb{G}$ would be determined by an element of the 3rd cohomology group of the G-module $K_0\mathbb{G}$ with coefficients in the G-module $K_1\mathbb{G}$. The approach we provide for such a cohomology theory $H^n_{G,s}(M,N)$, called *symmetric cohomology of G-modules*, is not too surprising, since a G-graded Picard category can be seen as a category fibered above the one-object category with automorphism group G, and it is a fact that the homotopy of spaces on which a fixed group G

acts is equivalent to the homotopy theory of spaces over the classifying space BG of the group G (see, for example, in a simplicial context [13]). Hence, we introduce the symmetric cohomology groups $H^n_{G,s}(M,N)$, of a G-module M with coefficients in a G-module N, as the reduced G-equivariant cohomology of the Eilenberg–Mac Lane minimal complex $K(M,3)$, with its natural G-action, or, in other words, to be the cohomology groups [28, §2, Section 5] of $K(M,3)$ in the homotopy category of pointed G-spaces, $\mathrm{Ho}\,\mathbf{S}^G_*$, with respect to the closed model structure where weak equivalences are those G-equivariant pointed maps that are weak equivalences on the underlying spaces, that is,

$$H^n_{G,s}(M,N) = \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{S}^G_*}\big(K(M,3), K(N,n+2)\big), \quad n \geqslant 0.$$

Our classification results point out the potential interest of graded Picard categories in equivariant homotopy theory. Indeed, for any triple (M,N,k), where M and N are G-modules and $k \in H^3_{G,s}(M,N)$, there is a pointed G-space $(X,*)$, unique up to equivariant weak equivalence, such that $\pi_3 X = M$, $\pi_4 X = N$, $\pi_i X = 0$ for all $i \neq 3,4$ and k is the (unique non-trivial) equivariant Postnikov invariant of X. Thus as a result, *the homotopy category of G-equivariant pointed 2-connected 4-types is equivalent to the homotopy category of G-graded Picard categories*. We give an illustration of this equivalence by constructing a simplicial nerve of a graded Picard category.

The plan of this paper, briefly, is as follows. After this introductory Section 1, the paper is organized in three sections, each with two subsections. The first subsection of Section 2 is devoted to briefly discussing some necessary fundamental aspects, results, and notations of the homotopy theory of G-spaces. The second subsection is dedicated to the definition and study of symmetric cohomology groups $H^n_{G,s}(M,N)$ of G-modules M, N. In Section 3 we first review some definitions and facts concerning symmetric graded tensor categories and graded Picard categories and, in the second subsection, we include our main results on the homotopy classification of G-graded Picard categories and their homomorphisms by means of the symmetric cohomology groups $H^3_{G,s}(M,N)$ and $H^2_{G,s}(M,N)$. In Section 4, the final one, help prepare the reader for the definition of nerves of graded Picard categories, we have first included a subsection where we review some facts concerning nerves and higher categorical structures. Last, in the second subsection, the simplicial nerve of a graded Picard category is defined and some of its main properties are shown.

2. Symmetric cohomology of G-modules

We shall begin by reviewing some facts concerning the homotopy and cohomology of G-spaces. We refer to [19] for background.

2.1. Preliminaries on the homotopy and cohomology of pointed G-spaces

Throughout, G is a fixed (discrete) group, $\mathbf{S}$ denotes the category of simplicial sets, and $\mathbf{S}^G_*$ is the category of all pointed simplicial sets $X = (X,*)$ with a (left) G-action by pointed simplicial automorphisms, hereafter referred to as pointed G-spaces, and equivariant pointed simplicial maps (or pointed G-maps, in short) between them.

There is a closed Quillen model category structure on $\mathbf{S}^G_*$ such that a pointed G-map is a weak equivalence, respectively a fibration, if the underlying map in $\mathbf{S}$ is a weak equivalence, respectively a (Kan) fibration, in $\mathbf{S}$. Furthermore, a pointed G-map $f : X \to Y$ is a cofibration if and only if it is injective and $Y \setminus f(X)$ is a free G-set. Thus, in this homotopy theory, a pointed G-space $(X,*)$ is fibrant whenever X is a Kan simplicial set, while $(X,*)$ is cofibrant if no non-identity element of G fixes a simplex different from the base point. As usual, let

$$\mathrm{Ho}\,\mathbf{S}_*^G$$

denote the corresponding homotopy category, that is, the localization of $\mathbf{S}_*^G$ with respect to the pointed equivariant weak equivalences.

It is a fact that the above-described homotopy theory of pointed G-spaces is equivalent, via the so-called Borel construction, to the homotopy theory of retractive spaces over BG, the classifying minimal complex of the group G. To be more precise in the statement of this result, let $\mathbf{S}_{BG}^{BG}$ be the double comma category of retractive spaces over BG

$$R \underset{r}{\overset{s}{\rightleftarrows}} BG\ , \quad rs = id,$$

and whose morphisms are those simplicial maps $f : R \to R'$ such that

$$r'f = r \quad \text{and} \quad fs = s'.$$

This category $\mathbf{S}_{BG}^{BG}$ has a closed model structure induced by the usual one of simplicial sets; that is, a map f in $\mathbf{S}_{BG}^{BG}$ is a weak equivalence, cofibration or fibration if and only if f is a weak equivalence, cofibration or fibration of simplicial sets, respectively.

For any G-space X, the associated Borel construction $E_G X = EG \times_G X$ is isomorphic to the homotopy colimit of the corresponding functor $X : G \to \mathbf{S}$ (see [19, IV, Example 1.10]). Thus, $E_G X$ can be described as the simplicial set whose set of n-simplices is $G^n \times X_n$ and whose face and degeneracy operators are given by those of BG and X, except d_0 which is defined by

$$d_0(\sigma_1, \dots, \sigma_n, x) = \big(\sigma_2, \dots, \sigma_n, {}^{\sigma_1^{-1}} d_0 x\big).$$

In particular, for $X = *$ we have $E_G * = BG$. This construction is functorial and, therefore, any pointed G-space $X = (X,*) = (X \leftrightarrows *)$ gives rise to a retractive space over BG

$$E_G X \leftrightarrows BG, \tag{1}$$

defining a functor

$$E_G : \mathbf{S}_*^G \to \mathbf{S}_{BG}^{BG}.$$

And we recall the following result:

Theorem 2.1. *The functor $E_G : \mathbf{S}_*^G \to \mathbf{S}_{BG}^{BG}$ is a right Quillen equivalence. Then, it induces an equivalence*

$$\mathrm{Ho}\,\mathbf{S}_*^G \simeq \mathrm{Ho}\,\mathbf{S}_{BG}^{BG}$$

between the associated homotopy categories.

For any G-module N, the Eilenberg–Mac Lane abelian group minimal complexes $K(N,n)$, $n \geqslant 0$, have an evident structure of pointed G-spaces and the G-equivariant cohomology groups of a pointed G-space X with coefficients in N [28], $H^n_G(X,N)$, are defined by

$$H^n_G(X,N) = \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{S}^G_*}\big(X, K(N,n)\big), \quad n \geqslant 0.$$

Since every object in $\mathbf{S}^{BG}_{BG}$ is cofibrant and, for any G-module N, the retractions (1) $E_G K(N,n) \to BG$ are fibrant (indeed, they are split minimal fibrations), it is a consequence of the above Theorem 2.1 that, for any pointed G-space X, there are natural isomorphisms

$$\begin{aligned} H^n_G(X,N) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{S}^{BG}_{BG}}\big(E_G X, E_G K(N,n)\big) \\ &= \big[E_G X, E_G K(N,n)\big]_{\mathbf{S}^{BG}_{BG}} \\ &\cong H^n(E_G X, BG; N), \end{aligned} \tag{2}$$

where $[E_G X, E_G K(N,n)]_{\mathbf{S}^{BG}_{BG}}$ is the abelian group of homotopy classes of maps and $H^n(E_G X, BG; N)$ is the ordinary cohomology of the simplicial set $E_G X$ relative to the subspace BG with local coefficients in the G-module N. The last isomorphism sends a homotopy class represented by a simplicial map $E_G X \xrightarrow{f} E_G K(N,n)$ to the cohomology class represented by the cocycle $G^n \times X_n \xrightarrow{f_n} G^n \times N \xrightarrow{pr} N$ (see [19, VI, Proposition 4.13]).

Furthermore, by the Eilenberg–Zilber theorem, the cochain complex

$$C^\bullet(E_G X, N) \cong \mathrm{Hom}_{\mathbb{Z}G}\big(\mathbb{Z}(EG \times X), N\big),$$

where G acts diagonally on $EG \times X$, is quasi-isomorphic to the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}EG \otimes \mathbb{Z}X, N),$$

with the diagonal G action on the tensor product complex. This quasi-isomorphism is natural on X, and by combining it with the corresponding one for the G-fixed base point $* \in X$, we get a quasi-isomorphism between the relative cochain complex $C^\bullet(E_G X, BG; N)$ and the cochain complex kernel of the cochain complex map

$$\begin{gathered} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}EG \otimes \mathbb{Z}X, N) \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}EG, N) \cong C^\bullet(BG, N), \\ \big(G^p \times X_q \xrightarrow{f} N\big) \longmapsto \big(G^p \xrightarrow{f|_{G^p \times *}} N\big). \end{gathered}$$

Hence, there are natural isomorphisms

$$H^n_G(X,N) \cong H^n\,\mathrm{Ker}\big(\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}EG \otimes \mathbb{Z}X, N) \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}EG, N)\big). \tag{3}$$

We shall end this preliminary subsection by recalling the universal coefficient spectral sequence

$$\mathrm{Ext}^p_{\mathbb{Z}G}\big(\widetilde{H}_q(X), N\big) \Rightarrow H^{p+q}_G(X,N), \tag{4}$$

which is functorially associated to any pointed G-space X and any G-module N, where $\widetilde{H}_q(X)$ denotes the ordinary reduced homology of X with its natural G-module structure induced by the G-action on X.

2.2. The symmetric cohomology groups $H^n_{G,s}(M,N)$

For any two abelian groups M and N, the groups $H^n_s(M,N) = H^{n+2}(K(M,3),N)$ define the (second level or symmetric) cohomology theory of the abelian group M with coefficients in the abelian group N [15]. Then, in a natural way, we establish the following:

Definition 2.2. The *symmetric* cohomology groups of a G-module M with coefficients in a G-module N are defined by

$$H^n_{G,s}(M,N) = H^{n+2}_G\big(K(M,3),N\big), \quad n \geqslant 1. \tag{5}$$

As occurs in the ordinary non-equivariant case, elements of these cohomology groups $H^n_{G,s}(M,N)$ represent certain equivariant homotopy types. What follows is actually only a small part of an equivariant Postnikov theory of k-invariants, but Theorem 2.3 below states everything we are going to use in this paper on equivariant weak homotopy types. First we introduce some notations:

Let

$$\mathbf{S}^{BG}_{BG}(3,4) \subset \mathbf{S}^{BG}_{BG}$$

be the full subcategory of all retractive spaces over BG, $R \leftrightarrows BG$, such that $\pi_i R = 0$ for all $i \neq 1,3,4$ and $\pi_1 R = G$. From Theorem 2.1, *the homotopy category*

$$\mathrm{Ho}\,\mathbf{S}^{BG}_{BG}(3,4)$$

is equivalent to the homotopy category of pointed 2*-connected G-equivariant homotopy* 4*-types*, that is of pointed G-spaces X such that $\pi_i X = 0$ for all $i \neq 3,4$.

Let

$$\mathcal{H}^3_{G,s} \tag{6}$$

be the category whose objects (M,N,k) consist of G-modules M, N and cohomology classes $k \in H^3_{G,s}(M,N)$. An arrow $(M,N,k) \xrightarrow{(p,q)} (M',N',k')$ is a pair of G-module homomorphisms $p: M \to M'$, $q: N \to N'$ such that $q_*k = p^*k'$,

$$H^3_{G,s}(M,N) \xrightarrow{q_*} H^3_{G,s}(M,N') \xleftarrow{p^*} H^3_{G,s}(M',N'),$$

$$k \mapsto q_*k = p^*k' \leftarrow k'.$$

Theorem 2.3. *There is a* classifying functor

$$\begin{aligned} \mathrm{cl}: \mathbf{S}_{BG}^{BG}(3,4) &\to \mathcal{H}_{G,s}^{3}, \\ R &\mapsto (\pi_3 R, \pi_4 R, k_R), \\ f &\mapsto (\pi_3 f, \pi_4 f), \end{aligned}$$

where k_R is referred to as the equivariant Postnikov invariant *of R, with the following properties*:

(i) *For any object $(M, N, k) \in \mathcal{H}_{G,s}^{3}$, there exists $R \in \mathbf{S}_{BG}^{BG}(3,4)$ with an isomorphism* $\mathrm{cl}(R) \cong (M, N, k)$.

(ii) *For any $f : R \to R'$ in $\mathbf{S}_{BG}^{BG}(3,4)$, $\mathrm{cl}(f)$ is an isomorphism if and only if f is a weak equivalence.*

(iii) *The induced* classifying functor

$$\mathrm{cl}: \mathrm{Ho}\,\mathbf{S}_{BG}^{BG}(3,4) \to \mathcal{H}_{G,s}^{3}$$

is full, but it is not faithful. Indeed, for any arrow $(p, q): \mathrm{cl}(R) \to \mathrm{cl}(R')$ in $\mathcal{H}_{G,s}^{3}$, there is a bijection

$$\bigl\{[f] \in \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{S}_{BG}^{BG}(3,4)}(R, R') \bigm| \mathrm{cl}[f] = (p, q)\bigr\} \cong H_{G,s}^{2}(\pi_3 R, \pi_4 R').$$

Proof. Let M and N be any two G-modules. The classification, by its weak homotopy type, of retractive spaces $R \in \mathbf{S}_{BG}^{BG}(3,4)$ with $\pi_3 R = M$ and $\pi_4 R = N$, is equivalent to the classification, up to isomorphisms, of minimal fibre sequences in $\mathbf{S}_{BG}^{BG}$

$$E_G K(N,4) \hookrightarrow R \to E_G K(M,3). \tag{7}$$

In effect, note that any retractive space in $\mathbf{S}_{BG}^{BG}(3,4)$ is weak homotopy equivalent to one given by a minimal fibre sequence with a cross section $F \hookrightarrow R \leftrightarrows BG$ such that $\pi_3 F = M$, $\pi_4 F = N$ and $\pi_i F = 0$ for all $i \neq 3, 4$. In the case when $N = 0$, such a split minimal fibre sequence is necessarily isomorphic to the split minimal fibre sequence $K(M,3) \hookrightarrow E_G K(M,3) \leftrightarrows BG$ and, similarly, when $M = 0$, such a split minimal fibre sequence is isomorphic to the split minimal fibre sequence $K(N,4) \hookrightarrow E_G K(N,4) \leftrightarrows BG$. Sequence (7) then represents the Postnikov tower of R in $\mathbf{S}_{BG}^{BG}$.

Now, to give a minimal fibre sequence (7) in $\mathbf{S}_{BG}^{BG}$ is the same as giving a minimal fibre sequence $K(N,4) \hookrightarrow R \to E_G K(M,3)$ with a cross section from BG,

$$\begin{array}{ccccc} & & & & BG \\ & & & \swarrow & \downarrow \\ K(N,4) & \hookrightarrow & R & \to & E_G K(M,3). \end{array}$$

Hence, all the statements in the theorem follow from the ordinary classification and obstruction theory for relative fibre sequences with fibre an Eilenberg–Mac Lane space, simply by taking into account the isomorphisms showed in (2), that is, $H_{G,s}^{3}(M,N) \cong H^5(E_G K(M,3), BG; N)$ and $H_{G,s}^{2}(M,N) \cong H^4(E_G K(M,3), BG; N)$. $\square$

The lower symmetric cohomology groups $H^n_{G,s}(M,N)$ have a closed relationship with the groups $\mathrm{Ext}^n_{\mathbb{Z}G}(M,N)$, which can be quickly deduced from the universal coefficient spectral sequence (4) by using the well-known Eilenberg–Mac Lane computation of the homology groups of spaces $K(M,3)$ in low dimensions [16, II, Theorems 20.3, 20.5 and 23.1], namely

$$\widetilde{H}_i\big(K(M,3)\big)=\begin{cases}0 & \text{for } i=0,1,2,\\ M & \text{for } i=3,\\ 0 & \text{for } i=4,\\ M/2M & \text{for } i=5.\end{cases}$$

Hence, spectral sequence (4) gives:

Theorem 2.4. *For any G-modules M and N there are natural isomorphisms*

$$H^1_{G,s}(M,N)\cong \mathrm{Hom}_{\mathbb{Z}G}(M,N),$$
$$H^2_{G,s}(M,N)\cong \mathrm{Ext}^1_{\mathbb{Z}G}(M,N),$$

and a five-term exact sequence

$$\begin{array}{c} 0\to \mathrm{Ext}^2_{\mathbb{Z}G}(M,N)\longrightarrow H^3_{G,s}(M,N)\longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(M/2M,N)\\ \swarrow\\ \mathrm{Ext}^3_{\mathbb{Z}G}(M,N)\longrightarrow H^4_{G,s}(M,N).\end{array}\tag{8}$$

Thanks to Theorem 2.4, we have a precise and understandable algebraic interpretation for the symmetric cohomology groups $H^1_{G,s}(M,N)$ and $H^2_{G,s}(M,N)$. When group G is trivial, since the $\mathrm{Ext}^n_{\mathbb{Z}}(M,N)$ groups vanish for $n>1$, we have the well-known Eilenberg–Mac Lane isomorphism $H^3_s(M,N)\cong \mathrm{Hom}_{\mathbb{Z}}(M/2M,N)$. However, for an arbitrary G, the above five-term exact sequence only gives a good approach to such an algebraic interpretation of the 3rd symmetric cohomology groups $H^3_{G,s}(M,N)$. To go farther looking for a precise algebraic interpretation of these symmetric cohomology groups, as we do in the next section, we need to use a specific complex to compute them. This is the aim of the next part.

In [15,16,24] Eilenberg and Mac Lane described complexes, for any abelian group M, $A(M,3)$:

$$\cdots\to\mathbb{Z}\big(M^4\cup\big(M^2\mid M\big)\cup\big(M\mid M^2\big)\cup(M\,||\,M)\big)\to\mathbb{Z}\big(M^3\cup(M\mid M)\big)\to\mathbb{Z}\big(M^2\big)\to\cdots$$

which are homologically equivalent with the complexes $K(M,3)$ [16, I, Theorem 20.3] and algebraically more lucid. Indeed, the Eilenberg–Mac Lane chain equivalence

$$A(M,3)\to\mathbb{Z}K(M,3),\tag{9}$$

provides the possibility of explicitly computing the symmetric cohomology groups of an abelian group M with coefficients in an abelian group N, as

$$H^n_s(M,N)=H^nC_s(M,N),$$

where $C^n_s(M,N):=\mathrm{Hom}(A(M,3)_{n+2},N)$.

The construction of the complexes $A(M,3)$ is functorial on the abelian group M. Therefore, for any G-module M the complex $A(M,3)$ is canonically a chain complex of G-modules and, by naturality, the chain equivalence (9) becomes a G-module homomorphism. Hence, it follows from isomorphisms (3) that, for any G-module of coefficients N, there are natural isomorphisms

$$H^n_{G,s}(M,N) \cong H^{n+2}\,\mathrm{Ker}\big(\mathrm{Hom}_{\mathbb{Z}G}\big(\mathbb{Z}EG \otimes \mathbb{Z}A(M,3), N\big) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}EG, N)\big),$$

which lead us to make the following definition:

Definition 2.5. Let M, N be two G-modules. The complex $C_{G,s}(M,N)$, referred to as the complex of *symmetric cochains* of the G-module M with coefficients in the G-module N, is defined to be

$$C^{\bullet}_{G,s}(M,N) = \mathrm{Ker}\big(\mathrm{Hom}_{\mathbb{Z}G}\big(\mathbb{Z}EG \otimes \mathbb{Z}A(M,3), N\big) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}EG, N)\big)^{\bullet+2}.$$

So that we have

Theorem 2.6. *For any G-modules M and N, there are natural isomorphisms*

$$H^n_{G,s}(M,N) \cong H^n C_{G,s}(M,N), \quad n \geqslant 0. \tag{10}$$

In this paper we are going to use only the symmetric cohomology groups $H^n_{G,s}(M,N)$ for $n \leqslant 3$. Therefore, for future reference we specify below the relevant truncated subcomplex of $C_{G,s}(M,N)$, namely

$$0 \to C^1_{G,s}(M,N) \xrightarrow{\partial} C^2_{G,s}(M,N) \xrightarrow{\partial} Z^3_{G,s}(M,N) \to 0,$$

in which $C^1_{G,s}(M,N)$ consists of all normalized maps

$$M \xrightarrow{f} N,$$

$C^2_{G,s}(M,N)$ consists of all normalized maps

$$M^2 \cup (M \times G) \xrightarrow{g} N,$$

and $Z^3_{G,s}(M,N)$, the abelian group of symmetric 3-cocycles, consists of all normalized maps

$$M^3 \cup (M \mid M) \cup \big(M^2 \times G\big) \cup \big(M \times G^2\big) \xrightarrow{h} N$$

satisfying the following 3-cocycle conditions:

$$h(y,z,t) + h(x,y+z,t) + h(x,y,z) = h(x+y,z,t) + h(x,y,z+t), \tag{11}$$

$$h(x \mid z) + h(y,x,z) + h(x \mid y) = h(y,z,x) + h(x \mid y+z) + h(x,y,z), \tag{12}$$

$$h(x \mid y) = -h(y \mid x), \tag{13}$$

$${}^{\sigma}h(x,y,z) + h(x+y,z,\sigma) + h(x,y,\sigma) = h\big({}^{\sigma}x, {}^{\sigma}y, {}^{\sigma}z\big) + h(y,z,\sigma) + h(x,y+z,\sigma), \tag{14}$$

$$^{\sigma}h(x \mid y) + h(y, x, \sigma) = h\left({}^{\sigma}x \mid {}^{\sigma}y\right) + h(x, y, \sigma), \tag{15}$$

$$^{\sigma}h(x, y, \tau) + h\left({}^{\tau}x, {}^{\tau}y, \sigma\right) + h(x + y, \sigma, \tau) = h(x, y, \sigma\tau) + h(x, \sigma, \tau) + h(y, \sigma, \tau), \tag{16}$$

$$^{\sigma}h(x, \tau, \gamma) + h(x, \sigma, \tau\gamma) = h(x, \sigma\tau, \gamma) + h\left({}^{\gamma}x, \sigma, \tau\right), \tag{17}$$

with the coboundary maps

$$(\partial f)(x, y) = f(x) - f(x + y) + f(y), \tag{18}$$

$$(\partial f)(x, \sigma) = {}^{\sigma}f(x) - f\left({}^{\sigma}x\right), \tag{19}$$

$$(\partial g)(x, y, z) = g(y, z) - g(x + y, z) + g(x, y + z) - g(x, y), \tag{20}$$

$$(\partial g)(x \mid y) = g(x, y) - g(y, x), \tag{21}$$

$$(\partial g)(x, y, \sigma) = {}^{\sigma}g(x, y) - g\left({}^{\sigma}x, {}^{\sigma}y\right) - g(y, \sigma) + g(x + y, \sigma) - g(x, \sigma), \tag{22}$$

$$(\partial g)(x, \sigma, \tau) = {}^{\sigma}g(x, \tau) - g(x, \sigma\tau) + g\left({}^{\tau}x, \sigma\right), \tag{23}$$

for every $x, y, z, t \in M$ and $\sigma, \tau, \gamma \in G$.

3. Homotopy classification of graded Picard categories

This second part of the paper is dedicated to establishing precise theorems on the homotopy classification of graded Picard categories and their homomorphisms by means of symmetric cohomology groups of modules, studied throughout the previous section.

We shall begin by recalling from [17] (see also [9,11,18]) some needed definitions and terminology about symmetric graded tensor categories and graded Picard categories.

3.1. Preliminaries on graded Picard categories

Hereafter we regard group G as a category with one object, say $*$, where the morphisms are elements of G and the composition is the group operation. The neutral element of G is denoted by e.

By a *G-grading* on a category $\mathbb{G}$ we shall mean a functor $\mathrm{gr}:\mathbb{G} \to G$, such that for any object X of $\mathbb{G}$ and any $\tau \in G$, there exists an isomorphism f with domain X and such that $\mathrm{gr}(f) = \tau$ or, in other words, such that gr is a cofibration in the sense of Grothendieck [21]. We refer to $\mathrm{gr}(f) = \tau$ as the *degree* of f and say that f is a *τ-morphism*. An e-morphism is usually called a morphism *of trivial degree*.

A *graded functor* $F:\mathbb{G} \to \mathbb{H}$ between G-graded categories is a functor commuting with the G-gradings. From [21, Corollary 6.12], it follows that every graded functor between G-graded categories is cocartesian. A *graded natural equivalence* of graded functors, $\theta: F \to F'$, is a natural equivalence such that all isomorphisms $\theta_X : FX \to F'X$ are of trivial degree.

For a G-graded category $\mathbb{G}$, let $\mathbb{G}_e$ (or $\mathrm{Ker}\,\mathbb{G}$, in the terminology of [17,18]) denote the subcategory consisting of all morphisms of trivial degree, that is, the fibre category over the unique object of G. Then, a graded functor $F:\mathbb{G} \to \mathbb{H}$ between G-graded categories is an equivalence if and only if the induced functor $F:\mathbb{G}_e \to \mathbb{H}_e$ is an equivalence of categories [21, Proposition 6.5].

A *G-graded symmetric tensor category* $\mathbb{G} := (\mathbb{G}, \mathrm{gr}, \otimes, \mathrm{I}, A, R, C)$ is a G-graded category $(\mathbb{G}, \mathrm{gr})$, together with graded functors

$$\otimes : \mathbb{G} \times_G \mathbb{G} \to \mathbb{G}, \qquad \mathrm{I} : G \to \mathbb{G},$$

where $\mathbb{G} \times_G \mathbb{G}$ denotes the G-graded pullback category with respect to the G-gradings, and graded natural equivalences A, R and C (called associativity, unit and symmetry constraints, respectively) defined by isomorphisms of trivial degree

$$A_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \qquad R_X : X \otimes \mathrm{I} \cong X, \qquad C_{X,Y} : X \otimes Y \cong Y \otimes X,$$

such that, for any objects X, Y, Z, T of $\mathbb{G}$, the following coherence conditions hold:

$$A_{X,Y,Z\otimes T} A_{X\otimes Y,Z,T} = (id_X \otimes A_{Y,Z,T}) A_{X,Y\otimes Z,T} (A_{X,Y,Z} \otimes id_T), \tag{24}$$

$$C_{Y,X} C_{X,Y} = id_{X\otimes Y}, \tag{25}$$

$$(id_Y \otimes C_{X,Z}) A_{Y,X,Z} (C_{X,Y} \otimes id_Z) = A_{Y,Z,X} C_{X,Y\otimes Z} A_{X,Y,Z}, \tag{26}$$

$$(id_X \otimes R_Y) A_{X,Y,\mathrm{I}} = R_{X\otimes Y}, \tag{27}$$

$$(id_X \otimes R_Y)(id_X \otimes C_{\mathrm{I},Y}) A_{X,\mathrm{I},Z} = R_X \otimes id_Y. \tag{28}$$

If $\mathbb{G}$, $\mathbb{H}$ are G-graded symmetric tensor categories, then a *graded symmetric tensor functor* $F := (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}$ consists of a graded functor $F : \mathbb{G} \to \mathbb{H}$, natural isomorphisms of trivial degree, $\Phi_{X,Y} : FX \otimes FY \xrightarrow{\sim} F(X \otimes Y)$ and an isomorphism of trivial degree (natural with respect to the elements of G) $\Phi_* = \mathrm{I} \to F\,\mathrm{I}$, such that, for any objects X, Y, Z of $\mathbb{G}$, the following coherence conditions hold:

$$\Phi_{X,Y\otimes Z}(id_{FX} \otimes \Phi_{Y,Z}) A_{FX,FY,FZ} = F(A_{X,Y,Z}) \Phi_{X\otimes Y,Z} (\Phi_{X,Y} \otimes id_{FZ}), \tag{29}$$

$$\Phi_{Y,X} C_{FX,FY} = F(C_{X,Y}) \Phi_{X,Y}, \tag{30}$$

$$F(R_X) \Phi_{X,\mathrm{I}} (id_{FX} \otimes \Phi_*) = R_{FX}. \tag{31}$$

A *homotopy* (or *graded symmetric tensor natural equivalence*) between two graded symmetric tensor functors $(F, \Phi, \Phi_*), (F', \Phi', \Phi'_*) : \mathbb{G} \to \mathbb{H}$ is a graded natural equivalence $\theta : F \to F'$ such that, for all objects X, Y of $\mathbb{G}$, the following coherence conditions hold:

$$\Phi'_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X\otimes Y} \Phi_{X,Y}, \qquad \theta_{\mathrm{I}} \Phi_* = \Phi'_*. \tag{32}$$

The following lemma will be useful in the sequel, whose proof is parallel to Lemma 1.1 in [11].

Lemma 3.1. *Every graded symmetric tensor functor $F = (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}$ is homotopic to a graded symmetric tensor functor $F' = (F', \Phi', \Phi'_*)$ with $F'\mathrm{I} = \mathrm{I}$ and $\Phi'_* = id$.*

We next recall those graded tensor categories we are going to work with.

Definition 3.2. A *G-graded Picard category* is a G-graded symmetric tensor category $\mathbb{G}$ in which every morphism is invertible, that is, $\mathbb{G}$ is a groupoid, and for any object X, there is an object X' with an arrow of trivial degree $X \otimes X' \to \mathrm{I}$.

Observe that for any G-graded symmetric tensor category $\mathbb{G}$, the kernel subcategory $\mathbb{G}_e$ inherits a symmetric tensor structure and, since $\mathbb{G}$ is a groupoid if and only if $\mathbb{G}_e$ is as well, $\mathbb{G}$ is a graded Picard category if and only if $\mathbb{G}_e$ is a Picard category (or symmetric categorical group, as a Picard category is called by Joyal and Street in [23]). Interesting instances of graded Picard categories are shown by Fröhlich and Wall (see [17,18]). Next, we shall present three relevant constructions of graded Picard categories.

Example 3.3 *(The discrete).* A G-graded Picard category is called *discrete* if all morphisms of trivial degree are identities. Every discrete G-graded Picard category is determined by a G-module M as follows. The objects are the elements of M. For $x, y \in M$, there is a σ-morphism $x \to y$ if and only if $y = {}^{\sigma}x$ and this morphism is the element $\sigma \in G$. Composition is the multiplication in G, the G-graded tensor product is defined by

$$\big(x \xrightarrow{\sigma} y\big) \otimes \big(x' \xrightarrow{\sigma} y'\big) = \big(x + x' \xrightarrow{\sigma} y + y'\big),$$

and the unit G-graded functor is $\mathrm{I}(\sigma) = 0 \xrightarrow{\sigma} 0$. We denote by

$$\mathrm{dis}_G\, M \tag{33}$$

this discrete G-graded Picard category.

Example 3.4 *(The reduced).* A G-graded Picard category with only one object is called *reduced*. Any reduced G-graded Picard category can be defined by a G-module N as follows. The underlying groupoid is the semidirect product group $N \rtimes G$ and grading is the projection $N \rtimes G \to G$, so that a σ-morphism is a pair (a, σ) with $a \in N$, $\sigma \in G$, and composition is given by

$$(a, \sigma)(b, \tau) = \big(a + {}^{\sigma}b, \sigma\tau\big).$$

The tensor product is defined by

$$(a, \sigma) \otimes (b, \sigma) = (a + b, \sigma)$$

and the unit G-graded functor is given by $\mathrm{I}(\sigma) = (0, \sigma)$. Let

$$\mathrm{red}_G\, N \tag{34}$$

denote this G-graded Picard category.

Both G-graded Picard categories $\mathrm{dis}_G\, M$ and $\mathrm{red}_G\, N$, described above, are actually instances of the following construction, which will play a crucial role in the proof of our classification results.

Example 3.5 *(The defined by a symmetric 3-cocycle).* Any pair of G-modules, M, N, together with a symmetric 3-cocycle $h \in Z^3_{G,s}(M, N)$, give rise to a G-graded Picard category

$$\mathbb{G}(h) \tag{35}$$

which is defined as follows.

The objects of $\mathbb{G}(h)$ are the elements of M. An arrow $x \to y$ in $\mathbb{G}(h)$ is a pair (a, σ), where $a \in N$ and $\sigma \in G$ such that ${}^{\sigma}x = y$.

The composition of two morphisms $x \xrightarrow{(a,\sigma)} y \xrightarrow{(b,\tau)} z$ is defined by

$$(b, \tau)(a, \sigma) = \big(b + {}^{\tau}a + h(x, \tau, \sigma), \tau\sigma\big), \tag{36}$$

which is unitary thanks to the normalization condition of h and is associative owing to the 3-cocycle condition (17). Note that every morphism is invertible, indeed

$$(a, \sigma)^{-1} = \big(-{}^{\sigma^{-1}}a - h\big(x, \sigma^{-1}, \sigma\big), \sigma^{-1}\big).$$

Hence, $\mathbb{G}(h)$ is a groupoid.

The G-grading is given by $\mathrm{gr}(a, \sigma) = \sigma$.

The graded tensor product $\otimes : \mathbb{G}(h) \times_G \mathbb{G}(h) \to \mathbb{G}(h)$ is defined by

$$\big(x \xrightarrow{(a,\sigma)} y\big) \otimes \big(x' \xrightarrow{(b,\sigma)} y'\big) = \big(x + x' \xrightarrow{(a+b+h(x,x',\sigma),\sigma)} y + y'\big), \tag{37}$$

which is a functor thanks to the cocycle condition (16) and the normalization condition of h.

The associativity isomorphisms are

$$A_{x,y,z} = \big(h(x, y, z), e\big) : (x + y) + z \longrightarrow x + (y + z), \tag{38}$$

which satisfy coherence condition (24) because of the cocycle condition in (11). The naturality here follows since (14).

The graded functor $\mathrm{I} : G \to \mathbb{G}(h)$ is defined by

$$\mathrm{I}(\sigma) = \big(0 \xrightarrow{(0,\sigma)} 0\big), \tag{39}$$

and the unit constraint is identity $R_x = (0, e) : x \to x$.

The symmetry isomorphisms are given by

$$C_{x,y} = \big(h(x \mid y), e\big) : x + y \longrightarrow y + x. \tag{40}$$

The cocycle conditions (13) and (12) amount precisely to the coherence conditions (25) and (26), respectively. The naturality of C follows since (15).

Thus, $\mathbb{G}(h)$ is a G-graded symmetric tensor groupoid, which is actually a G-graded Picard category since, for any object x of $\mathbb{G}(h)$, we have $x \otimes (-x) = x + (-x) = 0 = \mathrm{I}$.

For later reference, we shall note here the existence of a short exact sequence of G-graded Picard categories (cf. [9, Definition 29], [7, Definition 2.2])

$$\mathrm{red}_G\, N \overset{\mathrm{j}}{\hookrightarrow} \mathbb{G}(h) \overset{\mathrm{q}}{\to} \mathrm{dis}_G\, M, \tag{41}$$

in which the graded symmetric tensor functors j and q are given by

$$(a, \sigma) \overset{\mathrm{j}}{\mapsto} \big(0 \xrightarrow{(a,\sigma)} 0\big) \quad (a \in N,\ \sigma \in G),$$
$$\big(x \xrightarrow{(a,\sigma)} y\big) \overset{\mathrm{q}}{\mapsto} \big(x \xrightarrow{\sigma} y\big) \quad (a \in N,\ \sigma \in G,\ x, y \in M).$$

3.2. Homotopy classification results

Let $\mathbf{Pic}_G$ denote the category of G-graded Picard categories and graded symmetric tensor functors between them. Homotopy, defined in the previous section, is an equivalence relation among graded symmetric tensor functors and it is compatible with compositions. Therefore, we can define the *homotopy category of G-graded Picard categories*, $\mathrm{Ho}\,\mathbf{Pic}_G$, to be the category of all G-graded Picard categories and homotopy classes of graded symmetric tensor functors between them. A graded symmetric tensor functor inducing an isomorphism in the homotopy category is said to be a *graded symmetric tensor equivalence*.

The classification of G-graded Picard categories is our major objective. (The non-graded case was dealt with in [29], see also [17].) For this classification, two G-graded Picard categories that are connected by a graded symmetric tensor equivalence are considered the same. The problem arises of giving a complete invariant of this relation. To this end, to each G-graded Picard category $\mathbb{G}$ we will associate the algebraic data $K_0\mathbb{G}$, $K_1\mathbb{G}$ and $k_{\mathbb{G}}$, which are invariants under graded symmetric tensor equivalence. We next introduce the first two as the first invariants of the (non-graded) Picard category $\mathbb{G}_e$ considered by Sinh in [29]:

- $K_0\mathbb{G} := K_0\mathbb{G}_e$, the (abelian) group of trivial degree isomorphism classes of the objects in $\mathbb{G}$, where the group structure is induced by the G-graded symmetric tensor structure of $\mathbb{G}$.
- $K_1\mathbb{G} := K_1\mathbb{G}_e$, the (abelian) group of automorphisms of trivial degree of the unit object I of $\mathbb{G}$.

Note that $K_1\mathbb{G}$ is abelian since the multiplication $K_1\mathbb{G} \times K_1\mathbb{G} \to K_1\mathbb{G}$, $(a,b) \mapsto R_\mathrm{I}(a \otimes b)R_\mathrm{I}^{-1}$, is a group homomorphism. The group $K_0\mathbb{G}$ is also abelian because of the symmetry. Next we observe that both *$K_0\mathbb{G}$ and $K_1\mathbb{G}$ are G-modules*.

Since the grading on $\mathbb{G}$ is a cofibration, for each $\sigma \in G$ and $[X] \in K_0\mathbb{G}$ there exists a σ-morphism $f : X \to X'$ and we write

$$ {}^{\sigma}[X] = [X']. \tag{42} $$

If $f' : Y \to Y'$ is another σ-morphism and $g : X \to Y$ is an e-morphism, then $f'gf^{-1} : X' \to Y'$ is an e-morphism. Therefore, $[X'] = [Y'] \in K_0\mathbb{G}$ implying that the map $(\sigma, [X]) \mapsto {}^{\sigma}[X]$ is well defined. Now it is easy to see that (42) indeed determines a G-module structure on $K_0\mathbb{G}$.

The G-module structure on $K_1\mathbb{G}$ is given by

$$ {}^{\sigma}a = \mathrm{I}(\sigma)a\mathrm{I}(\sigma)^{-1}, \tag{43} $$

for any $\sigma \in G$ and any e-morphism $a : \mathrm{I} \to \mathrm{I}$ in $\mathbb{G}$.

It is instructive the existence of an extension, analogous to (41), associated to any G-graded Picard category $\mathbb{G}$, since it is intimately related to the algebraic invariants attached to $\mathbb{G}$: let us write M and N for the G-modules $K_0\mathbb{G}$ and $K_1\mathbb{G}$, respectively. Then, there is a short exact sequence of G-graded Picard categories

$$ \mathrm{red}_G\, N \overset{\mathrm{j}}{\hookrightarrow} \mathbb{G} \overset{\mathrm{q}}{\to} \mathrm{dis}_G\, M, \tag{44} $$

where

$$ \mathrm{j}(a,\sigma) = \big(\mathrm{I} \xrightarrow{a\mathrm{I}(\sigma)} \mathrm{I}\big), \qquad \mathrm{q}\big(X \xrightarrow{f} Y\big) = \big([X] \xrightarrow{\mathrm{gr}(f)} [Y]\big). $$

Proposition 3.6.

(i) *Every graded symmetric tensor functor $F:\mathbb{G}\to\mathbb{G}'$ between G-graded Picard categories induces homomorphisms of G-modules*

$$K_iF:K_i\mathbb{G}\to K_i\mathbb{G}',\quad i=0,1,$$

given by $K_0F:[X]\mapsto[FX]$, $K_1F:a\mapsto\Phi_*^{-1}F(a)\Phi_*$.

(ii) *Two homotopic graded symmetric tensor functors induce the same homomorphisms of G-modules.*

(iii) *A graded symmetric tensor functor F is a graded symmetric tensor equivalence if and only if the induced homomorphisms K_0F and K_1F are isomorphisms.*

Proof. Since the restriction $F:\mathbb{G}_e\to\mathbb{G}'_e$ is a symmetric tensor functor, which is a symmetric tensor equivalence if and only if $F:\mathbb{G}\to\mathbb{G}'$ is a symmetric graded equivalence, we deduce the assertion by using the similar facts in a non-graded case [29] and by a slight modification of the proof of [11, Proposition 1.3]. □

Recall now that a fibred category is skeletal when all its fibre categories are skeletal. In particular, a graded category is skeletal when any two objects isomorphic by an isomorphism of trivial degree are equal. We have the following lemma, which will be needed in the sequel.

Lemma 3.7. *Any G-graded Picard category $\mathbb{G}=(\mathbb{G},\mathrm{gr},\otimes,\mathrm{I},A,R,C)$ is equivalent to a skeletal one, say $\widehat{\mathbb{G}}=(\widehat{\mathbb{G}},\widehat{\mathrm{gr}},\widehat{\otimes},\widehat{\mathrm{I}},\widehat{A},\widehat{R},\widehat{C})$, in which unit $\widehat{\mathrm{I}}$ is strict in the sense that $\widehat{R}=id$.*

Proof. For each $x\in K_0\mathbb{G}$ let us choose an object $O_x\in x$ with $O_{[\mathrm{I}]}=\mathrm{I}$, and for any other $O\in x$ fix an e-morphism $\Phi_O:O\to O_x$ with $\Phi_{O_x}=id_{O_x}$ and $\Phi_{O_x\otimes\mathrm{I}}=R_{O_x}$. Let $\widehat{\mathbb{G}}$ be the full subcategory of $\mathbb{G}$ whose objects are all O_x, $x\in K_0\mathbb{G}$, with the G-grading $\widehat{\mathrm{gr}}=\mathrm{gr}|_{\widehat{\mathbb{G}}}:\widehat{\mathbb{G}}\to G$. Then, the inclusion functor $\widehat{\mathbb{G}}\hookrightarrow\mathbb{G}$ is a graded equivalence. Now, the G-graded Picard category structure of $\mathbb{G}$ can be transported to $\widehat{\mathbb{G}}$, in a unique way such that the inclusion functor $\widehat{\mathbb{G}}\hookrightarrow\mathbb{G}$, together with the isomorphisms $\Phi_{O_x,O_y}=\Phi_{O_x\otimes O_y}:O_x\otimes O_y\to O_x\mathbin{\widehat{\otimes}}O_y=O_{xy}$ and $\Phi_*=id_\mathrm{I}$, turns out to be a symmetric graded tensor equivalence. Clearly, the resulting G-graded Picard category $\widehat{\mathbb{G}}=(\widehat{\mathbb{G}},\widehat{\mathrm{gr}},\widehat{\otimes},\widehat{\mathrm{I}},\widehat{A},\widehat{R},\widehat{C})$ is skeletal and $\widehat{R}=id$. □

Note that the G-graded Picard category $\mathbb{G}(h)$, constructed in (35) from a pair of G-modules (M,N) and a 3-cocycle $h\in Z^3_{G,s}(M,N)$, is an example of a strictly unitary skeletal G-graded Picard category. Moreover, observe that its first invariants $K_0\mathbb{G}(h)$ and $K_1\mathbb{G}(h)$ are respectively isomorphic to the G-modules M and N, by means of the G-module isomorphisms

$$\alpha:K_0\mathbb{G}(h)\cong M,\qquad \beta:K_1\mathbb{G}(h)\cong N, \tag{45}$$

given by

$$\alpha(x)=x,\qquad \beta(a,e)=a.$$

We shall establish the following terminology.

Definition 3.8. A *G-graded Picard category of type* (M, N), where M and N are G-modules, is a triple $(\mathbb{G}, \alpha, \beta)$, in which $\mathbb{G}$ is a G-graded Picard category and $\alpha : K_0\mathbb{G} \cong M$ and $\beta : K_1\mathbb{G} \cong N$ are G-module isomorphisms.

If $(\mathbb{G}', \alpha', \beta')$ is a G-graded Picard category of type (M', N'), then a graded symmetric tensor functor $F : \mathbb{G} \to \mathbb{G}'$ is said to be *of type* (p, q), where $p : M \to M'$ and $q : N \to N'$ are G-module homomorphisms, whenever the two diagrams below commute

$$\begin{array}{ccc} K_0\mathbb{G} & \xrightarrow{\alpha} & M \\ {\scriptstyle K_0F}\downarrow & & \downarrow{\scriptstyle p} \\ K_0\mathbb{G}' & \xrightarrow{\alpha'} & M', \end{array} \qquad \begin{array}{ccc} K_1\mathbb{G} & \xrightarrow{\beta} & N \\ {\scriptstyle K_1F}\downarrow & & \downarrow{\scriptstyle q} \\ K_1\mathbb{G}' & \xrightarrow{\beta'} & N'. \end{array}$$

Two G-graded Picard categories of the same type (M, N), say $(\mathbb{G}, \alpha, \beta)$ and $(\mathbb{G}', \alpha', \beta')$, are *equivalent* if there exists a graded symmetric tensor functor (necessarily an equivalence by Proposition 3.6) $F : \mathbb{G} \to \mathbb{G}'$ of type (id_M, id_N), that is, such that $\alpha' K_0F = \alpha$ and $\beta' K_1F = \beta$.

The set of equivalence classes of G-graded Picard categories of type (M, N) is denoted by

$$\mathbf{Pic}_G(M, N).$$

Now we are ready to state and prove the classification results of G-graded Picard categories. The next theorem deals with the classification of graded symmetric tensor functors between G-graded Picard categories of the form $\mathbb{G}(h)$ (see (35)), and the last theorem shows that every G-graded Picard category is equivalent to $\mathbb{G}(h)$ for some h.

Theorem 3.9. *Let $h \in Z^3_{G,s}(M, N)$ and $h' \in Z^3_{G,s}(M', N')$ be symmetric 3-cocycles. Suppose $p : M \to M'$ and $q : N \to N'$ are any given G-module homomorphisms. Then, there exists a graded symmetric tensor functor $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (p, q) if and only if the symmetric 3-cocycles $p^*h', q_*h \in Z^3_{G,s}(M, N')$ represent the same cohomology class, that is,*

$$p^*[h'] = q_*[h] \in H^3_{G,s}(M, N').$$

Furthermore, when $p^*[h'] = q_*[h]$, then the set of homotopy classes of graded symmetric tensor functors $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (p, q) is in bijection with $H^2_{G,s}(M, N')$.

Proof. First suppose $p^*[h'] = q_*[h] \in H^3_{G,s}(M, N')$. Then, there exists a symmetric 2-cochain $g \in C^2_{G,s}(M, N')$ such that $q_*h = p^*h' + \partial g$, which determines a graded symmetric tensor functor $F_g = (F_g, \Phi, \Phi_*) : \mathbb{G}(h) \to \mathbb{G}(h')$ of type (p, q) by the following equalities

$$F_g\big(x \xrightarrow{(a,\sigma)} y\big) = \big(p(x) \xrightarrow{(q(a)+g(x,\sigma),\sigma)} p(y)\big),$$
$$\Phi_{x,y} = \big(g(x, y), e\big) : p(x) + p(y) \to p(x + y),$$
$$\Phi_* = id = (0, e) : p(0) \to 0,$$

for all $x, y \in M$, $a \in N$ and $\sigma \in G$. So defined, F_g is actually a functor thanks to the equality (23) and the normalization condition of g. The isomorphisms $\Phi_{x,y}$ define a graded natural equivalence $F_g(-)\otimes F_g(-)\to F_g(-\otimes -)$ owing to the coboundary condition (22). The coherence conditions (29) and (30) hold because of the equalities (20) and (21) respectively, whilst (31) is trivially verified. Since $\alpha K_0F_g(x)=p(x)=p\alpha(x)$ and $\beta K_1F_g(a,e)=q(a)=q\beta(a,e)$, we see that F_g is actually of type (p,q).

Conversely, suppose that $F=(F,\Phi,\Phi_*):\mathbb{G}(h)\to\mathbb{G}(h')$ is any graded symmetric tensor functor of type (p,q). By Lemma 3.1, there is no loss of generality in assuming that $\Phi_*=id_0=(0,e)$. Since $\alpha K_0F=p\alpha$, we have $F(x)=p(x)$, $x\in M$, whilst the equality $\beta K_1F=q\beta$ implies that

$$F\big(0\xrightarrow{(a,e)}0\big)=0\xrightarrow{(q(a),e)}0,$$

for any $a\in N$. Furthermore, by coherence condition (31) one has $\Phi_{x,0}=id_x=\Phi_{0,x}$ for all $x\in M$ and then, since every morphism of trivial degree, say $x\xrightarrow{(a,e)}x$, can be expressed in the form $x\xrightarrow{(a,e)}x=(0\xrightarrow{(a,e)}0)\otimes(x\xrightarrow{(0,e)}x)$, we deduce by naturality that

$$\begin{aligned}F\big(x\xrightarrow{(a,e)}x\big)&=F\big(0\xrightarrow{(a,e)}0\big)\otimes F\big(x\xrightarrow{(0,e)}x\big)=\big(0\xrightarrow{(q(a),e)}0\big)\otimes\big(p(x)\xrightarrow{(0,e)}p(x)\big)\\&=p(x)\xrightarrow{(q(a),e)}p(x).\end{aligned}$$

If we write for each $\sigma\in G$ and $x,y\in M$

$$\begin{aligned}F\big(x\xrightarrow{(0,\sigma)}{}^\sigma x\big)&=\big(p(x)\xrightarrow{(g(x,\sigma),\sigma)}{}^\sigma p(x)\big),\quad g(x,\sigma)\in N',\\ \Phi_{x,y}&=\big(x+y\xrightarrow{(g(x,y),e)}x+y\big),\quad g(x,y)\in N',\end{aligned}$$

we get a 2-cochain $g\in C^2_{G,s}(M,N')$, which determines F completely. Indeed, for any morphism $x\xrightarrow{(a,\sigma)}y$ in $\mathbb{G}(h)$ we have

$$\begin{aligned}F\big(x\xrightarrow{(a,\sigma)}y\big)&=F\big(y\xrightarrow{(a,e)}y\big)F\big(x\xrightarrow{(0,\sigma)}y\big)\\&=\big(p(y)\xrightarrow{(a,e)}p(y)\big)\big(p(x)\xrightarrow{(g(x,\sigma),\sigma)}p(y)\big)\\&=p(x)\xrightarrow{(a+g(x,\sigma),\sigma)}p(y).\end{aligned}\tag{46}$$

The condition that F is a graded symmetric tensor functor amounts precisely to the equality $q_*h=p^*h'+\partial g$. In effect, the equality $q_*h(x,y,z)=p^*h'(x,y,z)+\partial g(x,y,z)$ follows from the coherence condition (29); $q_*h(x\mid y)=p^*h'(x\mid y)+\partial g(x\mid y)$ is a consequence of (30); that $q_*h(x,y,\sigma)=p^*h'(x,y,\sigma)+\partial g(x,y,\sigma)$ is owing to the naturality of the isomorphisms $\Phi_{x,y}$ and $q_*h(x,\sigma,\tau)=p^*h'(x,\sigma,\tau)+\partial g(x,\sigma,\tau)$ is a direct consequence of F being a functor. Therefore, q_*h and p^*h' are cohomologous symmetric 3-cocycles of the G-module M with coefficients in N', as claimed.

To prove the second statement of the theorem, we stress that the above-constructed map $g\mapsto F_g$ induces a surjection from the set of those symmetric 2-cochains $g\in C^2_{G,s}(M,N')$ such that $q_*h=p^*h'+\partial g$ onto the set of homotopy classes of symmetric graded tensor functors $\mathbb{G}(h)\to\mathbb{G}(h')$ of type (p,q). Now we fix any $g_0\in C^2_{G,s}(M,N')$ satisfying $q_*h=p^*h'+\partial g_0$, which exists under the hypothesis $p^*[h']=q_*[h]$. Then, any other such symmetric 2-cochain necessarily has the form g_0+g, where $g\in Z^2_{G,s}(M,N')$.

Thus, to complete the proof, it remains to check that two symmetric graded tensor functors F_{g_0+g} and $F_{g_0+g'}$, where $g, g' \in Z^2_{G,s}(M, N')$, are homotopic if and only if g and g' are cohomologous.

Let $g = g' + \partial f$ for some $f \in C^1_{G,s}(M, N')$. Then the family of isomorphisms of trivial degree in $\mathbb{G}(h')$,

$$\theta_x : p(x) \xrightarrow{(f(x),e)} p(x), \quad x \in M,$$

defines a graded natural equivalence $\theta : F_{g_0+g} \to F_{g_0+g'}$ thanks to equality (19), which also satisfies condition (32) due to equality (18), which means that θ is a homotopy of graded symmetric tensor functors.

And conversely, if $\theta : F_{g_0+g} \to F_{g_0+g'}$ is a homotopy and we write $\theta_x = (f(x), e) : p(x) \to p(x)$ for a map $f : M \to N'$, then the condition $g = g' + \partial f$ amounts precisely to the condition of θ being a symmetric graded tensor equivalence. □

Corollary 3.10. *Let $h, h' \in Z^3_{G,s}(M, N)$ be two symmetric 3-cocycles of a G-module M with coefficients in a G-module N. Then, the G-graded Picard categories of type (M, N), $\mathbb{G}(h)$ and $\mathbb{G}(h')$ are equivalent if and only if h and h' are cohomologous.*

Proof. By definition, $\mathbb{G}(h)$ and $\mathbb{G}(h')$ are equivalent whenever there exists a symmetric graded tensor functor $\mathbb{G}(h) \to \mathbb{G}(h')$ of type (id_M, id_N), which, by Theorem 3.9, occurs if and only if $0 = [h'] - [h] \in H^3_{G,s}(M, N)$. □

Theorem 3.11. *For any G-modules M and N, the map that carries a symmetric 3-cocycle $h \in Z^3_{G,s}(M, N)$, to the G-graded Picard category $\mathbb{G}(h)$, induces a bijection*

$$H^3_{G,s}(M, N) \cong \mathbf{Pic}_G(M, N).$$

Proof. By Corollary 3.10 above, the correspondence $[h] \mapsto [\mathbb{G}(h)]$ is a correctly defined injective map $H^3_{G,s}(M, N) \to \mathbf{Pic}_G(M, N)$. Therefore it remains to prove that every G-graded Picard category of type (M, N), say $\mathbb{G} = (\mathbb{G}, \alpha, \beta)$, is equivalent to $\mathbb{G}(h)$ for some $h \in Z^3_{G,s}(M, N)$.

To simplify the notations, we can assume, without loss of generality, that the G-module isomorphisms α and β are identities, that is, $K_0\mathbb{G} = M$ and $K_1\mathbb{G} = N$. Moreover, by using Lemma 3.7 we suppose that $\mathbb{G}$ is a skeletal G-graded Picard category in which the unit constraint is identity.

Then $\mathbb{G}$ has the following properties:

- $\mathrm{Ob}\,\mathbb{G} = K_0\mathbb{G} = M$ and $x \otimes y = x + y$ for all $x, y \in M$.
- The (strict) unit object is $\mathrm{I} = 0$, $\mathrm{Aut}_e(0) = K_1\mathbb{G} = N$ and $a \otimes b = a + b$ for all $a, b \in N$. Moreover, for any $x \in M$, there is an isomorphism $N \cong \mathrm{Aut}_e(x)$, $a \mapsto a \otimes id_x$, and by the naturality of the symmetry constraint one has

$$a \otimes id_x = id_x \otimes a. \tag{47}$$

– For any $\sigma \in G$ and $x, y \in M$, there exists a σ-morphism $u : x \to y$ if and only if ${}^{\sigma}x = y$ (according to (42)) and in this case we have the composite bijection

$$N \cong \mathrm{Aut}_e(y) \overset{u^*}{\cong} \mathrm{Hom}_\sigma(x, y), \quad a \mapsto (a \otimes id_y)u, \tag{48}$$

where $\mathrm{Hom}_\sigma(x, y)$ is the set of all σ-morphisms from x to $y = {}^{\sigma}x$.
– For any $\sigma \in G$ and a σ-morphism $u : x \to y$, by naturality of the unit constraint one has

$$\mathrm{I}(\sigma) \otimes u = u = u \otimes \mathrm{I}(\sigma) \tag{49}$$

and then, for all $a \in N$ we have

$$u(a \otimes id_x) = \left({}^{\sigma}a \otimes id_y\right)u. \tag{50}$$

In effect,

$$\begin{aligned}\left({}^{\sigma}a \otimes id_y\right)u &\overset{(49)}{=} \left({}^{\sigma}a \otimes id_y\right)\left(\mathrm{I}(\sigma) \otimes u\right) = \left({}^{\sigma}a\,\mathrm{I}(\sigma) \otimes u\right) \\ &\overset{(43)}{=} \left(\mathrm{I}(\sigma)\, a \otimes u\right) = \left(\mathrm{I}(\sigma) \otimes u\right)(a \otimes id_x) \overset{(49)}{=} u(a \otimes id_x).\end{aligned}$$

Now, for each $\sigma \in G$ and $x \in M$ we choose a morphism in $\mathbb{G}$ with domain x and degree σ, say

$$u_{x,\sigma} : x \to y = {}^{\sigma}x, \quad \text{with } u_{0,\sigma} = I(\sigma) \text{ and } u_{x,e} = id_x,$$

and, for each $a \in N$, we shall write

$$x \xrightarrow{(a,\sigma)} y := x \xrightarrow{(a \otimes id_y)u_{x,\sigma}} y. \tag{51}$$

That is, we are denoting by $(a, \sigma) : x \to y$ the σ-morphism from x to y that corresponds to a by the bijection (48) for $u = u_{x,\sigma}$. Thus, for example, we have the equalities

$$\begin{gathered} x \xrightarrow{(0,\sigma)} y = x \xrightarrow{u_{x,\sigma}} y, \qquad x \xrightarrow{(a,e)} x = x \xrightarrow{a \otimes id_x} x, \\ x \xrightarrow{(0,e)} x = x \xrightarrow{id_x} x, \qquad 0 \xrightarrow{(0,\sigma)} 0 = 0 \xrightarrow{\mathrm{I}(\sigma)} 0, \end{gathered} \tag{52}$$

$$x \xrightarrow{(a,\sigma)} y \overset{(51)}{=} \left(y \xrightarrow{(a,e)} y\right)\left(x \xrightarrow{(0,\sigma)} y\right), \tag{53}$$

$$x \xrightarrow{({}^{\sigma}a,\sigma)} y \overset{(50)}{=} \left(x \xrightarrow{(0,\sigma)} y\right)\left(x \xrightarrow{(a,e)} y\right), \tag{54}$$

$$\left(x \xrightarrow{(a,e)} x\right)\left(x \xrightarrow{(b,e)} x\right) = \left(x \xrightarrow{(a+b,e)} x\right), \tag{55}$$

since $(a \otimes id_x)(b \otimes id_x) = (a + b) \otimes id_x$,

$$\left(x \xrightarrow{(a,e)} x\right) \otimes \left(y \xrightarrow{(b,e)} y\right) = \left(x + y \xrightarrow{(a+b,e)} x + y\right), \tag{56}$$

since $(a \otimes id_x) \otimes (b \otimes id_y) \overset{(47)}{=} (a + b) \otimes id_{x+y}$.

Now we are ready to determine a symmetric 3-cochain

$$h = h_{\mathbb{G}} : M^3 \cup (M \mid M) \cup \big(M^2 \times G\big) \cup \big(M \times G^2\big) \to N,$$

by the following four equations.

$$\big(y \xrightarrow{(0,\sigma)} z\big)\big(x \xrightarrow{(0,\tau)} y\big) = \big(x \xrightarrow{(h(x,\sigma,\tau),\sigma\tau)} z\big), \tag{57}$$

for $\sigma, \tau \in G$, $x \in M$, ${}^{\tau}x = y$, ${}^{\sigma}y = z$;

$$\big(x \xrightarrow{(0,\sigma)} y\big) \otimes \big(x' \xrightarrow{(0,\sigma)} y'\big) = \big(x + x' \xrightarrow{(h(x,x',\sigma),\sigma)} y + y'\big), \tag{58}$$

for $\sigma \in G$, $x, x' \in M$, ${}^{\sigma}x = y$, ${}^{\sigma}x' = y'$;

$$\big(x + y \xrightarrow{C_{x,y}} y + x\big) = \big(x + y \xrightarrow{(h(x|y),e)} y + x\big), \tag{59}$$

for $x, y \in M$;

$$\big((x + y) + z \xrightarrow{A_{x,y,z}} x + (y + z)\big) = \big((x + y) + z \xrightarrow{(h(x,y,z),e)} x + (y + z)\big), \tag{60}$$

for $x, y, z \in M$.

So defined, this 3-cochain h completely determines the G-graded (strictly unitary) symmetric tensor category structure of $\mathbb{G}$, since the following two equalities hold:

$$\big(y \xrightarrow{(b,\tau)} z\big)\big(x \xrightarrow{(a,\sigma)} y\big) = \big(x \xrightarrow{(b+{}^{\tau}a+h(x,\tau,\sigma),\tau\sigma)} z\big), \tag{61}$$

$$\big(x \xrightarrow{(a,\sigma)} y\big) \otimes \big(x' \xrightarrow{(b,\sigma)} y'\big) = \big(x + x' \xrightarrow{(a+b+h(x,x',\sigma),\sigma)} y + y'\big), \tag{62}$$

for all $\sigma, \tau \in G$, $x, x', y, y', z \in M$, ${}^{\sigma}x = y$, ${}^{\sigma}x' = y'$, ${}^{\tau}y = z$. In effect,

$$\begin{aligned}
(b,\tau)\,(a,\sigma) &\overset{(53)}{=} (b,e)(0,\tau)(a,e)(0,\sigma) \overset{(54)}{=} (b,e)\big({}^{\tau}a,\tau\big)(0,\sigma) \\
&\overset{(53)}{=} (b,e)\big({}^{\tau}a,e\big)(0,\tau)(0,\sigma) \overset{(55),(57)}{=} \big(b + {}^{\tau}a, e\big)\big(h(x,\tau,\sigma),\tau\sigma\big) \\
&\overset{(53)}{=} \big(b + {}^{\tau}a + h(x,\tau,\sigma),\tau\sigma\big), \\
(a,\sigma) \otimes (b,\sigma) &\overset{(53)}{=} \big((a,e)(0,\sigma)\big) \otimes \big((b,e)(0,\sigma)\big) \\
&= \big((a,e) \otimes (b,e)\big)\big((0,\sigma) \otimes (0,\sigma)\big) \overset{(56),(58)}{=} (a+b,e)\big(h(x,x',\sigma),\sigma\big) \\
&\overset{(53)}{=} \big(a + b + h(x,x',\sigma),\sigma\big).
\end{aligned}$$

It is now easy to conclude from equalities (52), (61) and (62) that h is actually a symmetric 3-cocycle of the G-module M with coefficients in N. Since the composition and tensor in $\mathbb{G}$ are unitary and I is a functor, the normalization of h follows from the equality (49), coherent condition (25) and [23, Proposition 1.1]. The cocycle condition (17) follows from the associativity law for morphisms in $\mathbb{G}$ and (16) is a consequence of the graded tensor product $\otimes : \mathbb{G} \times_G \mathbb{G} \to \mathbb{G}$ being functorial. The equality (11) holds because of the coherence condition (24), and (14) follows from the naturality of the associativity constraints. The cocycle condition (13) is a consequence of (25). Moreover, (12) is verified owing to the coherence condition (26) whilst, finally, the naturality of the symmetry implies (15).

Hence, $h_{\mathbb{G}} = h \in Z^3_{G,s}(M, N)$ and, by comparison equalities (36) with (61), (37) with (62), (39) with (52), (38) with (60) and (40) with (59) respectively, we see that $\mathbb{G}(h)$ and $\mathbb{G}$ are isomorphic G-graded Picard categories of type (M, N). This completes the proof of the theorem. □

A consequence of Theorem 3.11 is that the cohomology class of the symmetric 3-cocycle $h_{\mathbb{G}}$ depends only on the homotopy equivalence class of $\mathbb{G}$. We denote it by

$$k_{\mathbb{G}} = [h_{\mathbb{G}}] \in H^3_{G,s}\big(K_0(\mathbb{G}), K_1(\mathbb{G})\big)$$

and refer to it as the *Postnikov invariant of the G-graded Picard category* $\mathbb{G}$.

Let us now recall from (6) the category $\mathcal{H}^3_{G,s}$ of symmetric 3-cocycles of G-modules. Then, Theorem 3.9, Corollary 3.10 and Theorem 3.11 jointly give the classification theorem below.

Theorem 3.12. *There is a* classifying functor

$$\begin{aligned} \mathrm{cl} : \mathbf{Pic}_G &\to \mathcal{H}^3_{G,s}, \\ \mathbb{G} &\mapsto (K_0\mathbb{G}, K_1\mathbb{G}, k_{\mathbb{G}}), \\ F &\mapsto (K_0F, K_1F), \end{aligned}$$

which has the following properties:

(i) *For any object $(M, N, k) \in \mathcal{H}^3_{G,s}$, there exists a G-graded Picard category $\mathbb{G}$ with an isomorphism* $\mathrm{cl}(\mathbb{G}) \cong (M, N, k)$.
(ii) *For any G-graded symmetric tensor functor $F : \mathbb{G} \to \mathbb{H}$ in* $\mathbf{Pic}_G$, $\mathrm{cl}(F)$ *is an isomorphism if and only if F is a graded equivalence.*
(iii) *The induced* classifying functor

$$\mathrm{cl} : \mathrm{Ho}\,\mathbf{Pic}_G \to \mathcal{H}^3_{G,s}$$

is full, but it is not faithful. Indeed, for any arrow $(p, q) : \mathrm{cl}(\mathbb{G}) \to \mathrm{cl}(\mathbb{H})$ in $\mathcal{H}^3_{G,s}$, there is a bijection

$$\big\{[F] \in \mathrm{Hom}_{\mathrm{Ho}\mathbf{Pic}_G}(\mathbb{G}, \mathbb{H}) \,\big|\, \mathrm{cl}[F] = (p, q)\big\} \cong H^2_{G,s}(K_0\mathbb{G}, K_1\mathbb{H}).$$

A comparison of Theorems 2.3 and 3.12 gives

Theorem 3.13. *There is an equivalence of categories between the homotopy category of pointed* 2*-connected G-equivariant* 4*-types and the homotopy category of G-graded Picard categories.*

A G-graded Picard category $\mathbb{G} = (\mathbb{G}, \mathrm{gr}, \otimes, \mathrm{I}, A, R, C)$ is said to be *restricted* if its commutativity constraint satisfies

$$C_{X,X} = id_{X\otimes X}$$

for any object X of $\mathbb{G}$ (see [12,30] or [17]).

Suppose

$$\mathrm{res}\,\mathbf{Pic}_G$$

denotes the full subcategory of $\mathbf{Pic}_G$ of all restricted G-graded Picard categories.

If M and N are any two G-modules, then it is a fact that the homomorphism

$$H^3_{G,s}(M,N) \to \mathrm{Hom}_{\mathbb{Z}G}(M/2M, N)$$

in the five-term exact sequence (8) associates to the class of a symmetric 3-cocycle h the map $q_h : M/2M \to N$ given by $q_h([x]) = h(x \mid x)$, $x \in M$. Hence, its composition with the bijection $\mathbf{Pic}_G(M,N) \cong H^3_{G,s}(M,N)$ associates to any equivalence class of a G-graded Picard category $\mathbb{G}$ of type (M,N) the map $q_{\mathbb{G}} : M/2M \to N$ determined by the equality

$$q_{\mathbb{G}}\big([x]\big) \otimes id_{X\otimes X} = C_{X,X},$$

for any representative object $X \in x$ and any $x \in M = K_0\mathbb{G}$ (cf. [23, proof of Theorem 3.3]). Hence it follows from the exactness of the sequence (8) that the G-graded Picard category $\mathbb{G}$ satisfies the restricted condition if and only if its Postnikov invariant $k(\mathbb{G})$ belongs to $\mathrm{Ext}^2_{\mathbb{Z}G}(M,N)$. This means that the classifying functor cl in Theorem 3.12 restricts to a classifying functor for restricted G-graded Picard categories, that is, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{res}\,\mathbf{Pic}_G & \xrightarrow{\mathrm{cl}} & \mathcal{E}xt^2_{\mathbb{Z}G} \\ \big\downarrow & & \big\downarrow \\ \mathbf{Pic}_G & \xrightarrow{\mathrm{cl}} & \mathcal{H}^3_{G,s}, \end{array}$$

where

$$\mathcal{E}xt^2_{\mathbb{Z}G}$$

is the full subcategory of $\mathcal{H}^3_{G,s}$ whose objects are those triples (M,N,k) with $k \in \mathrm{Ext}^2_{\mathbb{Z}G}(M,N)$. Thus, as a consequence of Theorem 3.12 we obtain the following theorem, which should be ascribed to Deligne (see [12, Corollaire 1.4.17]):

Theorem 3.14 *(Deligne). There is a* classifying *functor*

$$\begin{aligned} \mathrm{cl} : \mathrm{res}\,\mathbf{Pic}_G &\to \mathcal{E}xt^2_{\mathbb{Z}G}, \\ \mathbb{G} &\mapsto (K_0\mathbb{G}, K_1\mathbb{G}, k_{\mathbb{G}}), \\ F &\mapsto (K_0F, K_1F), \end{aligned}$$

which has the following properties:

(i) *For any object* $(M,N,k) \in \mathcal{E}xt^2_{\mathbb{Z}G}$*, there exists a restricted* G*-graded Picard category* $\mathbb{G}$ *with an isomorphism* $\mathrm{cl}(\mathbb{G}) \cong (M,N,k)$.

(ii) *For any morphism* $F : \mathbb{G} \to \mathbb{H}$ *in* $\mathrm{res}\,\mathbf{Pic}_G$, $\mathrm{cl}(F)$ *is an isomorphism if and only if* F *is a graded equivalence.*

(iii) *The induced* classifying functor

$$\mathrm{cl} : \mathrm{Ho}\ \mathrm{res}\,\mathbf{Pic}_G \longrightarrow \mathcal{E}xt^2_{\mathbb{Z}G}$$

if full, but it is not faithful. Indeed, for any arrow $(p,q) : \mathrm{cl}(\mathbb{G}) \to \mathrm{cl}(\mathbb{H})$ *in* $\mathcal{E}xt^2_{\mathbb{Z}G}$, *there is a bijection*

$$\bigl\{[F] \in \mathrm{Hom}_{\mathrm{Ho}\ \mathrm{res}\,\mathbf{Pic}_G}(\mathbb{G},\mathbb{H}) \bigm| \mathrm{cl}[F] = (p,q)\bigr\} \cong \mathrm{Ext}^1_{\mathbb{Z}G}(K_0\mathbb{G}, K_1\mathbb{H}).$$

4. The nerve of a graded Picard category

As a consequence of Theorems 2.3 and 3.12 we have an equivalence of homotopy categories

$$\mathrm{Ho}\,\mathbf{Pic}_G \simeq \mathrm{Ho}\,\mathbf{S}^{BG}_{BG}(3,4).$$

The aim of this section is to realize this equivalence by means of the construction of the nerve $\mathcal{N}_G\mathbb{G}$ of a G-graded Picard category $\mathbb{G}$, which is a space endowed with a split fibration over BG representing the homotopy type of $\mathbb{G}$. Consequently, our main purpose here is to state and prove the following:

Theorem 4.1. *There is a* "nerve functor," $\mathcal{N}_G$, *such that the diagram*

$$\begin{array}{ccc} \mathbf{Pic}_G & \xrightarrow{\ \mathcal{N}_G\ } & \mathbf{S}^{BG}_{BG}(3,4) \\ & {\scriptstyle \mathrm{cl}}\searrow \quad \swarrow{\scriptstyle \mathrm{cl}} & \\ & \mathcal{H}^3_{G,s} & \end{array}$$

is commutative.

Due to its potential interest for category theorists, and to help motivate the reader about the definition of nerves of graded Picard categories given further below, we have first included a section (Section 4.1) where, without any claim to originality, we review some facts concerning nerves and higher categorical structures.

4.1. Nerves, pseudofunctors, and cocycles

Hereafter, we shall regard each ordered set $[n]$ as the category with exactly one arrow $j \to i$ if $i \leqslant j$. Then, a non-decreasing map $[n] \to [m]$ is the same as a functor, so that we see Δ, the category of finite ordinal numbers, as a full subcategory of **Cat**, the category of small categories.

In Quillen's development of K-theory, the higher K-groups are described as the homotopy groups of a topological "classifying space," $|\mathbb{G}|$, functorially associated to a small category $\mathbb{G}$, the so-called geometric realization of its (Grothendieck) nerve, $\mathrm{Ner}\,\mathbb{G}$. This nerve is an easily described simplicial set:

$$\mathrm{Ner}\,\mathbb{G} = \mathrm{Func}(-,\mathbb{G}) : \Delta^{op} \to \mathbf{Set},$$

which encodes the structure of the category $\mathbb{G}$ in terms of its faces and degeneracies. Its 0-simplices are the objects of $\mathbb{G}$ and, for $n \geqslant 1$, an n-simplex of $\mathrm{Ner}\,\mathbb{G}$ is a tuple

$$\big(Y_j \xrightarrow{f_{ij}} Y_i\big)_{0\leqslant i\leqslant j\leqslant n}$$

of morphisms of the category such that $f_{ij} f_{jk} = f_{ik}$ if $i \leqslant j \leqslant k$ and $f_{kk} = id_{Y_k}$, or equivalently a composable sequence of morphisms in $\mathbb{G}$,

$$Y_0 \xleftarrow{f_{01}} Y_1 \xleftarrow{f_{12}} \cdots \leftarrow Y_n.$$

When a group G is regarded as a category with only one object, then $\mathrm{Ner}\,G = BG$, the classifying minimal complex of the group.

Suppose now that $\mathbb{G}:\mathcal{B} \rightsquigarrow \mathbf{Cat}$, $(i \xrightarrow{\sigma} j) \mapsto (\mathbb{G}_i \xrightarrow{\sigma(\,)} \mathbb{G}_j)$, is any given (strictly unitary) pseudofunctor [21], where $\mathcal{B}$ is a small category. Then, a "*crossed functor*" or "1-*cocycle*" of $\mathcal{B}$ with coefficients in $\mathbb{G}$ is a system of data

$$(Y, f)$$

consisting of:

- for each object $i \in \mathcal{B}$, an object $Y_i \in \mathbb{G}_i$;
- for each arrow $i \xrightarrow{\sigma} j$ in $\mathcal{B}$, a morphism ${}^{\sigma}Y_i \xrightarrow{f_\sigma} Y_j$ in $\mathbb{G}_j$;

such that the following two conditions are satisfied:

- for any pair of composable morphisms in $\mathcal{B}$, $i \xrightarrow{\tau} j \xrightarrow{\sigma} k$, the diagram in $\mathbb{G}_k$

$$\begin{array}{ccc} {}^{\sigma}({}^{\tau}Y_i) & \xrightarrow{{}^{\sigma}f_\tau} & {}^{\sigma}Y_j \\ \wr\downarrow & & \downarrow f_\sigma \\ {}^{\sigma\tau}Y_i & \xrightarrow{f_{\sigma\tau}} & Y_k \end{array}$$

  is commutative (i.e., $f_{\sigma\tau} = f_\sigma \cdot {}^{\sigma}f_\tau$ up to canonical isomorphism);
- for any object k of $\mathcal{B}$, $f_{id_k} = id_{Y_k}$;

and the "*nerve of the pseudofunctor*" $\mathbb{G}:\mathcal{B} \rightsquigarrow \mathbf{Cat}$ is the simplicial set defined by

$$\mathrm{Ner}_{\mathcal{B}}\,\mathbb{G}:[n] \mapsto \bigcup_{\sigma:[n]\to\mathcal{B}} Z^1\big([n], \sigma^*\mathbb{G}\big), \tag{63}$$

where $\sigma:[n] \to \mathcal{B}$ is any functor, that is, an n-simplex of $\mathrm{Ner}\,\mathcal{B}$, and $Z^1([n], \sigma^*\mathbb{G})$ is the set of 1-cocycles of $[n]$ with coefficients in the pseudofunctor

$$[n] \xrightarrow{\sigma} \mathcal{B} \overset{\mathbb{G}}{\rightsquigarrow} \mathbf{Cat},$$

obtained by composition of $\mathbb{G}$ with σ.

Note that there is a natural bijection

$$Z^1(\mathcal{B},\mathbb{G}) \cong \mathrm{Hom}_{\mathbf{S}_{\mathrm{Ner}\,\mathcal{B}}}(\mathrm{Ner}\,\mathcal{B}, \mathrm{Ner}_{\mathcal{B}}\,\mathbb{G}),$$

between the set of 1-cocycles of $\mathcal{B}$ in $\mathbb{G}$ and the set of cross sections of the natural projection map $\mathrm{Ner}_{\mathcal{B}}\,\mathbb{G} \to \mathrm{Ner}\,\mathcal{B}$, $(\sigma, Y, f) \mapsto \sigma$.

It is a consequence of Thomason's homotopy colimit theorem [33] that the nerve (63) of a pseudodiagram of categories actually represents its homotopy type. To be more explicit here, recall that the "*Grothendieck construction*" on a pseudofunctor $\mathbb{G} : \mathcal{B} \rightsquigarrow \mathbf{Cat}$,

$$\int_{\mathcal{B}} \mathbb{G}, \tag{64}$$

is the category whose objects are pairs (X, i), where i is an object of $\mathcal{B}$ and X is one of $\mathbb{G}_i$; a morphism $(g,\tau) : (X,i) \to (Y,j)$ in $\int_{\mathcal{B}}\mathbb{G}$ is a pair of morphisms where $\tau : i \to j$ in $\mathcal{B}$ and $g : {}^{\tau}X \to Y$ in $\mathbb{G}_j$. The composite of (g,τ) with the morphism $(f,\sigma) : (Y,j) \to (Z,k)$ of $\int_{\mathcal{B}}\mathbb{G}$ is defined to be the morphism $(f \star g, \sigma\tau) : (X,i) \to (Z,k)$, where $f \star g$ is the composite

$${}^{\sigma\tau}X \xrightarrow{\sim} {}^{\sigma}\big({}^{\tau}X\big) \xrightarrow{{}^{\sigma}g} {}^{\sigma}Y \xrightarrow{f} Z$$

of $\mathbb{G}_k$. Then, a straightforward comparison shows the following:

Proposition 4.2. *For any pseudofunctor* $\mathbb{G} : \mathcal{B} \rightsquigarrow \mathbf{Cat}$,

$$\mathrm{Ner}_{\mathcal{B}}\,\mathbb{G} = \mathrm{Ner} \int_{\mathcal{B}} \mathbb{G}.$$

Now, the categories $\mathbb{G}$ that arise here often also have a tensor product-like multiplication on them $\otimes : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ which makes them into what is called a *tensor* (or monoidal) category $\mathbb{G} = (\mathbb{G}, \otimes, \mathrm{I}, A, L, R)$ [25]. The associated classifying space,

$$|\mathbb{G}|^t,$$

to such a tensor category is very interesting to take into account since it is a "delooping" of the classifying space of the underlying category $|\mathbb{G}|$, in the sense that the group completion of $|\mathbb{G}|$ is homotopy equivalent to $\Omega|\mathbb{G}|^t$, the loop space of $|\mathbb{G}|^t$.

The classifying space of a tensor category is more complicated to describe than the classifying space of the underlying category since, unfortunately, the tensor product involved is in general not associative and not unitary, but it is only so up to coherent isomorphisms. This all too frequently occurring "defect" has the effect of taking what would be the natural bisimplicial structure of a simplicial category and forcing one to deal with defining the geometrical realization of what is not simplicial but only "simplicial up to isomorphisms," the pseudo-simplicial category

$$\overline{W}\mathbb{G} : \Delta^{op} \rightsquigarrow \mathbf{Cat}, \quad [n] \mapsto \mathbb{G}^n,$$

that $\mathbb{G}$ defines by the familiar reduced bar construction. Thus, following Thomason [33] (see also [22]), the classifying space $|\mathbb{G}|^t$ can be defined through two different but homotopy-equivalent constructions: on one hand, $|\mathbb{G}|^t$ can be defined as the classifying space of the "rectified" simplicial category obtained by applying Street's first construction [31] to the pseudo-simplicial category $\overline{W}\mathbb{G}$; on the other hand, one can consider the Grothendieck construction (64) on $\overline{W}\mathbb{G}$ and then define $|\mathbb{G}|^t$ as the classifying space of the resulting category. In any case, this process is quite indirect and the CW-complex thus obtained has many cells and scant apparent intuitive connection with the original tensor category.

There is, however, a genuine simplicial set, $\mathrm{Ner}^t\,\mathbb{G}$, due to Duskin and Street, associated with any tensor category (and, more generally, with any weak 2-category or bicategory) called its "*geometrical nerve*" [14], whose definition is very natural: Since any tensor category $\mathbb{G}$ can be considered as a weak 2-category with only one object [32, Example 2] and any category is a weak 2-category whose 2-cells are all identities, it makes complete sense to consider strictly unitary lax functors from the categories defined by the ordered sets $[n]$ to $\mathbb{G}$. Thus, the geometric nerve of the tensor category is defined as the simplicial set

$$\mathrm{Ner}^t\,\mathbb{G} = \text{lax-Func}\big(-, \mathbb{G}^t\big) : \Delta^{op} \to \mathbf{Set}, \tag{65}$$

where $\mathbb{G}^t$ denotes the tensor category $\mathbb{G}$, regarded as a weak 2-category with only one object. Its n-simplices are therefore the (strictly unitary) lax functors $[n] \rightsquigarrow \mathbb{G}^t$.

This simplicial set $\mathrm{Ner}^t\,\mathbb{G}$ completely encodes all the structure of the tensor category and, in fact, tensor categories are effectively embedded in simplicial sets by their geometric nerves [14, §2]. Moreover, in [5] it is proved that the geometric realization of Duskin and Street's nerve of a tensor category $\mathbb{G}$ is indeed a "correct" simplicial set model for its homotopy type. That is, there is a homotopy equivalence

$$|\mathbb{G}|^t \simeq \big|\mathrm{Ner}^t\,\mathbb{G}\big|.$$

Suppose now that $\mathbb{G} : \mathcal{B} \rightsquigarrow \mathbf{MonCat}$, $(i \xrightarrow{\sigma} j) \mapsto (\mathbb{G}_i \xrightarrow{{}^{\sigma}(\,)} \mathbb{G}_j)$, is a (strictly unitary) pseudofunctor of tensor categories; that is, a pseudofunctor where each category $\mathbb{G}_i$ has a tensor structure, each functor ${}^{\sigma}(\,)$ is a (strictly unitary) tensor functor and, for any two composable arrows in $\mathcal{B}$, $i \xrightarrow{\tau} j \xrightarrow{\sigma} k$, the associated natural equivalence ${}^{\sigma}({}^{\tau}(\,)) \xrightarrow{\sim} {}^{\sigma\tau}(\,)$ is an isomorphism of tensor functors. Then, a "2-*cocycle*" of $\mathcal{B}$ in $\mathbb{G}$ is a system of data

$$(Y, f)$$

consisting of:

– for each arrow $i \xrightarrow{\sigma} j$ in $\mathcal{B}$, an object $Y_\sigma \in \mathbb{G}_j$;
– for each pair of composable arrows in $\mathcal{B}$, $i \xrightarrow{\tau} j \xrightarrow{\sigma} k$, a morphism in $\mathbb{G}_k$

$$Y_\sigma \otimes {}^{\sigma}Y_\tau \xrightarrow{f_{\sigma,\tau}} Y_{\sigma\tau};$$

such that the following two conditions are satisfied:

– for any three composable arrows in $\mathcal{B}$, $i \xrightarrow{\gamma} j \xrightarrow{\tau} k \xrightarrow{\sigma} l$, the diagram in $\mathbb{G}_l$

$$\begin{array}{ccccc}
Y_\sigma \otimes {}^\sigma(Y_\tau \otimes {}^\tau Y_\gamma) & \xrightarrow{\sim} & Y_\sigma \otimes ({}^\sigma Y_\tau \otimes {}^{\sigma\tau} Y_\gamma) & \xrightarrow{\sim} & (Y_\sigma \otimes {}^\sigma Y_\tau) \otimes {}^{\sigma\tau} Y_\gamma \\
{\scriptstyle id\otimes {}^\sigma f_{\tau,\gamma}} \downarrow & & & & \downarrow {\scriptstyle f_{\sigma,\tau}\otimes id} \\
Y_\sigma \otimes {}^\sigma Y_{\tau\gamma} & \xrightarrow{f_{\sigma,\tau\gamma}} & Y_{\sigma\tau\gamma} & \xleftarrow{f_{\sigma\tau,\gamma}} & Y_{\sigma,\tau} \otimes {}^{\sigma\tau} Y_{\sigma\tau}
\end{array}$$

(where the unnamed isomorphisms are canonical) is commutative;

– $Y_{id} = I$, $f_{id,\sigma} = L : I \otimes Y_\sigma \xrightarrow{\sim} Y_\sigma$, $f_{\sigma,id} = R : Y_\sigma \otimes I \xrightarrow{\sim} Y_\sigma$.

The “*nerve of the pseudofunctor*” $\mathbb{G} : \mathcal{B} \rightsquigarrow \mathbf{MonCat}$ is then defined to be the simplicial set

$$\mathrm{Ner}^t_{\mathcal{B}}\, \mathbb{G} : [n] \mapsto \bigcup_{\sigma : [n] \to \mathcal{B}} Z^2\big([n], \sigma^*\mathbb{G}\big), \tag{66}$$

where $\sigma : [n] \to \mathcal{B}$ is any functor and $Z^2([n], \sigma^*\mathbb{G})$ is the set of 2-cocycles of $[n]$ in the composite pseudofunctor

$$[n] \xrightarrow{\sigma} \mathcal{B} \overset{\mathbb{G}}{\rightsquigarrow} \mathbf{MonCat}.$$

Note that, when the pseudofunctor is a constant tensor category $\mathbb{G}$, then a 2-cocycle with coefficients in $\mathbb{G}$ is simply a lax functor to the weak 2-category $\mathbb{G}^t$, so that we have the equality $\mathrm{Ner}^t_{\mathcal{B}}\, \mathbb{G} = \mathrm{Ner}\, \mathcal{B} \times \mathrm{Ner}^t\, \mathbb{G}$. Moreover, and similarly what occurs for 1-cocycles, for any pseudofunctor $\mathbb{G} : \mathcal{B} \rightsquigarrow \mathbf{MonCat}$, there is a natural bijection

$$Z^2(\mathcal{B}, \mathbb{G}) \cong \mathrm{Hom}_{\mathbf{S}_{\mathrm{Ner}\,\mathcal{B}}}\big(\mathrm{Ner}\, \mathcal{B}, \mathrm{Ner}^t_{\mathcal{B}}\, \mathbb{G}\big),$$

between the set of 2-cocycles of $\mathcal{B}$ in $\mathbb{G}$ and the set of cross sections of the natural projection map $\mathrm{Ner}^t_{\mathcal{B}}\, \mathbb{G} \to \mathrm{Ner}\, \mathcal{B}$, $(\sigma, Y, f) \mapsto \sigma$.

Geometric nerves of categorical groups, that is, of tensor categories $\mathbb{G}$ in which every arrow is invertible and every object has a quasi-inverse with respect to the tensor product, have been studied in detail in [8,10]. Nerves of pseudofunctors of categorical groups have been studied in [9], where 2-cocycles in the above sense arise as a factor set theory for categorical torsors.

When a tensor category $\mathbb{G}$ is enriched with a braiding $\{C_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}$ [23], then its classifying space $|\mathbb{G}|^t$ can be delooped again. This fact, due to Stasheff and Fiedorowicz (see [3, Theorem 1.2] for a proof, and [2] for a more general result), can be easily proved in the particular case in which $\mathbb{G}$ is a braided categorical group by using the notion of “*nerve of a braided tensor category*,” $\mathrm{Ner}^b\, \mathbb{G}$, such as was done in [3] (for the strict case) or in [8]. This nerve construction gives a functor that fully embeds the category of braided tensor categories into the category of simplicial sets, and it has an entirely natural definition: as a braided tensor category, $\mathbb{G}$, can be regarded as a one-object, one-arrow weak 3-category (or tricategory) [20, Corollary 8.7] and each category as a weak 3-category whose 2-cells and 3-cells are all identities, one can consider strictly unitary lax functors from the categories $[n]$ to the weak 3-category $\mathbb{G}^b$ that the braided tensor category $\mathbb{G}$ defines. Thus, its nerve is the simplicial set

$$\mathrm{Ner}^b\, \mathbb{G} = \text{lax-Func}\big(-, \mathbb{G}^b\big) : \Delta^{op} \to \mathbf{Set}, \tag{67}$$

whose n-simplices are therefore all the (strictly unitary) lax functors $[n] \rightsquigarrow \mathbb{G}^b$.

In the particular case of braided categorical groups, we have the following known facts (see [8, Propositions 2.8, 2.9 and 2.10], [3, Propositions 2.10, 2.11 and Theorem 3.3]):

– For any braided categorical group $\mathbb{G}$, $\mathrm{Ner}^b\,\mathbb{G}$ is a 4-skeletal 1-reduced Kan complex. There is a natural isomorphism

$$\Omega\,\mathrm{Ner}^b\,\mathbb{G} \cong \mathrm{Ner}^t\,\mathbb{G},$$

where $\mathrm{Ner}^t\,\mathbb{G}$ is the nerve of the underlying categorical group.

– The functor $\mathbb{G} \mapsto \mathrm{Ner}^b\,\mathbb{G}$ induces an equivalence between the homotopy category of braided categorical groups and the homotopy category of pointed 1-connected 3-types.

Let us suppose now that $\mathbb{G} : \mathcal{B} \rightsquigarrow \mathbf{BrMonCat}$, $(i \xrightarrow{\sigma} j) \mapsto (\mathbb{G}_i \xrightarrow{\sigma(\,)} \mathbb{G}_j)$, is a (strictly unitary) pseudofunctor of braided tensor categories. Then, a "3-*cocycle*" of $\mathcal{B}$ with coefficients in $\mathbb{G}$ is a system of data

$$(Y, f)$$

consisting of:

– for each two composable arrows in $\mathcal{B}$, $i \xrightarrow{\tau} j \xrightarrow{\sigma} k$, an object $Y_{\sigma,\tau} \in \mathbb{G}_k$;
– for each three composable arrows in $\mathcal{B}$, $i \xrightarrow{\gamma} j \xrightarrow{\tau} k \xrightarrow{\sigma} l$, a morphism in $\mathbb{G}_l$

$$Y_{\sigma,\tau\gamma} \otimes {}^{\sigma}Y_{\tau,\gamma} \xrightarrow{f_{\sigma,\tau,\gamma}} Y_{\sigma\tau,\gamma} \otimes Y_{\sigma,\tau};$$

such that the following two conditions hold:

– for any four composable arrows in $\mathcal{B}$, $i \xrightarrow{\delta} j \xrightarrow{\gamma} k \xrightarrow{\tau} l \xrightarrow{\sigma} m$, the diagram in $\mathbb{G}_m$ below commutes.

$$Y_{\sigma,\tau\gamma\delta} \otimes {}^{\sigma}(Y_{\tau,\gamma\delta} \otimes {}^{\tau}Y_{\gamma,\delta}) \xrightarrow{\sim} Y_{\sigma,\tau\gamma\delta} \otimes {}^{\sigma}Y_{\tau,\gamma\delta} \otimes {}^{\sigma}({}^{\tau}Y_{\gamma,\delta}) \xrightarrow{\sim} Y_{\sigma,\tau\gamma\delta} \otimes {}^{\sigma}Y_{\tau,\gamma\delta} \otimes {}^{\sigma\tau}Y_{\gamma,\delta} \xrightarrow{f_{\sigma,\tau,\gamma\delta}\otimes id} Y_{\sigma\tau,\gamma\delta} \otimes Y_{\sigma,\tau} \otimes {}^{\sigma\tau}Y_{\gamma,\delta} \xrightarrow{(f_{\sigma\tau,\gamma,\delta}\otimes id)(id\otimes C)} Y_{\sigma\tau\gamma,\delta} \otimes Y_{\sigma\tau,\gamma} \otimes Y_{\sigma,\tau}$$

$$Y_{\sigma,\tau\gamma\delta} \otimes {}^{\sigma}(Y_{\tau,\gamma\delta} \otimes {}^{\tau}Y_{\gamma,\delta}) \xrightarrow{id\otimes {}^{\sigma}f_{\tau,\gamma,\delta}} Y_{\sigma,\tau\gamma\delta} \otimes {}^{\sigma}(Y_{\tau\gamma,\delta} \otimes Y_{\tau,\gamma}) \xrightarrow{\wr} Y_{\sigma,\tau\gamma\delta} \otimes {}^{\sigma}Y_{\tau\gamma,\delta} \otimes {}^{\sigma}Y_{\tau,\gamma} \xrightarrow{f_{\sigma,\tau\gamma,\delta}\otimes id} Y_{\sigma\tau\gamma,\delta} \otimes Y_{\sigma,\tau\gamma} \otimes {}^{\sigma}Y_{\tau,\gamma} \xrightarrow{id\otimes f_{\sigma,\tau,\gamma}} Y_{\sigma\tau\gamma,\delta} \otimes Y_{\sigma\tau,\gamma} \otimes Y_{\sigma,\tau}$$

(where the unnamed isomorphisms are canonical and we have omitted the associativity constraints);

- $Y_{id,\sigma} = I = Y_{\sigma,id}$, $f_{id,\sigma,\tau} = C_{I,Y_{\sigma,\tau}}$, $f_{\sigma,id,\tau} = id_{Y_{\sigma,\tau}\otimes I}$ and $f_{\sigma,\tau,id} = C_{Y_{\sigma,\tau},I}$.

We write $Z^3(\mathcal{B},\mathbb{G})$ for the set of these 3-cocycles on $\mathcal{B}$ with coefficients in the pseudofunctor $\mathbb{G}:\mathcal{B} \rightsquigarrow \mathbf{BrMonCat}$. The "nerve" of $\mathbb{G}$ is defined to be the simplicial set

$$\mathrm{Ner}^b_{\mathcal{B}}\,\mathbb{G}:[n] \mapsto \bigcup_{\sigma:[n]\to\mathcal{B}} Z^3\big([n],\sigma^*\mathbb{G}\big), \tag{68}$$

where $\sigma:[n]\to\mathcal{B}$ is any functor and $\sigma^*\mathbb{G}:[n] \rightsquigarrow \mathbf{BrMonCat}$ is the pseudofunctor obtained by composition of $\mathbb{G}$ with σ.

So defined, it follows that nerves represent 3-cocycles in the sense that there is a natural bijection

$$Z^3(\mathcal{B},\mathbb{G}) \cong \mathrm{Hom}_{\mathbf{S}_{\mathrm{Ner}\,\mathcal{B}}}\big(\mathrm{Ner}\,\mathcal{B}, \mathrm{Ner}^b_{\mathcal{B}}\,\mathbb{G}\big),$$

between the set of 3-cocycles of $\mathcal{B}$ in $\mathbb{G}$ and the set of cross sections of the natural projection map $\mathrm{Ner}^b_{\mathcal{B}}\,\mathbb{G} \to \mathrm{Ner}\,\mathcal{B}$, $(\sigma, Y, f) \mapsto \sigma$.

As a last stage, suppose that $\mathbb{G}$ is now a symmetric tensor category. This is a very interesting case in which the classifying space of the underlying category is an infinite loop space up to group completion [27]. When $\mathbb{G}$ is a strict Picard category, then the simplicial Ω-spectrum defined by $\mathbb{G}$

$$\mathrm{Ner}\,\mathbb{G}, \mathrm{Ner}^t\,\mathbb{G}, \mathrm{Ner}^b\,\mathbb{G}, \mathrm{Ner}^s\,\mathbb{G}, \ldots$$

has a handle description by means of the construction of the n-nerves of $\mathbb{G}$, $n \geqslant 1$, as was done in [6]. In fact, these n-nerves can be defined as the simplicial nerves of the weak n-categories with only one i-cell for all $i \leqslant n-2$ that $\mathbb{G}$ defines. Although, we should say that there does not yet exist an axiomatization for a definitive definition of weak n-category if $n \geqslant 4$. However, beyond all reasonable doubt, it is a fact that any of the various proposed definitions of weak n-category reduces to a symmetric tensor category, whenever it has only one i-cell for all $i \leqslant n-2$ (see [1]). In this paper we are only interested in the case $n = 4$, and we use the unpublished but rigorous definition of tetracategory given by Trimble in [34], so that we shall consider as a "*nerve of a symmetric tensor category*" $\mathbb{G}$ the simplicial set

$$\mathrm{Ner}^s\,\mathbb{G} = \text{lax-Func}\big(-, \mathbb{G}^s\big):\Delta^{op} \to \mathbf{Set}, \tag{69}$$

whose n-simplices are therefore all the (strictly unitary) lax functors $[n] \rightsquigarrow \mathbb{G}^s$, from the weak 4-category $[n]$, whose 2-, 3- and 4-cells are all identities, to the weak 4-category $\mathbb{G}^s$, with only one 0-, 1- and 2-cells, that $\mathbb{G}$ defines.

If $\mathbb{G}:\mathcal{B} \rightsquigarrow \mathbf{SymMonCat}$, $(i \xrightarrow{\sigma} j) \mapsto (\mathbb{G}_i \xrightarrow{\sigma()} \mathbb{G}_j)$, is a (strictly unitary) pseudofunctor of symmetric tensor categories, then a "4-*cocycle*" of $\mathcal{B}$ in $\mathbb{G}$ is a system of data

$$(Y, f)$$

consisting of:

– for each three composable arrows in $\mathcal{B}$, $i \xrightarrow{\gamma} j \xrightarrow{\tau} k \xrightarrow{\sigma} l$, an object $Y_{\sigma,\tau,\gamma} \in \mathbb{G}_l$;
– for each four composable arrows in $\mathcal{B}$, $i \xrightarrow{\delta} j \xrightarrow{\gamma} k \xrightarrow{\tau} l \xrightarrow{\sigma} u$, a morphism in $\mathbb{G}_u$

$$Y_{\sigma,\tau,\gamma} \otimes \left(Y_{\sigma,\tau\gamma,\delta} \otimes {}^{\sigma}Y_{\tau,\gamma,\delta}\right) \xrightarrow{f_{\sigma,\tau,\gamma,\delta}} Y_{\sigma\tau,\gamma,\delta} \otimes Y_{\sigma,\tau,\gamma\delta};$$

such that the following two conditions hold:

– for any five composable arrows in $\mathcal{B}$, $i \xrightarrow{\eta} j \xrightarrow{\delta} k \xrightarrow{\gamma} l \xrightarrow{\tau} u \xrightarrow{\sigma} v$, the diagram in $\mathbb{G}_v$ below commutes.

$$\begin{array}{c}
Y_{\sigma,\tau,\gamma} \otimes Y_{\sigma,\tau\gamma,\delta} \otimes Y_{\sigma,\tau\gamma\delta,\eta} \otimes {}^{\sigma}(Y_{\tau,\gamma,\delta} \otimes Y_{\tau,\gamma\delta,\eta} \otimes {}^{\tau}Y_{\gamma,\delta,\eta}) \\
\xrightarrow{id\otimes {}^{\sigma} f_{\tau,\gamma,\delta,\eta}} Y_{\sigma,\tau,\gamma} \otimes Y_{\sigma,\tau\gamma,\delta} \otimes Y_{\sigma,\tau\gamma\delta,\eta} \otimes {}^{\sigma}(Y_{\tau,\gamma\delta,\eta} \otimes Y_{\tau,\gamma,\delta\eta}) \\
\xrightarrow{\sim} Y_{\sigma,\tau,\gamma} \otimes Y_{\sigma,\tau\gamma,\delta} \otimes Y_{\sigma,\tau\gamma\delta,\eta} \otimes {}^{\sigma}Y_{\tau,\gamma\delta,\eta} \otimes {}^{\sigma}Y_{\tau,\gamma,\delta\eta} \\
\xrightarrow{id\otimes f_{\sigma,\tau\gamma,\delta,\eta}\otimes id} Y_{\sigma,\tau,\gamma} \otimes Y_{\sigma\tau\gamma,\delta,\eta} \otimes Y_{\sigma,\tau\gamma,\delta\eta} \otimes {}^{\sigma}Y_{\tau,\gamma,\delta\eta} \\
\xrightarrow{(id\otimes f_{\sigma,\tau,\gamma,\delta\eta})(C\otimes id)} Y_{\sigma\tau\gamma,\delta,\eta} \otimes Y_{\sigma\tau,\gamma,\delta\eta} \otimes Y_{\sigma,\tau,\gamma\delta\eta}
\end{array}$$

$$\begin{array}{c}
Y_{\sigma,\tau,\gamma} \otimes Y_{\sigma,\tau\gamma,\delta} \otimes Y_{\sigma,\tau\gamma\delta,\eta} \otimes {}^{\sigma}(Y_{\tau,\gamma,\delta} \otimes Y_{\tau,\gamma\delta,\eta} \otimes {}^{\tau}Y_{\gamma,\delta,\eta}) \\
\xrightarrow{\sim} Y_{\sigma,\tau,\gamma} \otimes Y_{\sigma,\tau\gamma,\delta} \otimes Y_{\sigma,\tau\gamma\delta,\eta} \otimes {}^{\sigma}Y_{\tau,\gamma,\delta} \otimes {}^{\sigma}Y_{\tau,\gamma\delta,\eta} \otimes {}^{\sigma\tau}Y_{\gamma,\delta,\eta} \\
\xrightarrow{(f_{\sigma,\tau,\gamma,\delta}\otimes id)(id\otimes C\otimes id)} Y_{\sigma\tau,\gamma,\delta} \otimes Y_{\sigma,\tau\gamma\delta} \otimes Y_{\sigma,\tau\gamma\delta,\eta} \otimes {}^{\sigma}Y_{\tau,\gamma\delta,\eta} \otimes {}^{\sigma\tau}Y_{\gamma,\delta,\eta} \\
\xrightarrow{id\otimes f_{\sigma,\tau,\gamma\delta,\eta}\otimes id} Y_{\sigma\tau,\gamma,\delta} \otimes Y_{\sigma\tau,\gamma\delta,\eta} \otimes Y_{\sigma,\tau,\gamma\delta\eta} \otimes {}^{\sigma\tau}Y_{\gamma,\delta,\eta} \\
\xrightarrow{(f_{\sigma\tau,\gamma,\delta,\eta}\otimes id)(id\otimes C)} Y_{\sigma\tau\gamma,\delta,\eta} \otimes Y_{\sigma\tau,\gamma,\delta\eta} \otimes Y_{\sigma,\tau,\gamma\delta\eta}
\end{array}$$

(where the unnamed isomorphisms are canonical and we have omitted the associativity constraints);
– $Y_{id,\tau,\gamma} = I = Y_{\sigma,id,\gamma} = Y_{\sigma,\tau,id}$,

$$f_{id,\tau,\gamma,\delta} = C(id \otimes RC): I \otimes (I \otimes Y_{\tau,\gamma,\delta}) \to Y_{\tau,\gamma,\delta} \otimes I,$$
$$f_{\sigma,id,\gamma,\delta} = C(id \otimes R): I \otimes (Y_{\sigma,\gamma,\delta} \otimes I) \to Y_{\sigma,\gamma,\delta} \otimes I,$$
$$f_{\sigma,\tau,id,\delta} = id \otimes R: I \otimes (Y_{\sigma,\tau,\delta} \otimes I) \to I \otimes Y_{\sigma,\tau,\delta},$$
$$f_{\sigma,\tau,\gamma,id} = C(id \otimes R): Y_{\sigma,\tau,\gamma} \otimes (I \otimes I) \to I \otimes Y_{\sigma,\tau,\gamma}.$$

The "*nerve of the pseudofunctor*" $\mathbb{G}:\mathcal{B}\rightsquigarrow$ **SymMonCat** is then defined to be the simplicial set

$$\mathrm{Ner}^s_{\mathcal{B}}\,\mathbb{G}:[n]\mapsto \bigcup_{\sigma:[n]\to\mathcal{B}} Z^4\big([n],\sigma^*\mathbb{G}\big), \tag{70}$$

where $\sigma:[n]\to\mathcal{B}$ is any functor and $Z^4([n],\sigma^*\mathbb{G})$ is the set of 4-cocycles of $[n]$ in the composite pseudofunctor

$$[n]\xrightarrow{\sigma}\mathcal{B}\overset{\mathbb{G}}{\rightsquigarrow}\textbf{SymMonCat}.$$

Note the natural bijection

$$Z^4(\mathcal{B},\mathbb{G})\cong \mathrm{Hom}_{\mathbf{S}_{\mathrm{Ner}\,\mathcal{B}}}\big(\mathrm{Ner}\,\mathcal{B},\mathrm{Ner}^s_{\mathcal{B}}\,\mathbb{G}\big), \tag{71}$$

between the set of 4-cocycles of $\mathcal{B}$ in $\mathbb{G}$ and the set of cross sections of the natural projection map $\mathrm{Ner}^s_{\mathcal{B}}\,\mathbb{G}\to\mathrm{Ner}\,\mathcal{B}$, $(\sigma,Y,f)\mapsto\sigma$.

When the category $\mathcal{B}$ is a group, say G, and a pseudofunctor $\mathbb{G}$ defined on it associates a restricted Picard category (i.e., when the symmetry satisfies the equalities $C_{X,X}=id$) to the unique object of G, then $\mathbb{G}$ is what is called a category with a coherent G-module structure. In this context, the above-considered associated sets of cocycles $Z^n(G,\mathbb{G})$, $1\leqslant n\leqslant 4$, also arise in Ulbrich cohomology theory for (restricted) Picard categories [35] and, after [36, Theorem (2.2)], as well as in Fröhlich and Wall cohomology for strictly coherent group-like G-monoidal categories [17, §7].

The simplicial nerves of pseudofunctors of Picard categories $G\rightsquigarrow$ **Pic** will be studied in some detail in the next subsection.

4.2. *Simplicial nerves of graded Picard categories*

The Grothendieck correspondence between pseudofunctors on a category and cofibrations defined over itself [21] underlies an equivalence between the category of G-graded Picard categories and the category of ordinary Picard categories equipped with a pseudo-action of the group G by symmetric tensor self equivalences, that is, of pseudofunctors $G\rightsquigarrow$ **Pic**. This equivalence has been made explicit by Ulbrich in [36] (although in the particular setting of restrictive Picard categories), and it works as follows:

Let $\mathbb{G}=(\mathbb{G},\mathrm{gr},\otimes,\mathrm{I},A,R,C)$ be any given G-graded Picard category. Since the grading is stable (i.e., a Grothendieck cofibration), we can choose a *normalized cocleavage* for it, that is, a system

$$\Gamma=\big(X\xrightarrow{\Gamma_{\sigma,X}}{}^{\sigma}X\big), \tag{72}$$

consisting of a σ-morphism $\Gamma_{\sigma,X}:X\to{}^{\sigma}X$, for any object X of $\mathbb{G}$ and $\sigma\in G$. Specifically, we choose

$$\Gamma_{e,X}=id_X,\qquad \Gamma_{\sigma,\mathrm{I}}=\mathrm{I}(\sigma):\mathrm{I}\to\mathrm{I}\quad (X\in\mathrm{Ob}\,\mathbb{G},\ \sigma\in G).$$

Using this cocleavage, $\mathbb{G}$ determines a normalized pseudofunctor

$$\mathbb{G}_e : G \rightsquigarrow \mathbf{Pic} \tag{73}$$

that associates $\mathbb{G}_e = \operatorname{Ker}\mathbb{G} \subseteq \mathbb{G}$, the subcategory of morphisms of trivial degree with its inherited Picard structure, to the unique object of G. For any $\sigma \in G$, the (strictly unitary) symmetric tensor equivalence

$$^{\sigma}(-) : \mathbb{G}_e \to \mathbb{G}_e,$$

carries each e-morphism of $f : X \to Y$ to the e-morphism $^{\sigma}f : {}^{\sigma}X \to {}^{\sigma}Y$ defined by the composition

$$^{\sigma}X \xrightarrow{\Gamma^{-1}_{\sigma,X}} X \xrightarrow{f} Y \xrightarrow{\Gamma_{\sigma,Y}} {}^{\sigma}Y,$$

and on which the e-morphisms $^{\sigma}X \otimes {}^{\sigma}Y \to {}^{\sigma}(X \otimes Y)$ are those given by the compositions

$$^{\sigma}X \otimes {}^{\sigma}Y \xrightarrow{\Gamma^{-1}_{\sigma,X} \otimes \Gamma^{-1}_{\sigma,Y}} X \otimes Y \xrightarrow{\Gamma_{\sigma,X\otimes Y}} {}^{\sigma}(X \otimes Y).$$

Note that $^{\sigma}\mathrm{I} = \mathrm{I}$ for all $\sigma \in G$ and $^{e}(\,) = id_{\mathbb{G}_e}$. For any two elements $\sigma, \tau \in G$, the symmetric tensor natural equivalence

$$^{\sigma}\big({}^{\tau}(\,)\big) \simeq {}^{\sigma\tau}(\,) \tag{74}$$

is defined by the compositions

$$^{\sigma}\big({}^{\tau}X\big) \xrightarrow{\Gamma^{-1}_{\sigma,{}^{\tau}X}} {}^{\tau}X \xrightarrow{\Gamma^{-1}_{\tau,X}} X \xrightarrow{\Gamma_{\sigma\tau,X}} {}^{\sigma\tau}X.$$

In the other direction, any given pseudofunctor, say $\mathbb{G}_e : G \rightsquigarrow \mathbf{Pic}$, $* \mapsto \mathbb{G}_e$, as above, determines a G-graded Picard category

$$\int_G \mathbb{G}_e = \Big(\int_G \mathbb{G}_e, \mathrm{gr}, \otimes, \mathrm{I}, A, R, C\Big), \tag{75}$$

in which category $\int_G \mathbb{G}$ is the Grothendieck construction (64) on the pseudofunctor, so that $\operatorname{Ob}\int_G \mathbb{G}_e = \operatorname{Ob}\mathbb{G}_e$ and a morphism $(f,\sigma) : X \to Y$ is a pair where $\sigma \in G$ and $f : {}^{\sigma}X \to Y$ is a morphism in $\mathbb{G}_e$; the degree of (f,σ) is σ and the graded tensor product of (f,σ) with the morphism $(f',\sigma) : X' \to Y'$ is defined to be the morphism $(f \,\square\, f', \sigma) : X \otimes X' \to Y \otimes Y'$, where $f \square f'$ is the composite

$$^{\sigma}(X \otimes X') \xrightarrow{\sim} {}^{\sigma}X \otimes {}^{\sigma}X' \xrightarrow{f \otimes f'} Y \otimes Y'.$$

The graded functor $\mathrm{I} : G \to \int_G \mathbb{G}$ is defined by $\mathrm{I}(\sigma) = (id, \sigma) : \mathrm{I} \to \mathrm{I}$, and the associativity, symmetry and unit constraints are given by the corresponding ones of $\mathbb{G}_e$, that is, (A, e), (C, e) and (R, e), respectively.

We are now ready to introduce our main simplicial construction:

Definition 4.3. The nerve of a G-graded Picard category $\mathbb{G}$, denoted by $\mathcal{N}_G\mathbb{G}$, is the nerve of its associated normalized pseudofunctor $\mathbb{G}_e : G \rightsquigarrow \mathbf{Pic}$ (73), that is,

$$\mathcal{N}_G\mathbb{G} = \mathrm{Ner}^s_G\, \mathbb{G}_e : \Delta^{op} \to \mathbf{Set}$$

is the simplicial set defined by (see (70))

$$[n] \mapsto \bigcup_{\sigma : [n] \to G} Z^4\big([n], \sigma^*\mathbb{G}_e\big),$$

where $\sigma : [n] \to G$ is any functor, that is, an n-simplex of BG, and $Z^4([n], \sigma^*\mathbb{G}_e)$ is the set of 4-cocycles of $[n]$ in the composite pseudofunctor

$$[n] \xrightarrow{\sigma} G \overset{\mathbb{G}_e}{\rightsquigarrow} \mathbf{Pic}.$$

To give a more explicit description of these nerves, let us recall the (simplicial) standard n-simplices $\Delta[n] = \mathrm{Hom}_\Delta(-, [n]) = \mathrm{Ner}[n]$, whose m-simplices are those sequences $\mathbf{u} = (u_0, \dots, u_m)$ of integers u_i such that $0 \leqslant u_0 \leqslant \cdots \leqslant u_m \leqslant n$, with faces and degeneracies

$$d_i\mathbf{u} = (u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_m),$$
$$s_i\mathbf{u} = (u_0, \dots, u_i, u_i, \dots, u_m).$$

For later use, we denote by ι_n the unique non-degenerate n-simplex of $\Delta[n]$, that is,

$$\iota_n = (0, 1, \dots, n) \in \Delta[n]_n.$$

Then, an n-simplex, say ξ, of the nerve of a G-graded Picard category $\mathbb{G}$, $\mathcal{N}_G\mathbb{G}$, can be described as a triple of maps

$$\xi = (\sigma, Y, f), \tag{76}$$

where

$$\sigma : \Delta[n]_1 \to G,$$
$$Y : \Delta[n]_3 \to \mathrm{Ob}\,\mathbb{G},$$
$$f : \Delta[n]_4 \to \mathrm{Mor}\,\mathbb{G},$$

such that the following four conditions are satisfied:

(i) For any 2-simplex $\mathbf{u} = (u_0, u_1, u_2) \in \Delta[n]_2$,

$$\sigma_{d_2\mathbf{u}}\sigma_{d_0\mathbf{u}} = \sigma_{d_1\mathbf{u}} \quad \text{(i.e., } \sigma_{u_0u_1}\sigma_{u_1u_2} = \sigma_{u_0u_2}\text{)},$$
$$Y_{s_i\mathbf{u}} = I, \quad 0 \leqslant i \leqslant 2.$$

(ii) For any 4-simplex $\mathbf{u} = (u_0, u_1, u_2, u_3, u_4) \in \Delta[n]_4$,

$$Y_{d_4\mathbf{u}} \otimes \left(Y_{d_2\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}} Y_{d_0\mathbf{u}}\right) \xrightarrow{f_{\mathbf{u}}} Y_{d_1\mathbf{u}} \otimes Y_{d_3\mathbf{u}}$$

is a morphism in $\mathbb{G}$ of trivial degree.

(iii) For any 5-simplex $\mathbf{u} = (u_0, u_1, u_2, u_3, u_4, u_5) \in \Delta[n]_5$, the diagram $(\xi_{\mathbf{u}})$ below, in which the arrows denoted by $\xrightarrow{\sim}$ are canonical and where we have omitted the associativity constraints, is commutative.

$$(\xi_{\mathbf{u}}): \tag{77}$$

$$\begin{array}{l}
Y_{d_4d_5\mathbf{u}} \otimes Y_{d_2d_5\mathbf{u}} \otimes Y_{d_2d_3\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}}(Y_{d_0d_5\mathbf{u}} \otimes Y_{d_0d_3\mathbf{u}} \otimes {}^{\sigma_{u_1u_2}} Y_{d_0d_1\mathbf{u}}) \\
\quad \downarrow \wr \\
Y_{d_4d_5\mathbf{u}} \otimes Y_{d_2d_5\mathbf{u}} \otimes Y_{d_2d_3\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}} Y_{d_0d_5\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}} Y_{d_0d_3\mathbf{u}} \otimes {}^{\sigma_{u_0u_2}} Y_{d_0d_1\mathbf{u}} \\
\quad \downarrow (f_{d_5\mathbf{u}}\otimes id)(id\otimes C\otimes id) \\
Y_{d_1d_5\mathbf{u}} \otimes Y_{d_3d_5\mathbf{u}} \otimes Y_{d_2d_3} \otimes {}^{\sigma_{u_0u_1}} Y_{d_0d_3\mathbf{u}} \otimes {}^{\sigma_{u_0u_2}} Y_{d_0d_1\mathbf{u}} \\
\quad \downarrow id\otimes f_{d_3\mathbf{u}}\otimes id \\
Y_{d_1d_5\mathbf{u}} \otimes Y_{d_1d_3\mathbf{u}} \otimes Y_{d_3d_4\mathbf{u}} \otimes {}^{\sigma_{u_0u_2}} Y_{d_0d_1\mathbf{u}} \\
\quad \searrow (f_{d_1\mathbf{u}}\otimes id)(id\otimes C) \\
Y_{d_1d_2\mathbf{u}} \otimes Y_{d_1d_4\mathbf{u}} \otimes Y_{d_3d_4\mathbf{u}}.
\end{array}$$

$$\begin{array}{l}
Y_{d_4d_5\mathbf{u}} \otimes Y_{d_2d_5\mathbf{u}} \otimes Y_{d_2d_3\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}}(Y_{d_0d_5\mathbf{u}} \otimes Y_{d_0d_3\mathbf{u}} \otimes {}^{\sigma_{u_1u_2}} Y_{d_0d_1\mathbf{u}}) \\
\quad \searrow id\otimes {}^{\sigma_{u_0u_1}} f_{d_0\mathbf{u}} \\
Y_{d_4d_5\mathbf{u}} \otimes Y_{d_2d_5\mathbf{u}} \otimes Y_{d_2d_3\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}}(Y_{d_0d_2\mathbf{u}} \otimes Y_{d_0d_4\mathbf{u}}) \\
\quad \downarrow \wr \\
Y_{d_4d_5\mathbf{u}} \otimes Y_{d_2d_5\mathbf{u}} \otimes Y_{d_2d_3\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}} Y_{d_0d_2\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}} Y_{d_0d_4\mathbf{u}} \\
\quad \downarrow id\otimes f_{d_2\mathbf{u}}\otimes id \\
Y_{d_4d_5\mathbf{u}} \otimes Y_{d_1d_2\mathbf{u}} \otimes Y_{d_2d_4\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}} Y_{d_0d_4\mathbf{u}} \\
\quad \downarrow (id\otimes f_{d_4\mathbf{u}})(C\otimes id) \\
Y_{d_1d_2\mathbf{u}} \otimes Y_{d_1d_4\mathbf{u}} \otimes Y_{d_3d_4\mathbf{u}}.
\end{array}$$

(iv) For any 3-simplex $\mathbf{u} \in \Delta[n]_3$ and $0 \leqslant i \leqslant 3$,

$$f_{s_i\mathbf{u}} = \begin{cases} C(id \otimes RC) : \mathrm{I} \otimes (\mathrm{I} \otimes Y_{\mathbf{u}}) \to Y_{\mathbf{u}} \otimes \mathrm{I} & \text{if } i = 0, \\ C(id \otimes R) : \mathrm{I} \otimes (Y_{\mathbf{u}} \otimes \mathrm{I}) \to Y_{\mathbf{u}} \otimes \mathrm{I} & \text{if } i = 1, \\ id \otimes R : \mathrm{I} \otimes (Y_{\mathbf{u}} \otimes \mathrm{I}) \to \mathrm{I} \otimes Y_{\mathbf{u}} & \text{if } i = 2, \\ C(id \otimes R) : Y_{\mathbf{u}} \otimes (\mathrm{I} \otimes \mathrm{I}) \to \mathrm{I} \otimes Y_{\mathbf{u}} & \text{if } i = 3. \end{cases}$$

When group G acts on a Picard category $\mathbb{G}_e$ in the strict sense that the isomorphisms (74) are identities; that is, when $\mathbb{G}_e : G \to \mathbf{Pic}$ is a functor, then, by the composition

$$G \xrightarrow{\mathbb{G}_e} \mathbf{Pic} \xrightarrow{\mathcal{N}} \mathbf{S},$$

the nerve of the underlying Picard category $\mathcal{N}\mathbb{G}_e$ ($= \mathrm{Ner}^s\,\mathbb{G}_e$) is equipped with a structure of G-space. Therefore, it is natural to consider the Borel construction on it, $E_G\mathcal{N}\mathbb{G}_e$. In the next proposition, we compare this simplicial set with the nerve of the associated G-graded Picard category $\mathcal{N}_G \int_G \mathbb{G}_e$.

Note that, by the equivalence between G-graded Picard categories and pseudofunctors $G \rightsquigarrow$ **Pic**, given by the constructions (73), $\mathbb{G} \mapsto \mathbb{G}_e$, and (75), $\mathbb{G}_e \mapsto \int_G \mathbb{G}_e$, functors $\mathbb{G}_e : G \to$ **Pic** correspond to "*split G-graded Picard categories*," that is, G-graded Picard categories that allow a closed-under-composition cocleavage. This condition on a cocleavage (72) of a G-graded Picard category $\mathbb{G}$ means that the equalities

$$\Gamma_{\sigma\tau,X} = \Gamma_{\sigma,{}^{\tau}X}\Gamma_{\tau,X}$$

hold for all $\sigma, \tau \in G$ and $X \in \mathrm{Ob}\,\mathbb{G}$ (see [21, §9. Example]).

Proposition 4.4. *For any split G-graded Picard category $\mathbb{G}$, there is a natural isomorphism*

$$\mathcal{N}_G\mathbb{G} \cong E_G\mathcal{N}\mathbb{G}_e.$$

Proof. The isomorphism is defined on the n-simplex level by the bijection

$$\bigcup_{\sigma:[n]\to G} Z^4\big([n], \sigma^*\mathbb{G}_e\big) \cong (BG)_n \times Z^4\big([n], \mathbb{G}_e\big),$$

$$(\sigma, Y, f) \mapsto (\sigma, \widetilde{Y}, \tilde{f}),$$

where, for any 3-simplex $\mathbf{u} = (u_0, u_1, u_2, u_3) \in \Delta[n]_3$, $\widetilde{Y}_{\mathbf{u}}$ is given by

$$\widetilde{Y}_{\mathbf{u}} = {}^{\sigma_{0u_0}}Y_{\mathbf{u}},$$

and, for any 4-simplex $\mathbf{u} = (u_0, u_1, u_2, u_3, u_4) \in \Delta[n]_4$, the morphism

$$\tilde{f}_{\mathbf{u}} : \widetilde{Y}_{d_4\mathbf{u}} \otimes (\widetilde{Y}_{d_2\mathbf{u}} \otimes \widetilde{Y}_{d_0\mathbf{u}}) \longrightarrow \widetilde{Y}_{d_1\mathbf{u}} \otimes \widetilde{Y}_{d_3\mathbf{u}}$$

is the composition of ${}^{\sigma_{0u_0}}f_{\mathbf{u}}$ with the appropriate canonical isomorphisms, namely:

$$\begin{array}{ccc}
{}^{\sigma_{0u_0}}Y_{d_4\mathbf{u}} \otimes ({}^{\sigma_{0u_0}}Y_{d_2\mathbf{u}} \otimes {}^{\sigma_{0u_1}}Y_{d_0\mathbf{u}}) & \xrightarrow{\ \sim\ } & {}^{\sigma_{0u_0}}(Y_{d_4\mathbf{u}} \otimes (Y_{d_2\mathbf{u}} \otimes {}^{\sigma_{u_0u_1}}Y_{d_0\mathbf{u}})) \\
& \swarrow {\scriptstyle {}^{\sigma_{0u_0}}f_{\mathbf{u}}} & \\
{}^{\sigma_{0u_0}}(Y_{d_1\mathbf{u}} \otimes Y_{d_3\mathbf{u}}) & \xrightarrow{\ \sim\ } & {}^{\sigma_{0u_0}}Y_{d_1\mathbf{u}} \otimes {}^{\sigma_{0u_0}}Y_{d_3\mathbf{u}}.
\end{array}$$ □

The following two particular cases are worthy of special note. Recall the discrete (33) and reduced (34) G-graded Picard categories in Examples 3.3 and 3.4.

Proposition 4.5. *For any G-module M, there are natural isomorphisms*

$$\mathcal{N}_G \operatorname{dis}_G M \cong E_G K(M,3), \tag{78}$$

$$\mathcal{N}_G \operatorname{red}_G M \cong E_G K(M,4). \tag{79}$$

Proof. Since both $\operatorname{dis}_G M$ and $\operatorname{red}_G M$ are split G-graded Picard categories, Proposition 4.4 applies, giving isomorphisms $\mathcal{N}_G \operatorname{dis}_G M \cong E_G \mathcal{N} \operatorname{dis} M$ and $\mathcal{N}_G \operatorname{red}_G M \cong E_G \mathcal{N} \operatorname{red} M$. Furthermore, simply by specifying the description (76) of what an n-simplex of the nerve of a Picard category is, we see that, for any $n \geqslant 0$,

$$(\mathcal{N} \operatorname{dis} M)_n = Z^4\big([n], \operatorname{dis} M\big) = Z^3\big(\Delta[n], M\big) = K(M,3)_n,$$

$$(\mathcal{N} \operatorname{red} M)_n = Z^4\big([n], \operatorname{red} M\big) = Z^4\big(\Delta[n], M\big) = K(M,4)_n,$$

from whence the equalities

$$\mathcal{N} \operatorname{dis} M = K(M,3), \qquad \mathcal{N} \operatorname{red} M = K(M,4) \tag{80}$$

follow. This completes the proof. □

The nerve construction on graded Picard categories performs well with regard to fibrations, as we show in Theorem 4.7 below. Previously, we shall prove the following lemma.

Lemma 4.6. *Let $\mathbb{G}$ be a G-graded Picard category. For any integers k, n, where $0 \leqslant k \leqslant n$ and $n \geqslant 5$, every (k,n)-horn in $\mathcal{N}_G\mathbb{G}$ has a unique filler in $\mathcal{N}_G\mathbb{G}$; that is, every extension problem*

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathcal{N}_G\mathbb{G} \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

has a solution and it is unique.

Proof. From their description (76), it follows that two simplices $\Delta[n] \to \mathcal{N}_G\mathbb{G}$ that coincide on their 4-skeleton are equal, so any n-simplex of $\mathcal{N}_G\mathbb{G}$ is completely determined by its 4-dimensional faces. If $n \geqslant 6$, then $sk_4\Lambda^k[n] = sk_4\Delta[n]$ and therefore any (k,n)-horn $\Lambda^k[n] \to \mathcal{N}_G\mathbb{G}$ extends uniquely to $\Delta[n]$, defining the required filler. Indeed, this is trivial when $n \geqslant 7$, since then $sk_5\Lambda^k[n] = sk_5\Delta[n]$ and the necessary commutativity of the diagrams $(\xi_{\mathbf{u}})$ in (77), $\mathbf{u} \in \Delta[n]_5$, hold. In the case when $n = 6$, the commutativity of diagram $(\xi_{d_k\iota_6})$, where $\iota_6 = (0,1,2,3,4,5,6)$, does not hold a priori. However, this commutativity follows by naturality and coherence from the commutativity of the other diagrams $(\xi_{d_j\iota_6})$, $j \neq k$, corresponding to the remaining 5-dimensional faces of ι_6, which belong to $\Lambda^k[6]$.

Suppose now that $n = 5$ and that a $(k,5)$-horn in $\mathcal{N}_G\mathbb{G}$ is given. Since $sk_3\Lambda^k[5] = sk_3\Delta[5]$, we therefore have a map $\sigma : \Delta[5]_1 \to G$, satisfying $\sigma_{d_2\mathbf{u}}\sigma_{d_0\mathbf{u}} = \sigma_{d_1\mathbf{u}}$ for all $\mathbf{u} \in \Delta[5]_2$, a map

$Y:\Delta[5]_3 \to \mathrm{Ob}\,\mathbb{G}$, with $Y_{s_i\mathbf{u}} = I$ for $0 \leqslant i \leqslant 2$ and any $\mathbf{u} \in \Delta[5]_2$, and also five morphisms of trivial degree in $\mathbb{G}$

$$f_{d_j\iota_5} : Y_{d_4 d_j \iota_5} \otimes \left(Y_{d_2 d_j \iota_5} \otimes {}^{\sigma_{d_2^3 d_j \iota_5}} Y_{d_0\mathbf{u}}\right) \longrightarrow Y_{d_1 d_j \iota_5} \otimes Y_{d_3 d_j \iota_5},$$

one for each $0 \leqslant j \leqslant 5$, $j \neq k$, corresponding to the faces of $\iota_5 = (0,1,2,3,4,5)$ that belong to $\Lambda^k[5]$. Nonetheless, these data extends uniquely to a 5-simplex of $\mathcal{N}_G\mathbb{G}$, $\xi = (\sigma, Y, f)$, as in (76), since the necessary commutativity of diagram (ξ_{ι_5}), (77), gives an equation for the e-morphism $f_{d_k\iota_5}$, corresponding to the missing face, which can be uniquely solved thanks to the invertibility of elements σ's in G and objects and arrows in $\mathbb{G}$. □

Theorem 4.7. *Let $F:\mathbb{G} \to \mathbb{G}'$ be a surjective on objects graded symmetric tensor Grothendieck cofibration between G-graded Picard categories. Then, the induced simplicial map*

$$\mathcal{N}_G F : \mathcal{N}_G\mathbb{G} \to \mathcal{N}_G\mathbb{G}'$$

is a Kan fibration.

Proof. To prove the extension-lifting Kan condition for $\mathcal{N}_G F$ at dimensions $n \leqslant 2$ is trivial, since $sk_2\mathcal{N}_G\mathbb{G} = sk_2 BG = sk_2\mathcal{N}_G\mathbb{G}'$ and $\mathcal{N}_G F$ is the identity map on n-simplices of the nerves for $n \leqslant 2$. When $n = 3$, the Kan condition for $\mathcal{N}_G F$ follows from the hypothesis of the functor F being surjective on objects. To visualize this, simply note that a 3-simplex of $\mathcal{N}_G\mathbb{G}$ is a pair (σ, Y) where σ is a 3-simplex of BG and Y $(= Y_{\iota_3})$ is an object of $\mathbb{G}$ whose faces are those of σ in BG, and $\mathcal{N}_G F(\sigma, Y) = (\sigma, FY)$.

The proof for $n = 4$ is as follows. Let us observe that a 4-simplex $\xi = (\sigma, Y, f)$ of $\mathcal{N}_G\mathbb{G}$ consists of a 4-simplex $\sigma \in BG$, five objects Y_j $(= Y_{d_j\iota_4})$, $0 \leqslant j \leqslant 4$, and a morphism of trivial degree f $(= f_{\iota_4})$ of $\mathbb{G}$

$$Y_4 \otimes \left(Y_2 \otimes {}^{\sigma_{01}} Y_0\right) \xrightarrow{f} Y_1 \otimes Y_3,$$

whose faces are $d_j(\xi) = (d_j\sigma, Y_j)$. The simplicial map $\mathcal{N}_G F$ carries such a 4-simplex to the 4-simplex of $\mathcal{N}_G\mathbb{G}'$

$$\mathcal{N}_G F(\xi) = (\sigma, FY, \widetilde{Ff}),$$

where $(FY)_j = FY_j$, $0 \leqslant j \leqslant 4$, and $\widetilde{Ff}$ is the composite morphism of trivial degree of $\mathbb{G}'$

$$FY_4 \otimes \left(FY_2 \otimes {}^{\sigma_{01}} FY_0\right) \xrightarrow{\sim} F\left(Y_4 \otimes \left(Y_2 \otimes {}^{\sigma_{01}} Y_0\right)\right) \xrightarrow{Ff} F(Y_1 \otimes Y_3) \xrightarrow{\sim} FY_1 \otimes FY_3.$$

Therefore, the verification of the Kan fibration condition at level four consists in proving that, for any given morphism of trivial degree in $\mathbb{G}'$ of the form

$$Y'_4 \otimes \left(Y'_2 \otimes {}^{\sigma} Y'_0\right) \xrightarrow{f'} Y'_1 \otimes Y'_3,$$

where the Y_i' are objects of $\mathbb{G}'$ and $\sigma \in G$, an integer k, where $0 \leqslant k \leqslant 4$, and objects Y_i of $\mathbb{G}$, $0 \leqslant i \leqslant 4$, $i \neq k$, such that $F(Y_i) = Y_i'$, there exists an object Y_k and a morphism of trivial degree in $\mathbb{G}$

$$Y_4 \otimes \big(Y_2 \otimes {}^{\sigma} Y_0\big) \xrightarrow{f} Y_1 \otimes Y_3,$$

such that $F(Y_k) = Y_k'$ and the diagram below commutes

$$\begin{array}{ccc} F(Y_4 \otimes (Y_2 \otimes {}^{\sigma} Y_0)) & \xrightarrow{Ff} & F(Y_1 \otimes Y_3) \\ \wr\downarrow & & \downarrow\wr \\ Y_4' \otimes (Y_2' \otimes {}^{\sigma} Y_0') & \xrightarrow{f'} & Y_1' \otimes Y_3', \end{array}$$

where the vertical isomorphisms are canonical. Before doing so, first note that $F : \mathbb{G} \to \mathbb{G}'$ is also a Grothendieck fibration since both $\mathbb{G}$ and $\mathbb{G}'$ are groupoids. Consequently, we are going to discuss only the case when $k = 0$ since the other cases are similar:

Let us choose quasi-inverses in $\mathbb{G}$ of Y_2 and Y_4, say Y_2^* and Y_4^* respectively, and fix corresponding morphisms of trivial degree $Y_2^* \otimes Y_2 \xrightarrow{\sim} I$ and $Y_4^* \otimes Y_4 \xrightarrow{\sim} I$. Then, the given morphism of trivial degree f' determines another, say g', by the commutativity of the diagram

$$\begin{array}{ccc} {}^{\sigma^{-1}}[FY_2^* \otimes (FY_4^* \otimes (Y_4' \otimes (Y_2' \otimes {}^{\sigma} Y_0')))] & \xrightarrow{{}^{\sigma^{-1}}[id \otimes (id \otimes f')]} & {}^{\sigma^{-1}}[FY_2^* \otimes (FY_4^* \otimes (Y_1' \otimes Y_3'))] \\ \wr\downarrow & & \downarrow\wr \\ Y_0' & \xrightarrow{g'} & F[{}^{\sigma^{-1}}(Y_2^* \otimes (Y_4^* \otimes (Y_1 \otimes Y_3)))], \end{array}$$

where the vertical arrows represent the morphisms of trivial degree canonically obtained from the fixed ones above. Now, since F is a fibration, there exists a morphism of trivial degree in $\mathbb{G}$

$$g : Y_0 \to {}^{\sigma^{-1}}\big(Y_2^* \otimes \big(Y_4^* \otimes (Y_1 \otimes Y_3)\big)\big),$$

such that $Fg = g'$. In particular, we have $FY_0 = Y_0'$ and, furthermore, we find the required morphism of trivial degree f by taking it to be that given by the dotted arrow in the commutative triangle

$$\begin{array}{ccc} Y_4 \otimes (Y_2 \otimes {}^{\sigma} Y_0) & \overset{f}{\dashrightarrow} & Y_1 \otimes Y_3 \\ & {\scriptstyle id \otimes (id \otimes {}^{\sigma} g)} \searrow \quad \nearrow {\scriptstyle \sim} & \\ & Y_4 \otimes (Y_2 \otimes {}^{\sigma}({}^{\sigma^{-1}}(Y_2^* \otimes (Y_4^* \otimes (Y_1 \otimes Y_3))))). & \end{array}$$

Finally, the verification of the Kan fibration condition for $\mathcal{N}_G F$ at dimensions $n \geqslant 5$ is trivial after Lemma 4.6. □

The nerve of a G-graded Picard category is actually a retractive space over BG, that is, an object of $\mathbf{S}^{BG}_{BG}$. An easy way to see that is as follows: View the group of grades G as the final object in the category of G-graded Picard category, that is, G is the strict G-graded Picard category whose underlying category is G with grading the identity map on G; the graded tensor functor $\otimes : G \times_G G \to G$ and the graded unit functor $\mathrm{I} : G \to G$ being the unique ones: $\sigma \otimes \sigma = \sigma$, $I(\sigma) = \sigma$. Then, any G-graded Picard category $\mathbb{G} = (\mathbb{G}, \mathrm{gr}, \otimes, \mathrm{I}, A, R, C)$ determines a retraction diagram of G-graded Picard categories

$$\mathbb{G} \underset{\mathrm{gr}}{\overset{\mathrm{I}}{\rightleftarrows}} G$$

which, by applying the functor $\mathcal{N}_G$, gives the retraction diagram of simplicial sets

$$\mathcal{N}_G\mathbb{G} \underset{\rho}{\overset{\epsilon}{\rightleftarrows}} \mathcal{N}_G G = BG \ , \quad \rho\epsilon = id, \tag{81}$$

in which $\rho = \mathcal{N}_G\mathrm{gr}$ is precisely the projection map $(\sigma, Y, f) \mapsto \sigma$ and $\epsilon = \mathcal{N}_G\mathrm{I}$ is given by $\epsilon(\sigma) = (\sigma, \mathrm{I}, R)$, where, for any $n \geqslant 0$, $\mathrm{I}_{\mathbf{u}} = \mathrm{I}$, for all $\mathbf{u} \in \Delta[n]_3$, and $R_{\mathbf{u}} = id \otimes R : \mathrm{I} \otimes (\mathrm{I} \otimes \mathrm{I}) \to \mathrm{I} \otimes \mathrm{I}$, for all $\mathbf{u} \in \Delta[n]_4$.

Since the grading $\mathrm{gr} : \mathbb{G} \to G$ is a surjective on objects graded symmetric tensor Grothendieck cofibration, from Theorem 4.7 we obtain the following consequence:

Corollary 4.8. *For any G-graded categorical group $\mathbb{G}$, the projection map*

$$\rho : \mathcal{N}_G\mathbb{G} \to BG, \quad (\sigma, Y, f) \mapsto \sigma,$$

is a split Kan fibration.

In the next theorem we recognize $\mathcal{N}_G\mathbb{G}$ as a fibrant object of the category $\mathbf{S}^{BG}_{BG}(3,4)$ and describe its homotopy groups.

Theorem 4.9. *The nerve $\mathcal{N}_G\mathbb{G}$ of a G-graded categorical group $\mathbb{G}$ is a reduced Kan complex, whose homotopy groups are*

$$\pi_i\mathcal{N}_G\mathbb{G} = \begin{cases} 0 & \text{for } i \neq 1, 3, 4, \\ G & \text{for } i = 1, \\ K_0\mathbb{G} & \text{for } i = 3, \\ K_1\mathbb{G} & \text{for } i = 4. \end{cases}$$

Proof. That $\mathcal{N}_G\mathbb{G}$ is a Kan complex is a consequence of $\rho : \mathcal{N}_G\mathbb{G} \to BG$ being a Kan fibration and BG a Kan complex. From Lemma 4.6, it follows that $\pi_i\mathcal{N}_G\mathbb{G} = 0$ for all $i \geqslant 5$, and also that the homotopy relation between 4-simplices is trivial; then,

$$\pi_4\mathcal{N}_G\mathbb{G} = \{ f : \mathrm{I} \otimes (\mathrm{I} \otimes \mathrm{I}) \to \mathrm{I} \otimes \mathrm{I},\ \mathrm{gr}(f) = 1 \} \cong \{ f \in \mathrm{Aut}_{\mathbb{G}}(\mathrm{I}),\ \mathrm{gr}(f) = 1 \} = K_1\mathbb{G}.$$

For $i = 3$, elements of $\pi_3\mathcal{N}_G\mathbb{G}$ are represented by the objects of $\mathbb{G}$ and a homotopy between two objects, say X and Y, is a morphism of trivial degree of the form $f : X \otimes (\mathrm{I} \otimes \mathrm{I}) \to \mathrm{I} \otimes Y$, which clearly exists if and only if X and Y are in the same trivial degree isomorphism class of objects of $\mathbb{G}$. Thus, we see that $\pi_3\mathcal{N}_G\mathbb{G} = K_0\mathbb{G}$.

The remainder of the proof follows from the fact that the retraction map $\rho:\mathcal{N}_G\mathbb{G} \to BG$ is the identity on i-simplices for $i \leqslant 2$. That is,

$$\pi_i\mathcal{N}_G\mathbb{G} = \pi_i BG = \begin{cases} 0 & \text{if } i = 0, 2, \\ G & \text{if } i = 1. \end{cases} \qquad \square$$

As a consequence of Proposition 3.6 and Theorem 4.9, we obtain the following:

Theorem 4.10. *A symmetric graded tensor functor between G-graded Picard categories $F:\mathbb{G} \to \mathbb{G}'$ is an equivalence if and only if the simplicial map induced on nerves $\mathcal{N}_G F:\mathcal{N}_G\mathbb{G} \to \mathcal{N}_G\mathbb{G}'$ is a homotopy equivalence.*

We are now ready to complete the proof of Theorem 4.1, that is:

Proposition 4.11. *The diagram*

$$\begin{array}{ccc} \mathbf{Pic}_G & \xrightarrow{\ \mathcal{N}_G\ } & \mathbf{S}_{BG}^{BG}(3,4) \\ & {\scriptstyle \mathrm{cl}}\searrow \quad \swarrow{\scriptstyle \mathrm{cl}} & \\ & \mathcal{H}^3_{G,s} & \end{array}$$

is commutative.

Proof. Let $\mathbb{G}$ be any given G-graded Picard category, and let us write M and N for the G-modules $K_0\mathbb{G}$ and $K_1\mathbb{G}$, respectively. Since, from Theorem 4.9, we have $\pi_3\mathcal{N}_G\mathbb{G} = M$ and $\pi_4\mathcal{N}_G\mathbb{G} = N$, it only remains to prove that the cohomological invariants attached to $\mathbb{G}$ and to $\mathcal{N}_G\mathbb{G}$ coincide.

By Theorems 3.11 and 4.10, we can assume that $\mathbb{G} = \mathbb{G}(h)$ (recall it from (35)) for some symmetric 3-cocycle $h \in Z^3_{G,s}(M,N)$, and then prove that h is also a representative of the Postnikov invariant of $\mathcal{N}_G\mathbb{G}(h)$ in

$$H^3_{G,s}(M,N) = H^5_G\big(K(M,3),N\big).$$

To this end, we shall note that the cohomology class of h in $H^3C_{G,s}(M,N)$ corresponds, by the composition of the isomorphisms

$$\begin{aligned} H^3C_{G,s}(M,N) &\overset{(10)}{\cong} H^3_{G,s}(M,N) = H^5_G\big(K(M,3),N\big) \overset{(2)}{\cong} H^5\big(E_GK(M,3),BG;N\big) \\ &\overset{(78)}{\cong} H^5(\mathcal{N}_G\,\mathrm{dis}_G\,M, BG; N), \end{aligned}$$

to the cohomology class in $H^5(\mathcal{N}_G\,\mathrm{dis}_G\,M, BG; N)$ represented by the 5-cocycle

$$\tilde{h}:(\mathcal{N}_G\,\mathrm{dis}_G\,M)_5 \to N$$

given by

$$\begin{aligned}
\tilde{h}(\sigma, Y) = {}^{\sigma_{01}{}^{-1}}\big[&h\big(Y_{0145}, {}^{\sigma_{01}}Y_{1345}, {}^{\sigma_{01}}Y_{1235}\big) - h\big(Y_{0345}, Y_{0135}, {}^{\sigma_{01}}Y_{1235}\big) \\
&+ h\big(Y_{0245}, Y_{0125}, {}^{\sigma_{02}}Y_{2345}\big) - h\big(Y_{0134}, {}^{\sigma_{01}}Y_{1234}, Y_{0145} + {}^{\sigma_{01}}Y_{1245} + {}^{\sigma_{02}}Y_{2345}\big) \\
&- h\big(Y_{0124}, Y_{0145}, {}^{\sigma_{01}}Y_{1245}\big) + h\big({}^{\sigma_{01}}Y_{1234}, Y_{0145}, {}^{\sigma_{01}}Y_{1245} + {}^{\sigma_{02}}Y_{2345}\big) \\
&+ h(Y_{0345}, Y_{0235}, Y_{0125}) + h\big(Y_{0123}, Y_{0134} + {}^{\sigma_{01}}Y_{1234}, Y_{0145} + {}^{\sigma_{01}}Y_{1245} + {}^{\sigma_{02}}Y_{2345}\big) \\
&- h\big(Y_{0123}, Y_{0345}, Y_{0135} + {}^{\sigma_{01}}Y_{1235}\big) - h\big(Y_{0345}, Y_{0123}, Y_{0135} + {}^{\sigma_{01}}Y_{1235}\big) \\
&- h\big(Y_{0145}, {}^{\sigma_{01}}Y_{1234}, {}^{\sigma_{01}}Y_{1245} + {}^{\sigma_{02}}Y_{2345}\big) - h\big(Y_{0245}, {}^{\sigma_{02}}Y_{2345}, Y_{0125}\big) \\
&- h\big(Y_{0145}, {}^{\sigma_{01}}Y_{1245}, {}^{\sigma_{02}}Y_{2345}\big) + h\big(Y_{0134}, Y_{0145} + {}^{\sigma_{01}}Y_{1345}, {}^{\sigma_{01}}Y_{1235}\big) \\
&+ h\big(Y_{0234}, Y_{0245} + {}^{\sigma_{02}}Y_{2345}, Y_{0125}\big) + h\big(Y_{0234}, Y_{0124} + Y_{0145} + {}^{\sigma_{01}}Y_{1245}, {}^{\sigma_{02}}Y_{2345}\big) \\
&+ h\big(Y_{0145} \,\big|\, {}^{\sigma_{01}}Y_{1234}\big) - h(Y_{0123} \mid Y_{0345}) + h\big(Y_{0125} \,\big|\, {}^{\sigma_{02}}Y_{2345}\big) \\
&- h\big(Y_{1234}, Y_{1245} + {}^{\sigma_{12}}Y_{2345}, \sigma_{01}\big) + h(Y_{1345}, Y_{1235}, \sigma_{01}) - h\big(Y_{1245}, {}^{\sigma_{12}}Y_{2345}, \sigma_{01}\big) \\
&+ h(Y_{2345}, \sigma_{01}, \sigma_{12})\big].
\end{aligned}$$

Now, recall that the G-graded Picard category $\mathbb{G}(h)$ has an associated exact sequence (41), (44)

$$\mathrm{red}_G\, N \overset{\mathrm{j}}{\hookrightarrow} \mathbb{G}(h) \overset{\mathrm{q}}{\to} \mathrm{dis}_G\, M.$$

By applying the nerve construction to it, we obtain a fibre sequence in $\mathbf{S}_{BG}^{BG}$

$$\mathcal{N}_G\, \mathrm{red}_G\, N \xhookrightarrow{\mathcal{N}_G \mathrm{j}} \mathcal{N}_G \mathbb{G}(h) \xrightarrow{\mathcal{N}_G \mathrm{q}} \mathcal{N}_G\, \mathrm{dis}_G\, M, \tag{82}$$

where $\mathcal{N}_G\mathrm{q}$ is a Kan fibration by Theorem 4.7, since q is a surjective on objects Grothendieck cofibration. Note that, by Proposition 4.5, there are natural isomorphisms $\mathcal{N}_G\, \mathrm{red}_G\, N \cong E_G K(N, 4)$ and $\mathcal{N}_G\, \mathrm{dis}_G\, M \cong E_G K(M, 3)$.

The fibre sequence (82) actually represents a principal minimal fibration in $\mathbf{S}_{BG}^{BG}$, in which the structural group $\mathcal{N}_G\, \mathrm{red}_G\, N$ operates on $\mathcal{N}_G \mathbb{G}(h)$ by the map

$$\begin{gathered}
\mathcal{N}_G\, \mathrm{red}_G\, N \times_{BG} \mathcal{N}_G \mathbb{G}(h) \to \mathcal{N}_G \mathbb{G}(h), \\
\big((\sigma, g), (\sigma, Y, f)\big) \mapsto (\sigma, Y, g \otimes f),
\end{gathered}$$

where, for each $\mathbf{u} \in \Delta[n]_4$,

$$(g \otimes f)_{\mathbf{u}} = g_{\mathbf{u}} \otimes f_{\mathbf{u}} : Y_{d_4\mathbf{u}} + Y_{d_2\mathbf{u}} + {}^{\sigma_{u_0 u_1}} Y_{d_0\mathbf{u}} \to Y_{d_1\mathbf{u}} + Y_{d_3\mathbf{u}}.$$

Indeed, we recognize the cohomology class of $\tilde{h}$ as the Postnikov invariant of $\mathcal{N}_G \mathbb{G}(h)$ since this principal fibration (82) has a pseudo-cross section $s : \mathcal{N}_G\, \mathrm{dis}_G\, M \to \mathcal{N}_G \mathbb{G}(h)$, which is the identity on n-simplices for $n \leqslant 4$ and is defined on 5-simplices by the formula

$$s(\sigma, Y) = \big(\sigma, Y, \tilde{h}^{\sigma, Y}\big),$$

where $\tilde{h}^{\sigma,Y}$ means the mapping $\tilde{h}^{\sigma,Y} : \Delta[5]_4 \to \operatorname{Mor}\mathbb{G}(h)$ defined by

$$\tilde{h}^{\sigma,Y}_{\mathbf{u}} = \begin{cases} (\tilde{h}(\sigma,Y), e) : Y_{d_4\mathbf{u}} + Y_{d_2\mathbf{u}} + {}^{\sigma_{12}}Y_{d_0\mathbf{u}} \to Y_{d_1\mathbf{u}} + Y_{d_3\mathbf{u}} & \text{if } \mathbf{u} = d_0\iota_5, \\ id = (0,e) : Y_{d_4\mathbf{u}} + Y_{d_2\mathbf{u}} + {}^{\sigma_{u_0u_1}}Y_{d_0\mathbf{u}} \to Y_{d_1\mathbf{u}} + Y_{d_3\mathbf{u}} & \text{if } \mathbf{u} = d_j\iota_5,\ j \neq 0. \end{cases}$$

To be more precise, the fibre sequence in $\mathbf{S}$, with a cross section from BG,

$$\begin{array}{ccccc} & & & & BG \\ & & & \swarrow & \downarrow \\ K(N,4) \overset{(80)}{=} \mathcal{N}\operatorname{red} N & \hookrightarrow & \mathcal{N}_G\mathbb{G}(h) & \xrightarrow{\mathcal{N}_G \mathrm{q}} & \mathcal{N}_G \operatorname{dis}_G M \end{array}$$

may be identified with the principal twisted cartesian product with base $\mathcal{N}_G \operatorname{dis}_G M$, local group $K(N,4)$ and whose twisting function $\mathcal{N}_G \operatorname{dis}_G M \to K(N,5)$ is defined by the relative 5-cocycle $\tilde{h} : (\mathcal{N}_G \operatorname{dis}_G M)_5 \to N$, by means of the isomorphism

$$\varphi : K(N,4) \times_{\tilde{h}} \mathcal{N}_G \operatorname{dis}_G M \xrightarrow{\sim} \mathcal{N}_G\mathbb{G}(h),$$

which is the identity on n-simplices for $n \leqslant 4$ and on 5-simplices is defined by the formula

$$\varphi\big(g, (\sigma, Y)\big) = (\sigma, Y, f),$$

where $f : \Delta[5]_4 \to \operatorname{Mor}\mathbb{G}(h)$ is the mapping defined by

$$f_{\mathbf{u}} = \begin{cases} ({}^{\sigma_{01}^{-1}} g_{\mathbf{u}} + \tilde{h}(\sigma,Y), e) : Y_{d_4\mathbf{u}} + Y_{d_2\mathbf{u}} + {}^{\sigma_{12}}Y_{d_0\mathbf{u}} \to Y_{d_1\mathbf{u}} + Y_{d_3\mathbf{u}} & \text{if } \mathbf{u} = d_0\iota_5, \\ ({}^{\sigma_{0u_0}^{-1}} g_{\mathbf{u}}, e) : Y_{d_4\mathbf{u}} + Y_{d_2\mathbf{u}} + {}^{\sigma_{u_0u_1}}Y_{d_0\mathbf{u}} \to Y_{d_1\mathbf{u}} + Y_{d_3\mathbf{u}} & \text{if } \mathbf{u} = d_j\iota_5,\ j \neq 0. \end{cases}$$

Since nerves of graded Picard categories are 5-coskeletal, this 5-truncated simplicial map extends uniquely to the fully defined isomorphism φ, whence $k_{\mathcal{N}_G\mathbb{G}(h)} = [\tilde{h}] = [h] = k_{\mathbb{G}(h)}$. $\square$

As a last comment, we shall remark that the nerve of a G-graded Picard category, $\mathcal{N}_G\mathbb{G}$, is actually a commutative H-group in the category $\mathbf{S}_{BG}$ of spaces over BG. The multiplication $\mathcal{N}_G\mathbb{G} \times_{BG} \mathcal{N}_G\mathbb{G} \to \mathcal{N}_G\mathbb{G}$ is induced by the tensor product $\otimes : \mathbb{G} \times_G \mathbb{G} \to \mathbb{G}$, since $\otimes$ is indeed a graded symmetric tensor functor and we have the equality $\mathcal{N}_G(\mathbb{G} \times_G \mathbb{G}) = \mathcal{N}_G\mathbb{G} \times_{BG} \mathcal{N}_G\mathbb{G}$. In particular, the set $\operatorname{Hom}_{\operatorname{Ho}\mathbf{S}_{BG}}(BG, \mathcal{N}_G\mathbb{G})$ of fibre homotopy classes of cross-sections of the fibration $\mathcal{N}_G\mathbb{G} \to BG$ has a natural abelian group structure. When $\mathbb{G}$ is a restricted G-graded Picard category, then this structure is precisely the one that realizes an abelian group isomorphism

$$H^4(G, \mathbb{G}) \cong \operatorname{Hom}_{\operatorname{Ho}\mathbf{S}_{BG}}(BG, \mathcal{N}_G\mathbb{G}),$$

where $H^4(G, \mathbb{G})$ is the fourth Fröhlich–Wall–Ulbrich cohomology group of G with coefficients in $\mathbb{G}$ [17,35,36], which is induced by the natural bijection (71)

$$Z^4(G, \mathbb{G}) \cong \operatorname{Hom}_{\mathbf{S}_{BG}}(BG, \mathcal{N}_G\mathbb{G}).$$

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