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Higher Hopf formula for homology of Leibniz n-algebras

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ABSTRACT

We fit the homology with trivial coefficients of Leibniz n-algebras into the context of Quillen homology and provide the Hopf type formula for the higher homology.

1. Introduction

The idea of the generalization of Lie structures by extending the binary bracket to an n-ary bracket comes from the formalism of Nambu mechanics [1], where the Poisson bracket on the algebra of smooth functions on a manifold is replaced by an n-linear skew-symmetric bracket. Independently from this idea, the theory of Lie n-algebras was introduced within the framework of geometry [2] and further developed in some papers [3,4].

Recently, the non-commutative version of Lie n-algebras, the so-called Leibniz n-algebras, was introduced in [5] which, at the same time, generalizes the notion of Leibniz algebras [6,7] from the case $n = 2$ to the case $n \geq 3$. In the last few years, a number of papers were dedicated to the investigation of properties of these new algebraic objects (see [8–11,3] and related references given there).

In [9], the homology with trivial coefficients of Leibniz n-algebras is constructed as the homology of an explicit chain complex and the first homology is interpreted by means of a Hopf formula. In [10] we introduced crossed modules of Leibniz n-algebras, proved that they are equivalent to internal categories in Leibniz n-algebras and described the second cohomology of Leibniz n-algebras [5] via crossed extensions.

In this paper, we continue our investigation in [9,10] on (co)homological properties of Leibniz n-algebras. We fit the homology with trivial coefficients of Leibniz n-algebras developed in [9] into the context of Quillen homology [12]. As the main result, we obtain Hopf type formulas for higher dimensional homology of Leibniz n-algebras, which are similar to Brown and Ellis formulas [13] for the higher homology of groups. As the main tool for our investigation, we develop the theory of higher dimensional crossed modules, crossed m-cubes of Leibniz n-algebras, and we use the method of m-fold Čech derived functors developed in [14,15].

In a recent paper by Everaert, Gran and Van der Linden [16], a conceptual proof of the higher Hopf formula is given in a very general framework, for semi-abelian categories [17], and so may be applied to the category of Leibniz n-algebras. In spite of our different approach, the main result in the present paper can confirm the categorical result of [16]. Nevertheless

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it is not straightforward to establish a relationship between the Čech derived functors and the categorical approaches, and this problem will be the subject of a further work.

1.1. Organization

After the introductory Section 1, the paper is organized in four sections. Section 2 is devoted to recalling from [10,5] some necessary definitions about Leibniz n-algebras, their actions, crossed modules and simplicial Leibniz n-algebras. The simplicial nerve of a crossed module of Leibniz n-algebras is also constructed and some needed standard facts are given. In Section 3 we prove that the homology of Leibniz n-algebras with trivial coefficients developed in [9] is the same as the Quillen homology for Leibniz n-algebras (Theorem 4). In Section 4 the notions of crossed m-cubes of Leibniz n-algebras and cat^m-Leibniz n-algebras are introduced and their equivalence is shown (Theorem 8). The abelianization and the diagonal of the multinerve of crossed m-cubes of Leibniz n-algebras are also investigated. Section 5 is the main one. Here the mth m-fold Čech derived functor of the abelianization functor from Leibniz n-algebras to vector spaces is calculated (Theorem 15), implying the description of the mth homology of a Leibniz n-algebra by a Hopf type formula (Theorem 17).

1.2. Notations and conventions

We fix $\mathbf{k}$ as a ground field. All vector spaces, tensor products and direct sums are considered over $\mathbf{k}$. By a linear map we mean a $\mathbf{k}$-linear map. For a non-negative integer m we denote by $\langle m \rangle$ the set of first m natural numbers $\{1, \dots, m\}$. When it is not necessary, we write arguments of maps without brackets ( ). By $[-, \dots, -]$ both the Leibniz n-bracket (see the definition immediately below) and the action of a Leibniz n-algebra (see Section 2.2) will be denoted similarly.

2. Preliminaries

2.1. Leibniz n-algebras

A *Leibniz n-algebra* [5] is a vector space $\mathcal{L}$ equipped with an n-ary bracket (n-bracket) $[-, \dots, -] : \mathcal{L}^{\otimes n} \to \mathcal{L}$ satisfying the following fundamental identity

$$[[x_1, \dots, x_n], y_1, \dots, y_{n-1}] = \sum_{i \in \langle n \rangle} [x_1, \dots, x_{i-1}, [x_i, y_1, \dots, y_{n-1}], x_{i+1}, \dots, x_n]. \tag{1}$$

A homomorphism of Leibniz n-algebras $\mathcal{L} \to \mathcal{L}'$ is a linear map preserving the n-bracket. The respective category of Leibniz n-algebras will be denoted by $_{\mathbf{n}}\mathbf{Lb}$.

A Leibniz 2-algebra $\mathcal{L}$ is simply a *Leibniz algebra* [6] and it is a *Lie algebra* if the condition $[l, l] = 0$ is fulfilled for all $l \in \mathcal{L}$. Similarly, for $n \geq 3$, a Leibniz n-algebra is a Lie n-algebra [2] if $[l_1, \dots, l_i, l_{i+1}, \dots, l_n] = 0$ holds as soon as $l_i = l_{i+1}$ for some $i \in \langle n - 1 \rangle$.

Any Leibniz algebra is also Leibniz n-algebra with respect to the n-bracket

$$[x_1, x_2, \dots, x_n] = [x_1, [x_2, \dots, [x_{n-1}, x_n] \cdots]]$$

(see [5]) and conversely, the Daletskii's functor [3] assigns to a Leibniz n-algebra $\mathcal{L}$ the Leibniz algebra $\mathcal{D}_{n-1}(\mathcal{L}) = \mathcal{L}^{\otimes n-1}$ with the bracket

$$[l_1 \otimes \cdots \otimes l_{n-1}, l'_1 \otimes \cdots \otimes l'_{n-1}] = \sum_{i \in \langle n-1 \rangle} l_1 \otimes \cdots \otimes [l_i, l'_1, \dots, l'_{n-1}] \otimes \cdots \otimes l_{n-1}.$$

A subalgebra $\mathcal{L}'$ of a Leibniz n-algebra $\mathcal{L}$ is said to be an *n-sided ideal* if $[l_1, \dots, l_n] \in \mathcal{L}'$ as soon as $l_i \in \mathcal{L}'$ for some $i \in \langle n \rangle$.

For any n-sided ideals $\mathcal{L}_1, \dots, \mathcal{L}_n$ of a Leibniz n-algebra $\mathcal{L}$, we denote by $[\mathcal{L}_1, \dots, \mathcal{L}_n]$ the vector subspace of $\mathcal{L}$ spanned by the brackets $[l_1, \dots, l_n]$, where $l_i \in \mathcal{L}_i$, $i \in \langle n \rangle$. Clearly $[\mathcal{L}_1, \dots, \mathcal{L}_n]$ is an n-sided ideal of $\bigcap_{i \in \langle n \rangle} \mathcal{L}_i$.

For any two n-sided ideals $\mathcal{L}'$ and $\mathcal{L}''$ of a Leibniz n-algebra $\mathcal{L}$, we denote by $[\mathcal{L}', \mathcal{L}'', \mathcal{L}^{n-2}]$ the vector subspace of $\mathcal{L}$ spanned by the brackets $[l_1, \dots, l_n]$, where necessarily $l_i \in \mathcal{L}'$ and $l_j \in \mathcal{L}''$ for some $i, j \in \langle n \rangle$, $i \neq j$. If $\mathcal{L}'' = \mathcal{L}$, then we use the notation $[\mathcal{L}', \mathcal{L}^{n-1}]$ instead of $[\mathcal{L}', \mathcal{L}, \mathcal{L}^{n-2}]$. Note that $[\mathcal{L}', \mathcal{L}^{n-1}]$ is an n-sided ideal of $\mathcal{L}$. In particular, $[\mathcal{L}, \dots, \mathcal{L}]$ is called the *commutator n-sided ideal* of $\mathcal{L}$.

Abelian group objects in $_{\mathbf{n}}\mathbf{Lb}$ are *abelian Leibniz n-algebras*, that is, Leibniz n-algebras with the trivial n-bracket, or just vector spaces. Their category will be denoted by $\mathbf{Vect}$. *The abelianization functor*

$$\mathfrak{Ab} : {_{\mathbf{n}}\mathbf{Lb}} \to \mathbf{Vect},$$

which is left adjoint to the inclusion functor $\mathbf{Vect} \hookrightarrow {_{\mathbf{n}}\mathbf{Lb}}$, is given by $\mathfrak{Ab}(\mathcal{L}) = \mathcal{L}/[\mathcal{L}, \dots, \mathcal{L}]$.

2.2. Actions and semi-direct product

Let $\mathcal{L}$ and $\mathcal{P}$ be Leibniz n-algebras. We will say that $\mathcal{P}$ acts on $\mathcal{L}$ [10] if $2^n - 2$ linear maps (of n variables)

$$[-, \dots, -] : \mathcal{L}^{\otimes i_1} \otimes \mathcal{P}^{\otimes j_1} \otimes \cdots \otimes \mathcal{L}^{\otimes i_m} \otimes \mathcal{P}^{\otimes j_m} \to \mathcal{L}$$

are given, where $m \in \langle n-1 \rangle$, $\sum_{k\in\langle m\rangle}(i_k + j_k) = n$, $0 \le i_k \le n-1$ and at least one $i_k \neq 0$, $0 \le j_k \le n-1$ and at least one $j_k \neq 0$, such that $2^{2n-1} - 2$ equalities hold which are obtained from (1) by taking exactly i of the variables $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ in $\mathcal{L}$ and all the others in $\mathcal{P}$ $\left(\binom{2n-1}{i} \text{ equalities}\right)$ and by changing $i = 1, \dots, 2n-2$.

For example, if $\mathcal{L}$ is an n-sided ideal of a Leibniz n-algebra $\mathcal{P}$, then the Leibniz n-bracket in $\mathcal{P}$ yields an action of $\mathcal{P}$ on $\mathcal{L}$.

Let us fix $i_1, j_1, \dots, i_m, j_m$ with the properties as above. Then the image of the corresponding map $[-, \dots, -]$ is the vector subspace of $\mathcal{L}$ spanned by elements of the form

$$[l_1^1, \dots, l_{i_1}^1, p_1^1, \dots, p_{j_1}^1, \dots, l_1^m, \dots, l_{i_m}^m, p_1^m, \dots, p_{j_m}^m],$$

where $l_1^k, \dots, l_{i_k}^k \in \mathcal{L}$, $p_1^k, \dots, p_{j_k}^k \in \mathcal{P}$, $k \in \langle m \rangle$. This vector subspace of $\mathcal{L}$ will be denoted by $[\mathcal{L}^{i_1}, \mathcal{P}^{j_1}, \dots, \mathcal{L}^{i_m}, \mathcal{P}^{j_m}]$.

Given a Leibniz n-algebra $\mathcal{P}$ acting on a Leibniz n-algebra $\mathcal{L}$, we can form their *semi-direct product*, $\mathcal{L} \rtimes \mathcal{P}$, with underlying vector space $\mathcal{L} \oplus \mathcal{P}$ and n-bracket

$$[(l_1, p_1), \dots, (l_n, p_n)] = ([l_1, \dots, l_n] + \Sigma\{l_1, \dots, l_n, p_1, \dots, p_n\}, [p_1, \dots, p_n]),$$

here $\Sigma\{l_1, \dots, l_n, p_1, \dots, p_n\}$ denotes the sum in $\mathcal{L}$ of $2^n - 2$ elements of the type $[x_1, \dots, x_n]$, where $x_k = l_k$ or $x_k = p_k$, $k \in \langle n \rangle$, exactly i of the variables $x_1, \dots, x_n$ are taken in $\mathcal{L}$ and $n - i$ are taken in $\mathcal{P}$, $i = 1, \dots, n-1$.

Remark 1. If a Leibniz n-algebra $\mathcal{P}$ acts on a Leibniz n-algebra $\mathcal{L}$, then there is also an action of $\mathcal{L} \rtimes \mathcal{P}$ on $\mathcal{L}$, given by the n-bracket in $\mathcal{L} \rtimes \mathcal{P}$, where $\mathcal{L}$ is considered as an n-sided ideal of $\mathcal{L} \rtimes \mathcal{P}$ via the natural inclusion $\mathcal{L} \hookrightarrow \mathcal{L} \rtimes \mathcal{P}$.

2.3. Crossed module and its nerve

A crossed module [10] is a homomorphism of Leibniz n-algebras $\mu : \mathcal{L} \to \mathcal{P}$ together with an action of $\mathcal{P}$ on $\mathcal{L}$ satisfying the following three conditions:

(CM1) μ is compatible with the action of $\mathcal{P}$ on $\mathcal{L}$, that is,

$$\begin{aligned}&\mu[l_1^1, \dots, l_{i_1}^1, p_1^1, \dots, p_{j_1}^1, \dots, l_1^m, \dots, l_{i_m}^m, p_1^m, \dots, p_{j_m}^m]\\ &\quad = [\mu l_1^1, \dots, \mu l_{i_1}^1, p_1^1, \dots, p_{j_1}^1, \dots, \mu l_1^m, \dots, \mu l_{i_m}^m, p_1^m, \dots, p_{j_m}^m];\end{aligned}$$

(CM2) The n-bracket in $\mathcal{L}$

$$(*) \quad [l_1, l_2, \dots, l_n]$$

is equal to any expression obtained from $(*)$ by replacing exactly i of the variables l's by μl's, for every $i \in \langle n-1 \rangle$;

(CM3) If $\sum_{k\in\langle m\rangle} i_k \ge 2$, then the expression

$$(**) \quad [l_1^1, \dots, l_{i_1}^1, p_1^1, \dots, p_{j_1}^1, \dots, l_1^m, \dots, l_{i_m}^m, p_1^m, \dots, p_{j_m}^m]$$

is equal to any expression obtained from $(**)$ by replacing exactly one of l's (and so i of l's, for every $1 \le i \le \sum_{k\in\langle m\rangle} i_k - 1$) by μl.

Lemma 2. *Let $\mu : \mathcal{L} \to \mathcal{P}$ be a crossed module of Leibniz n-algebras. Then*

$$\sum_{i_1, j_1, \dots, i_m, j_m} [\mathcal{L}^{i_1}, \mathcal{P}^{j_1}, \dots, \mathcal{L}^{i_m}, \mathcal{P}^{j_m}] = [\mathcal{L}, \mathcal{P}^{n-1}] \supseteq [\mathcal{L}, \dots, \mathcal{L}],$$

where the sum is taken over all $i_1, j_1, \dots, i_m, j_m$ such that $\sum_{k\in\langle m\rangle}(i_k + j_k) = n$, $0 \le i_k \le n-1$ and at least one $i_k \neq 0$, $0 \le j_k \le n-1$ and at least one $j_k \neq 0$, $m \in \langle n-1 \rangle$.

Proof. The required equality follows directly from the condition (CM3), whilst the inclusion is a consequence of the condition (CM2). $\square$

Recall that the nerve of a small category $\mathbf{C}$ with source and target maps $s, t : \mathbf{C} \to Ob(\mathbf{C})$ is the simplicial set $\mathfrak{Ner}(\mathbf{C}, s, t)_*$, with $\mathfrak{Ner}(\mathbf{C}, s, t)_k = \mathbf{C} \times_{Ob(\mathbf{C})} \cdots \times_{Ob(\mathbf{C})} \mathbf{C}$ (k factors), that is, k-simplices are the sequences of composable morphisms $c_0 \to c_1 \to \cdots \to c_k$. The ith face (resp. ith degeneracy) of such a k-simplex is obtained by deleting c_i (resp. inserting the identity morphism $c_i \to c_i$).

Given a crossed module of Leibniz n-algebras $\mu : \mathcal{L} \to \mathcal{P}$, consider the semi-direct product $\mathcal{L} \rtimes \mathcal{P}$, and the homomorphisms of Leibniz n-algebras $s, t : \mathcal{L} \rtimes \mathcal{P} \to \mathcal{P}$ given by $s(l, p) = p$ and $t(l, p) = \mu l + p$. According to [10], $(\mathcal{L} \rtimes \mathcal{P}, s, t)$ has an 1-fold internal category structure within the category ${}_{\mathbf{n}}\mathbf{Lb}$. The objects are the elements of $\mathcal{P} = \mathrm{Im}(s) =$

$\mathrm{Im}(t)$, the morphisms are the elements of $\mathcal{L} \rtimes \mathcal{P}$, the source and target maps are s and t, respectively. The morphisms (l, p) and (l', p') are composable if $\mu l + p = p'$ and their composite is $(l', p') \circ (l, p) = (l + l', p)$. The nerve of this category structure forms the simplicial Leibniz n-algebra $\mathfrak{Ner}(\mathcal{L} \rtimes \mathcal{P}, s, t)_*$, where $\mathfrak{Ner}(\mathcal{L} \rtimes \mathcal{P}, s, t)_k = (\mathcal{L} \rtimes \mathcal{P}) \times_{\mathcal{P}} \cdots \times_{\mathcal{P}} (\mathcal{L} \rtimes \mathcal{P})$ (k factors of $\mathcal{L} \rtimes \mathcal{P}$). Thus the k-simplices are k-tuples of the form

$$\big((l_1, p), (l_2, \mu l_1 + p), \ldots, (l_k, \mu l_{k-1} + \cdots + \mu l_1 + p)\big).$$

Now it is routine and we left to the reader to check that, for any $k \geq 1$, there is a natural isomorphism of Leibniz n-algebras

$$(\mathcal{L} \rtimes \mathcal{P}) \times_{\mathcal{P}} \cdots \times_{\mathcal{P}} (\mathcal{L} \rtimes \mathcal{P}) \xrightarrow{\cong} \mathcal{L} \rtimes (\cdots \rtimes (\mathcal{L} \rtimes \mathcal{P}) \cdots)$$

given by $\big((l_1, p), (l_2, \mu l_1 + p), \ldots, (l_k, \mu l_{k-1} + \cdots + \mu l_1 + p)\big) \mapsto (l_1, l_2, \ldots, l_k, p)$.

By using this isomorphism, from $\mathfrak{Ner}(\mathcal{L} \rtimes \mathcal{P}, s, t)_*$ we obtain the simplicial Leibniz n-algebra which is called the *nerve of the crossed module* $\mu : \mathcal{L} \to \mathcal{P}$ and it will be denoted by $E^{(1)}(\mathcal{L} \xrightarrow{\mu} \mathcal{P})_*$. Thus $E^{(1)}(\mathcal{L} \xrightarrow{\mu} \mathcal{P})_k = \mathcal{L} \rtimes (\cdots \rtimes (\mathcal{L} \rtimes \mathcal{P}) \cdots)$ with k semi-direct factors of $\mathcal{L}$, and face and degeneracy homomorphisms are given by

$$\begin{aligned}
&d_0(l_1, \ldots, l_k, p) = (l_2, \ldots, l_k, p),\\
&d_i(l_1, \ldots, l_k, p) = (l_1, \ldots, l_i + l_{i+1}, \ldots, l_k, p), \quad i \in \langle k-1 \rangle,\\
&d_k(l_1, \ldots, l_k, p) = (l_1, \ldots, l_{k-1}, \mu l_k + p),\\
&s_i(l_1, \ldots, l_k, p) = (l_1, \ldots, l_i, 0, l_{i+1}, \ldots, l_k, p), \quad 0 \leq i \leq k.
\end{aligned}$$

2.4. Homotopy of simplicial Leibniz n-algebras

Given a simplicial Leibniz n-algebra $\mathcal{L}_* = (\mathcal{L}_*, d_i^*, s_i^*)$, its Moore complex is the chain complex of Leibniz n-algebras $(N\mathcal{L}_*, \partial_*)$ given by

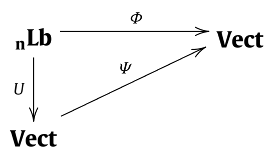
$$N\mathcal{L}_k = \bigcap_{i \in \langle k \rangle} \mathrm{Ker}\, d_{i-1}^k \quad \text{and} \quad \partial_k = d_k^k|_{N\mathcal{L}_k}.$$

Note that the Moore complex of the nerve of a crossed module of Leibniz n-algebras $\mu : \mathcal{L} \to \mathcal{P}$ is trivial in dimension ≥ 2 and it is just the original crossed module up to isomorphism with $\mathcal{L}$ in dimension 1 and $\mathcal{P}$ in dimension 0.

The image $d_{k+1}^{k+1}(N\mathcal{L}_{k+1})$ is an n-sided ideal of $\mathcal{L}_k$ and the kth homotopy of the simplicial Leibniz n-algebra $\mathcal{L}_*$ is defined as $\pi_k(\mathcal{L}_*) = H_k(N\mathcal{L}_*, \partial_*) = \mathrm{Ker}\, \partial_k / \mathrm{Im}\, \partial_{k+1}$. Note that in any homotopy π_k, $k \geq 1$, the n-bracket induced by that of $\mathcal{L}_k$ vanishes. We say that an augmented simplicial Leibniz n-algebra $(\mathcal{L}_*, d_0^0, \mathcal{L})$ is *aspherical* if $\pi_k(\mathcal{L}_*) = 0$ for all $k \geq 1$ and d_0^0 induces an isomorphism of Leibniz n-algebras $\pi_0(\mathcal{L}_*) \xrightarrow{\cong} \mathcal{L}$.

The following lemma will be useful in the sequel

Lemma 3. *Let $(\mathcal{L}_*, d_0^0, \mathcal{L})$ be an aspherical augmented simplicial Leibniz n-algebra. Suppose $\Phi : {}_{\mathbf{n}}\mathbf{Lb} \to \mathbf{Vect}$ and $\Psi : \mathbf{Vect} \to \mathbf{Vect}$ are functors such that the diagram*



commutes, where U is the forgetful functor from the category ${}_{\mathbf{n}}\mathbf{Lb}$ *to the category* **Vect**. *Then the augmented simplicial vector space $(\Phi(\mathcal{L}_*), \Phi(d_0^0), \Phi(\mathcal{L}))$ is acyclic.*

Proof. Straightforward from the fact that an acyclic augmented simplicial vector space $(\mathcal{U}(\mathcal{L}_*), \mathcal{U}(d_0^0), \mathcal{U}(\mathcal{L}))$ has a linear left (right) contraction. □

3. Homology as derived functors

In [9] the homology with trivial coefficients ${}_nHL_*(\mathcal{L})$ of a Leibniz n-algebra $\mathcal{L}$ is introduced as the homology of an explicit chain complex ${}_nCL_*(\mathcal{L})$, which is the Leibniz complex [7] associated to the Leibniz algebra $\mathcal{D}_{n-1}(\mathcal{L})$ and its co-representation $\mathcal{L}$. Let us briefly recall the construction of ${}_nHL_*(\mathcal{L})$.

In [7] the homology $HL_*(\mathfrak{g}, M)$ of a Leibniz algebra $\mathfrak{g}$ with coefficients in a co-representation M of $\mathfrak{g}$ is computed to be the homology of the Leibniz complex $CL_*(\mathfrak{g}, M)$ given by

$$CL_k(\mathfrak{g}, M) = M \otimes \mathfrak{g}^{\otimes k}, \quad k \geq 0,$$

with the boundary map $\partial_k : CL_k(\mathfrak{g}, M) \to CL_{k-1}(\mathfrak{g}, M)$ defined by

$$\partial_k(m, x_1, \ldots, x_k) = ([m, x_1], x_2, \ldots, x_k) + \sum_{2\le i\le k} (-1)^i ([x_i, m], x_i, \ldots, \hat{x}_i, \ldots, x_k) + \sum_{1\le i\le j\le k} (-1)^{j+1} (m, x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_k).$$

An essential fact for the construction of the complex ${}_nCL_*(\mathcal{L})$ in [9] is that any Leibniz n-algebra $\mathcal{L}$ can be considered as a co-representation of the Leibniz algebra $\mathcal{D}_{n-1}(\mathcal{L})$ using the following bilinear maps

$$[-,-] : \mathcal{L} \times \mathcal{D}_{n-1}(\mathcal{L}) \to \mathcal{L}, \qquad [l, l_1 \otimes \cdots \otimes l_{n-1}] = [l, l_1, \ldots, l_{n-1}];$$
$$[-,-] : \mathcal{D}_{n-1}(\mathcal{L}) \times \mathcal{L} \to \mathcal{L}, \qquad [l_1 \otimes \cdots \otimes l_{n-1}, l] = -[l, l_1, \ldots, l_{n-1}].$$

Then the complex ${}_nCL_*(\mathcal{L})$ is defined to be $CL_*(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})$. Thus

$${}_nHL_*(\mathcal{L}) = H_*({}_nCL_*(\mathcal{L})) = HL_*(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L}).$$

Note that when $\mathcal{L}$ is a Leibniz 2-algebra, that is, a Leibniz algebra, then we have

$${}_2CL_k(\mathcal{L}) = CL_k(\mathcal{L}, \mathcal{L}) = CL_{k+1}(\mathcal{L})$$

for all $k \ge 0$. Hence

$${}_2HL_k(\mathcal{L}) = HL_{k+1}(\mathcal{L}).$$

In the sequel we shall need the following easily verified equality

$${}_nHL_0(\mathcal{L}) = HL_0(\mathcal{L}^{\otimes n-1}, \mathcal{L}) = \mathrm{Coker}(\partial_1 : \mathcal{L}^{\otimes n} \to \mathcal{L}) = \mathfrak{Ab}(\mathcal{L})$$

and the fact that ${}_nHL_k(\mathcal{F}) = 0$, $k \ge 1$, if $\mathcal{F}$ is a free Leibniz n-algebra [9].

Now we show that the homology of Leibniz n-algebras is fitted in the context of homology theory developed by Quillen in a very general framework [12]. Let us recall that the Quillen homology of an object X in an algebraic category **C** is defined as the derived functors of the abelianization functor $\mathfrak{Ab} : \mathbf{C} \to \mathfrak{Ab}\mathbf{C}$ from **C** to the abelian category $\mathfrak{Ab}\mathbf{C}$ of abelian group objects in **C**. This theory can be applied for Leibniz n-algebras. Given a Leibniz n-algebra $\mathcal{L}$, Quillen homology of $\mathcal{L}$ is defined by

$$H_k^Q(\mathcal{L}) = H_k(\mathfrak{Ab}(\mathcal{F}_*)), \quad k \ge 0,$$

where $\mathcal{F}_* \to \mathcal{L}$ is an aspherical augmented simplicial Leibniz n-algebra such that each component $\mathcal{F}_k$, $k \ge 0$, is a free Leibniz n-algebra. Here $\mathfrak{Ab}(\mathcal{F}_*)$ is the simplicial vector space obtained by applying the functor $\mathfrak{Ab} : {}_{\mathbf{n}}\mathbf{Lb} \to \mathbf{Vect}$ dimensionwise to $\mathcal{F}_*$.

Theorem 4. *Let $\mathcal{L}$ be a Leibniz n-algebra. Then there is an isomorphism*

$$H_k^Q(\mathcal{L}) \cong {}_nHL_k(\mathcal{L}), \quad k \ge 0.$$

Proof. Since $\mathcal{F}_* \to \mathcal{L}$ is an aspherical simplicial Leibniz n-algebra, it is a consequence of Lemma 3 that ${}_nCL_k(\mathcal{F}_*) \to {}_nCL_k(\mathcal{L})$ is an acyclic simplicial vector space. Using the facts that ${}_nHL_k(\mathcal{F}_q) = 0$ and ${}_nHL_0(\mathcal{F}_q) = \mathfrak{Ab}(\mathcal{F}_q)$ for $k \ge 1$, $q \ge 0$, it follows that both spectral sequences for the bicomplex ${}_nCL_*(\mathcal{F}_*)$ degenerate and give the required isomorphism. □

4. Crossed *m*-cubes and *cat^m*-Leibniz *n*-algebras

4.1. Definitions and equivalence

The following notion of a crossed m-cube of Leibniz n-algebras is derived from the definition of crossed m-cube of algebras [18] by considering h-functions of n arguments satisfying the fundamental identity (1).

Definition 5. A crossed m-cube of Leibniz n-algebras $\{\mathcal{M}_A : A \subseteq \langle m\rangle, \mu_i, h\}$ is a family of Leibniz n-algebras $\{\mathcal{M}_A\}$ together with homomorphisms $\mu_i : \mathcal{M}_A \to \mathcal{M}_{A\setminus\{i\}}$ for $i \in \langle m\rangle$, $A \subseteq \langle m\rangle$ and n-linear functions $h : \mathcal{M}_{A_1} \times \cdots \times \mathcal{M}_{A_n} \longrightarrow \mathcal{M}_{A_1\cup\cdots\cup A_n}$ for $A_1, \ldots, A_n \subseteq \langle m\rangle$, such that for all $a \in \mathcal{M}_A$, $a_1 \in \mathcal{M}_{A_1}, \ldots, a_{2n-1} \in \mathcal{M}_{A_{2n-1}}$, $i, j \in \langle m\rangle$, $2 \le k \le n$ and $j_1, \ldots, j_k \in \langle n\rangle$ the following conditions hold:

(x1) $\mu_i a = a \quad \text{if } i \notin A$;

(x2) $\mu_i\mu_j a = \mu_j\mu_i a$;

(x3) $\mu_i h(a_1, \ldots, a_n) = h(\mu_i a_1, \ldots, \mu_i a_n)$;

(x4) $h(a_1, \ldots, a_{j_1}, \ldots, a_{j_k}, \ldots, a_n) = h(a_1, \ldots, \mu_i a_{j_1}, \ldots, a_{j_k}, \ldots, a_n) = \cdots = h(a_1, \ldots, a_{j_1}, \ldots, \mu_i a_{j_k}, \ldots, a_n) \quad \text{if } i \in A_{j_1} \cap \cdots \cap A_{j_k}$;

(x5) $h(a_1, \ldots, a_n) = [a_1, \ldots, a_n] \quad \text{if } A_1 = \cdots = A_n$;

(x6) $h(h(a_1, \ldots, a_n), a_{n+1}, \ldots, a_{2n-1}) = \sum_{k\in\langle n\rangle} h(a_1, \ldots, h(a_k, a_{n+1}, \ldots, a_{2n-1}), \ldots, a_n)$.

A morphism of crossed m-cubes of Leibniz n-algebras, $\{\mathcal{M}_A\} \to \{\mathcal{M}'_A\}$, is a family $\{f_A : \mathcal{M}_A \to \mathcal{M}'_A,\ A \subseteq \langle m\rangle\}$ of homomorphisms of Leibniz n-algebras commuting with the μ_i and the h-functions. The resultant category of crossed m-cubes of Leibniz n-algebras will be denoted by ${}_{\mathbf{n}}\mathbf{Lb}\mathfrak{X}^{\mathbf{m}}$.

Example 6. Let $\mathcal{L}$ be a Leibniz n-algebra and $\mathcal{I}_1, \ldots, \mathcal{I}_m$ be n-sided ideals of $\mathcal{L}$. Let $\mathcal{M}_A = \bigcap_{j\in A} \mathcal{I}_j$ for $A \subseteq \langle m\rangle$ (here $\mathcal{M}_\emptyset$ is understood to mean $\mathcal{L}$); given $i \in \langle m\rangle$, define $\mu_i : \mathcal{M}_A \to \mathcal{M}_{A\setminus\{i\}}$ to be the inclusion; let $h : \mathcal{M}_{A_1} \times \cdots \times \mathcal{M}_{A_n} \to \mathcal{M}_{A_1\cup\cdots\cup A_n}$, for $A_1, \ldots, A_n \subseteq \langle m\rangle$, be given by the n-bracket in $\mathcal{L}$: $h(a_1, \ldots, a_n) = [a_1, \ldots, a_n]$. Then $\{\mathcal{M}_A\}$ is a crossed m-cube of Leibniz n-algebras, called the inclusion crossed m-cube given by the Leibniz n-algebra $\mathcal{L}$ and its n-sided ideals $\mathcal{I}_1, \ldots, \mathcal{I}_m$.

Note that, given a crossed m-cube of Leibniz n-algebras $\{\mathcal{M}_A\}$, if $A_i = A$, $i \in B$ and $A_j = A \setminus A'$, $j \in \langle n\rangle \setminus B$ for some $\emptyset \neq B \subseteq \langle n-1\rangle$ and $A' \subseteq A \subseteq \langle m\rangle$, then the functions $h : \mathcal{M}_{A_1} \times \cdots \times \mathcal{M}_{A_n} \longrightarrow \mathcal{M}_A$ define an action of the Leibniz n-algebra $\mathcal{M}_{A\setminus A'}$ on $\mathcal{M}_A$. Moreover, every homomorphism $\mu_i : \mathcal{M}_A \to \mathcal{M}_{A\setminus\{i\}}$, together with such an action of $\mathcal{M}_{A\setminus\{i\}}$ on $\mathcal{M}_A$, is a crossed module of Leibniz n-algebras. In particular, for $m = 1$ we find that a crossed 1-cube is the same as a crossed module of Leibniz n-algebras.

According to [10] the category of crossed modules of Leibniz n-algebras is equivalent to that of cat^1-Leibniz n-algebras. Below we prove the higher dimensional version of this result, similarly to the case of groups [19] and algebras [18]. First, by close analogy with Loday's original notion of cat^m-groups [20], we give the definition of a cat^m-Leibniz n-algebra, which is equivalent to an m-fold category object in ${}_{\mathbf{n}}\mathbf{Lb}$.

Definition 7. A cat^m-Leibniz n-algebra $(\mathcal{N}, s_i, t_i)$ is a Leibniz n-algebra $\mathcal{N}$ together with $2m$ endomorphisms $s_i, t_i : \mathcal{N} \to \mathcal{N}$, $i \in \langle m\rangle$, such that

(c1) $t_i s_i = s_i, \quad s_i t_i = t_i,$

(c2) $s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$ for $i \neq j$,

(c3) $[\operatorname{Ker} s_i, \operatorname{Ker} t_i, \mathcal{N}^{n-2}] = 0.$

A morphism of cat^m-Leibniz n-algebras $(\mathcal{N}, s_i, t_i) \to (\mathcal{N}', s'_i, t'_i)$ is a homomorphism of Leibniz n-algebras $\varphi : \mathcal{N} \to \mathcal{N}'$ such that $\varphi s_i = s'_i \varphi$, $\varphi t_i = t'_i \varphi$ for all $i \in \langle m\rangle$. The resultant category of cat^m-Leibniz n-algebras will be denoted by ${}_{\mathbf{n}}\mathbf{Lb}\mathfrak{C}^{\mathbf{m}}$.

Theorem 8. *The categories* ${}_{\mathbf{n}}\mathbf{Lb}\mathfrak{X}^{\mathbf{m}}$ *and* ${}_{\mathbf{n}}\mathbf{Lb}\mathfrak{C}^{\mathbf{m}}$ *are equivalent.*

Proof. To any cat^m-Leibniz n-algebra $(\mathcal{N}, s_i, t_i)$ we correspond a crossed m-cube of Leibniz n-algebras $\{\mathcal{M}_A : A \subseteq \langle m\rangle, \mu_i, h\}$ defined as follows:

$$\mathcal{M}_A = \bigcap_{i\in A} \operatorname{Ker} s_i \cap \bigcap_{i\notin A} \operatorname{Im} s_i;$$

$$\mu_i(a) = t_i(a), \quad a \in \mathcal{M}_A;$$

$$h(a_1, \ldots, a_n) = [a_1, \ldots, a_n], \quad a_1 \in \mathcal{M}_{A_1}, \ldots, a_n \in \mathcal{M}_{A_n}.$$

Straightforward calculations show that $\mathcal{M}_A$ indeed is a crossed m-cube of Leibniz n-algebras. For instance, the equality

$$h(a_1, \ldots, a_{j_1}, \ldots, a_{j_k}, \ldots, a_n) = h(a_1, \ldots, \mu_i a_{j_1}, \ldots, a_{j_k}, \ldots, a_n)$$

in (x4) is a consequence of (c3). In fact,

$$[a_1, \ldots, a_{j_1}, \ldots, a_{j_k}, \ldots, a_n] - [a_1, \ldots, \mu_i a_{j_1}, \ldots, a_{j_k}, \ldots, a_n] = [a_1, \ldots, a_{j_1} - \mu_i a_{j_1}, \ldots, a_{j_k}, \ldots, a_n] = 0$$

since $a_{j_1} - \mu_i a_{j_1} \in \operatorname{Ker} t_i$ and $a_{j_k} \in \operatorname{Ker} s_i$.

Conversely, given a crossed m-cube of Leibniz n-algebras $\{\mathcal{M}_A\}$, choose an ordering of the subsets of $\langle m\rangle$ and define a cat^m-Leibniz n-algebra $(\mathcal{N}, s_i, t_i)$ with underlying vector space $\mathcal{N} = \bigoplus_{A\subseteq\langle m\rangle} \mathcal{M}_A$. Thus any element of $\mathcal{N}$ can be written uniquely as $\sum_{A\subseteq\langle m\rangle} a_A$ with $a_A \in \mathcal{M}_A$. Then $\mathcal{N}$ has an n-linear bracket given by

$$\left[\sum_{A_1\subseteq\langle m\rangle} a_{A_1}, \ldots, \sum_{A_n\subseteq\langle m\rangle} a_{A_n}\right] = \sum_{A_1,\ldots,A_n\subseteq\langle m\rangle} h(a_{A_1}, \ldots, a_{A_n}).$$

The equality (x6) amounts exactly that the fundamental identity (1) holds. The endomorphisms $s_i, t_i : \mathcal{N} \to \mathcal{N}$, $i \in \langle m\rangle$, are

$$s_i \sum_{A\subseteq\langle m\rangle} a_A = \sum_{\substack{A\subseteq\langle m\rangle \\ i\notin A}} a_A, \qquad t_i \sum_{A\subseteq\langle m\rangle} a_A = \sum_{A\subseteq\langle m\rangle} \mu_i a_A.$$

Obviously s_i indeed is a homomorphism of Leibniz n-algebras, whilst t_i is a homomorphism because of the equality (x3). It is easy to see that (x1) and (x2) imply that all equalities in (c1) and (c2) hold. It remains to check the condition (c3). Let

$$\left[\sum_{A_1\subseteq\langle m\rangle} a_{A_1}, \ldots, \sum_{A_k\subseteq\langle m\rangle} a_{A_k}, \ldots, \sum_{A_l\subseteq\langle m\rangle} a_{A_l}, \ldots, \sum_{A_n\subseteq\langle m\rangle} a_{A_n}\right] \in [\operatorname{Ker} s_i, \operatorname{Ker} t_i, \mathcal{N}^{n-2}]$$

and suppose $\sum_{A_k\subseteq\langle m\rangle} a_{A_k} \in \operatorname{Ker} s_i$, $\sum_{A_l\subseteq\langle m\rangle} a_{A_l} \in \operatorname{Ker} t_i$. Then $a_{A_k} = 0$ if $i \notin A_k$ and $\mu_i a_{A_l} = -a_{A_l\setminus\{i\}}$ if $i \in A_l$. Respectively $h(a_{A_1}, \dots, a_{A_k}, \dots, a_{A_l}, \dots, a_{A_n}) = 0$ if $i \notin A_k$ and

$$h(a_{A_1}, \dots, a_{A_k}, \dots, a_{A_l}, \dots, a_{A_n}) = -h(a_{A_1}, \dots, a_{A_k}, \dots, a_{A_l\setminus\{i\}}, \dots, a_{A_n})$$

if $i \in A_l$. Then, by (x4) we have

$$\begin{aligned}
&\left[\sum_{A_1\subseteq\langle m\rangle} a_{A_1}, \dots, \sum_{A_k\subseteq\langle m\rangle} a_{A_k}, \dots, \sum_{A_l\subseteq\langle m\rangle} a_{A_l}, \dots, \sum_{A_n\subseteq\langle m\rangle} a_{A_n}\right] \\
&\quad = \sum_{A_1,\dots,A_n\subseteq\langle m\rangle} h(a_{A_1}, \dots, a_{A_k}, \dots, a_{A_l}, \dots, a_{A_n}) \\
&\quad = \sum_{\substack{A_1,\dots,A_n\subseteq\langle m\rangle \\ i\in A_k\cap A_l}} h(a_{A_1}, \dots, a_{A_k}, \dots, a_{A_l}, \dots, a_{A_n}) + \sum_{\substack{A_1,\dots,A_n\subseteq\langle m\rangle \\ i\in A_k, i\notin A_l}} h(a_{A_1}, \dots, a_{A_k}, \dots, a_{A_l}, \dots, a_{A_n}) \\
&\quad = \sum_{\substack{A_1,\dots,A_n\subseteq\langle m\rangle \\ i\in A_k\cap A_l}} h(a_{A_1}, \dots, a_{A_k}, \dots, \mu_i a_{A_l}, \dots, a_{A_n}) \\
&\qquad + \sum_{\substack{A_1,\dots,A_n\subseteq\langle m\rangle \\ i\in A_k, i\notin A_l}} h(a_{A_1}, \dots, a_{A_k}, \dots, a_{A_l}, \dots, a_{A_n}) = 0.
\end{aligned}$$

Thus $(\mathcal{N}, s_i, t_i)$ is a cat^m-Leibniz n-algebra.

The above constructed assignments $\{\mathcal{M}_A\} \rightleftarrows (\mathcal{N}, s_i, t_i)$ are clearly functorial. Moreover, if $(\mathcal{N}, s_i, t_i)$ is a cat^m-Leibniz n-algebra and $\mathcal{M}_A = \operatorname{Ker}_{i\in A} \bigcap s_i \cap \bigcap_{i\notin A} \operatorname{Im} s_i$, then the canonical homomorphism $\bigoplus_{A\subseteq\langle m\rangle} \mathcal{M}_A \to \mathcal{N}$ is an isomorphism. This implies that the assignments are quasi-inverses to each other. □

Note that Theorem 8 in the case $m = 1$ recovers Theorem 10 in [10].

4.2. Functors $E^{(m)}$ and $\mathfrak{Ab}^{(m)}$

If $\mathcal{M}$ is a crossed m-cube of Leibniz n-algebras, the associated cat^m-Leibniz n-algebra is endowed with m compatible category structures. Then by applying the crossed module nerve structure $E^{(1)}$ in the m-independent directions, this construction leads naturally to an m-simplicial Leibniz n-algebra, called the *multinerve* of $\mathcal{M}$. Taking the diagonal of this m-simplicial Leibniz n-algebra gives a simplicial Leibniz n-algebra denoted by $E^{(m)}(\mathcal{M})_*$.

Let $\mathbf{Vect}\mathfrak{X}^{\mathbf{m}}$ denote the subcategory of the category of abelian crossed m-cubes of groups (for the definition we refer the reader to [21,14]) consisting of those abelian crossed m-cubes $\{\mathcal{G}_A : A \subseteq \langle m\rangle,\ \mu_i,\ h\}$ in which each abelian group $\mathcal{G}_A$ has a structure of vector space and each μ_i is a homomorphism of vector spaces. Then we define *the abelianization functor*

$$\mathfrak{Ab}^{(m)} : {}_{\mathbf{n}}\mathbf{Lb}\mathfrak{X}^{\mathbf{m}} \to \mathbf{Vect}\mathfrak{X}^{\mathbf{m}}$$

as follows: for any crossed m-cube of Leibniz n-algebras $\{\mathcal{M}_A : A \subseteq \langle m\rangle,\ \mu_i,\ h\}$

$$\mathfrak{Ab}^{(m)}(\mathcal{M})_A = \frac{\mathcal{M}_A}{\sum\limits_{A_1\cup\cdots\cup A_n=A} D(A_1, \dots, A_n)},$$

where $D(A_1, \dots, A_n)$ is the subspace of $\mathcal{M}_A$ generated by the elements $h(a_1, \dots, a_n)$, for $h : \mathcal{M}_{A_1} \times \cdots \times \mathcal{M}_{A_n} \to \mathcal{M}_A$ and $a_j \in \mathcal{M}_{A_j}, j \in \langle n\rangle$. The homomorphism

$$\widetilde{\mu}_i : \mathfrak{Ab}^{(m)}(\mathcal{M})_A \to \mathfrak{Ab}^{(m)}(\mathcal{M})_{A\setminus\{i\}}, \quad A \subseteq \langle m\rangle,\ i \in \langle m\rangle,$$

is induced by the homomorphism μ_i and the function

$$\widetilde{h} : \mathfrak{Ab}^{(m)}(\mathcal{M})_{A_1} \times \cdots \times \mathfrak{Ab}^{(m)}(\mathcal{M})_{A_n} \to \mathfrak{Ab}^{(m)}(\mathcal{M})_{A_1\cup\cdots\cup A_n}, \quad A_1, \dots, A_n \subseteq \langle m\rangle,$$

is induced by h and therefore is the trivial map.

Here we point out that, under the equivalence described in Theorem 8, the functor $\mathfrak{Ab}^{(m)}$ assigns to any cat^m-Leibniz n-algebra $(\mathcal{N}, s_i, t_i)$ the abelian cat^m-group (vector space) $(\mathcal{N}/[\mathcal{N}, \dots, \mathcal{N}], \widetilde{s}_i, \widetilde{t}_i)$, where $\widetilde{s}_i$ and $\widetilde{t}_i$ are induced by s_i and t_i.

The following assertion establishes the commutativity relation between the functors $\mathfrak{Ab}^{(m)}$ and $E^{(m)}$, which plays an essential role to obtain Hopf type formulas for the homology of Leibniz n-algebras.

Proposition 9. *Let $\mathcal{M}$ be a crossed m-cube of Leibniz n-algebras and $m \geq 1$. Then there is an isomorphism of simplicial vector spaces*

$$\mathfrak{Ab}\big(E^{(m)}(\mathcal{M})_*\big) \cong E^{(m)}\big(\mathfrak{Ab}^{(m)}(\mathcal{M})\big)_*$$

Proof. The proof will be done by induction on m.

For $m = 1$, given a crossed module of Leibniz n-algebras $\mathcal{M} = (\mathcal{L} \xrightarrow{\mu} \mathcal{P})$, we have to show an isomorphism of simplicial vector spaces

$$\mathfrak{Ab}\big(E^{(1)}(\mathcal{L}\xrightarrow{\mu}\mathcal{P})_*\big)\cong E^{(1)}\big(\mathfrak{Ab}^{(1)}(\mathcal{L}\xrightarrow{\mu}\mathcal{P})\big)_*.$$

By Lemma 2 we get

$$\mathfrak{Ab}^{(1)}\big(\mathcal{L}\xrightarrow{\mu}\mathcal{P}\big)=\big(\mathcal{L}/[\mathcal{L},\mathcal{P}^{n-1}]\xrightarrow{\widetilde{\mu}}\mathcal{P}/[\mathcal{P},\ldots,\mathcal{P}]\big).$$

It is easy to see that the homomorphisms of vector spaces

$$\big(\mathcal{L}\rtimes\mathcal{P}\big)/[\mathcal{L}\rtimes\mathcal{P},\ldots,\mathcal{L}\rtimes\mathcal{P}]\underset{\beta}{\overset{\alpha}{\rightleftarrows}}\mathcal{L}/[\mathcal{L},\mathcal{P}^{n-1}]\times\mathcal{P}/[\mathcal{P},\ldots,\mathcal{P}],$$

given by $\alpha\overline{(l,p)}=(\bar{l},\bar{p})$ and $\beta(\bar{l},\bar{p})=\overline{(l,p)}$, where the bar denotes a coset, are well defined and inverses to each other. Using isomorphism α we have

$$\begin{aligned}\mathfrak{Ab}\big(E^{(1)}(\mathcal{L}\xrightarrow{\mu}\mathcal{P})_2\big)&\cong\mathcal{L}/[\mathcal{L},(\mathcal{L}\rtimes\mathcal{P})^{n-1}]\times\big(\mathcal{L}\rtimes\mathcal{P}\big)/[\mathcal{L}\rtimes\mathcal{P},\ldots,\mathcal{L}\rtimes\mathcal{P}]\\&\cong\mathcal{L}/[\mathcal{L},\mathcal{P}^{n-1}]\times\mathcal{L}/[\mathcal{L},\mathcal{P}^{n-1}]\times\mathcal{P}/[\mathcal{P},\ldots,\mathcal{P}]=E^{(1)}\big(\mathfrak{Ab}^{(1)}(\mathcal{L}\xrightarrow{\mu}\mathcal{P})\big)_2,\end{aligned}$$

since $[\mathcal{L},(\mathcal{L}\rtimes\mathcal{P})^{n-1}]=[\mathcal{L},\mathcal{P}^{n-1}]$. Indeed, $[\mathcal{L},(\mathcal{L}\rtimes\mathcal{P})^{n-1}]$ is generated by the elements

$$\begin{aligned}&[(l_1,p_1),\ldots,(l_{k-1},p_{k-1}),l,(l_{k+1},p_{k+1}),\ldots,(l_n,p_n)]=[l_1,\ldots,l_{k-1},l,l_{k+1},\ldots,l_n]\\&\quad+\Sigma\{l_1,\ldots,l_{k-1},l,l_{k+1},\ldots,l_n,p_1,\ldots,p_{k-1},0,p_{k+1},\ldots,p_n\}\\&=[\mu l_1,\ldots,\mu l_{k-1},l,\mu l_{k+1},\ldots,\mu l_n]\\&\quad+\Sigma\{\mu l_1,\ldots,\mu l_{k-1},l,\mu l_{k+1},\ldots,\mu l_n,p_1,\ldots,p_{k-1},0,p_{k+1},\ldots,p_n\}\in[\mathcal{L},\mathcal{P}^{n-1}].\end{aligned}$$

By similar computations we get isomorphisms between higher terms of simplicial vector spaces $\mathfrak{Ab}\big(E^{(1)}(\mathcal{L}\xrightarrow{\mu}\mathcal{P})_*\big)$ and $E^{(1)}\big(\mathfrak{Ab}^{(1)}(\mathcal{L}\xrightarrow{\mu}\mathcal{P})\big)_*$, which are compatible with face and degeneracy maps.

Proceeding by induction, we suppose that the assertion is true for $m-1$ and we will prove it for m.

Given a crossed m-cube of Leibniz n-algebras $\mathcal{M}$, by applying the nerve $E^{(1)}$ to $m-1$ directions we obtain an $(m-1)$-simplicial object in the category of crossed modules. Its diagonal, $E^{(m-1)}(\mathcal{M})_*$, is a simplicial crossed module of Leibniz n-algebras. As a consequence of Theorem 8, $E^{(m-1)}(\mathcal{M})_*$ is just a simplicial Leibniz n-algebra endowed with 1-fold category structure induced by some structural endomorphisms s_j, t_j of the corresponding to $\mathcal{M}$ cat^m-Leibniz n-algebra. Since the abelianization of a cat^1-Leibniz n-algebra is just the abelianization of the underlying Leibniz n-algebra endowed with induced structural endomorphisms, the inductive hypothesis implies the isomorphism

$$\mathfrak{Ab}^{(1)}\big(E^{(m-1)}(\mathcal{M})_*\big)\cong E^{(m-1)}\big(\mathfrak{Ab}^{(m)}(\mathcal{M})\big)_*. \tag{2}$$

On the other hand, by construction, $E^{(m)}(\mathcal{M})_*$ is the diagonal of the bisimplicial Leibniz n-algebra obtained by applying the crossed module nerve construction $E^{(1)}$ to the simplicial crossed module $E^{(m-1)}(\mathcal{M})_*$, that is, $E^{(m)}(\mathcal{M})_k=E^{(1)}(E^{(m-1)}(\mathcal{M})_k)_k$, for $k\geq 0$. Since the assertion is true for $m=1$, applying the abelianization functor to this equality and using (2) we have

$$\begin{aligned}\mathfrak{Ab}\big(E^{(m)}(\mathcal{M})_k\big)&=\mathfrak{Ab}\big(E^{(1)}\big(E^{(m-1)}(\mathcal{M})_k\big)_k\big)\cong E^{(1)}\big(\mathfrak{Ab}^{(1)}\big(E^{(m-1)}(\mathcal{M})\big)_k\big)_k\\&\cong E^{(1)}\big(E^{(m-1)}\big(\mathfrak{Ab}^{(m)}(\mathcal{M})\big)_k\big)_k=E^{(m)}\big(\mathfrak{Ab}^{(m)}(\mathcal{M})\big)_k.\quad\square\end{aligned}$$

5. Hopf type formulas

5.1. *m-fold Čech derived functors*

The diagonal of the multinerve of crossed m-cubes of Leibniz n-algebras is closely related to the m-fold Čech derived functors of functors from the category ${}_{\mathbf{n}}\mathbf{Lb}$ to the category of vector spaces, which we consider immediately below, whilst the general situation has been dealt with in [15].

Let us consider the set $\langle m\rangle$. The subsets of $\langle m\rangle$ are ordered by inclusion. This ordered set determines in the usual way a category $\mathbf{C_m}$. For every pair (A,B) of subsets with $A\subseteq B\subseteq\langle m\rangle$, there is the unique morphism $\rho_B^A:A\to B$ in $\mathbf{C_m}$. Any morphism in $\mathbf{C_m}$, not an identity, is generated by $\rho_{A\cup\{j\}}^A$ for all $A\subseteq\langle m\rangle$, $A\neq\langle m\rangle$ and $j\in\langle m\rangle\setminus A$.

An *m-cube of Leibniz n-algebras* is a functor $\mathfrak{F}:\mathbf{C_m}\to{}_{\mathbf{n}}\mathbf{Lb}$. A *morphism between m-cubes* $\mathfrak{F}$, $\mathfrak{F}'$ is a natural transformation $\kappa:\mathfrak{F}\to\mathfrak{F}'$.

Example 10. Let $(\mathcal{L}_*,d_0^0,\mathcal{L})$ be an augmented simplicial Leibniz n-algebra. A natural m-cube of Leibniz n-algebras $\mathfrak{F}^{(m)}=\mathfrak{F}^{(m)}(\mathcal{L}_*,d_0^0,\mathcal{L}):\mathbf{C_m}\to{}_{\mathbf{n}}\mathbf{Lb}$, $m\geq 1$, is defined as follows:

$$\begin{aligned}\mathfrak{F}^{(m)}(A)&=\mathcal{L}_{m-1-|A|}\quad\text{for all }A\subseteq\langle m\rangle,\\\mathfrak{F}^{(m)}(\rho_{A\cup\{j\}}^A)&=d_{k-1}^{m-1-|A|}\quad\text{for all }A\neq\langle m\rangle,\ j\in\langle m\rangle\setminus A,\end{aligned}$$

where $|A|$ denotes the cardinality of A, $\mathcal{L}_{-1} = \mathcal{L}$ and $k \in \langle m - |A|\rangle$ is the preimage of j for the unique monotone bijection $\langle m - |A|\rangle \xrightarrow{\approx} \langle m\rangle \setminus A$ between the subsets $\langle m - |A|\rangle$ and $\langle m\rangle \setminus A$ of positive integers.

Given an m-cube of Leibniz n-algebras $\mathfrak{F}$, there is a natural homomorphism of Leibniz n-algebras $\mathfrak{F}(A) \xrightarrow{\alpha_A} \lim\limits_{B \supset A} \mathfrak{F}(B)$ for any $A \subseteq \langle m\rangle, A \neq \langle m\rangle$.

Definition 11. An m-cube of Leibniz n-algebras $\mathfrak{F}$ will be called an m-presentation of a Leibniz n-algebra $\mathcal{L}$ if $\mathfrak{F}(\langle m\rangle) = \mathcal{L}$. An m-presentation $\mathfrak{F}$ of $\mathcal{L}$ is called free if $\mathfrak{F}(A)$ is a free Leibniz n-algebra for all $A \neq \langle m\rangle$ and it is called exact if α_A is an epimorphism for all $A \neq \langle m\rangle$.

Note that a free exact 1-presentation of a Leibniz n-algebra $\mathcal{L}$ is the same as the free presentation of $\mathcal{L}$ in [9].

The following lemma is straightforward.

Lemma 12. *An augmented simplicial Leibniz n-algebra* $(\mathcal{L}_*, d_0^0, \mathcal{L})$, *with* $\pi_0(\mathcal{L}_*) \cong \mathcal{L}$, *is aspherical if and only if the m-cube of Leibniz n-algebras* $\mathfrak{F}^{(m)}(\mathcal{L}_*, d_0^0, \mathcal{L})$ *is an exact m-presentation of* $\mathcal{L}$ *for all* $m \geq 1$.

Given a homomorphism of Leibniz n-algebras $\alpha : \mathcal{R} \to \mathcal{L}$, the *Čech augmented complex* for α is the augmented simplicial Leibniz n-algebra $(\check{C}(\alpha)_*, \alpha, \mathcal{L})$ given by

$$\check{C}(\alpha)_k = \underbrace{\mathcal{R} \times_{\mathcal{L}} \cdots \times_{\mathcal{L}} \mathcal{R}}_{(k+1)-\text{times}} = \{(r_0, \ldots, r_k) \in \mathcal{R}^{k+1} \mid \alpha(r_0) = \cdots = \alpha(r_k)\},$$

$$d_i^k(r_0, \ldots, r_k) = (r_0, \ldots, \hat{r}_i, \ldots, r_k),$$

$$s_i^k(r_0, \ldots, r_k) = (r_0, \ldots, r_i, r_i, r_{i+1}, \ldots, r_k)$$

for $k \geq 0, 0 \leq i \leq k$.

Now let $\mathfrak{F}$ be an m-presentation of the Leibniz n-algebra $\mathcal{L}$. Applying $\check{C}$ in the m-independent directions, this construction leads naturally to an augmented m-simplicial Leibniz n-algebra. Taking the diagonal we obtain an augmented simplicial Leibniz n-algebra $(\check{C}^{(m)}(\mathfrak{F})_*, \alpha, \mathcal{L})$ called an *augmented m-fold Čech complex* for $\mathfrak{F}$, where $\alpha = \mathfrak{F}(\rho^{\emptyset}_{\langle m\rangle}) : \mathfrak{F}(\emptyset) \to \mathcal{L}$. In case $\mathfrak{F}$ is a free exact m-presentation of $\mathcal{L}$, then $(\check{C}^{(m)}(\mathfrak{F})_*, \alpha, \mathcal{L})$ will be called an *m-fold Čech resolution* of $\mathcal{L}$.

Definition 13. Let $T : {}_{\mathbf{n}}\mathbf{Lb} \to \mathbf{Vect}$ be a functor. Define the kth *m-fold Čech derived functor* $L_k^{(m)}T : {}_{\mathbf{n}}\mathbf{Lb} \to \mathbf{Vect}$, $k \geq 0$, of the functor T by choosing a free exact m-presentation $\mathfrak{F}$ for each Leibniz n-algebra $\mathcal{L}$, and setting

$$L_k^{(m)}T(\mathcal{L}) = \pi_k(T\check{C}^{(m)}(\mathfrak{F})_*),$$

where $(\check{C}^{(m)}(\mathfrak{F})_*, \alpha, \mathcal{L})$ is the m-fold Čech resolution of the Leibniz n-algebra $\mathcal{L}$ for the free exact m-presentation $\mathfrak{F}$ of $\mathcal{L}$.

Note that thanks to [15] the m-fold Čech derived functors are well defined. Furthermore, it follows directly from the definition that, for $k \geq 1$, the value of the kth m-fold Čech derived functor on a free Leibniz n-algebra is trivial.

Lemma 14. *Let* $\mathfrak{F}$ *be an m-presentation of a Leibniz n-algebra* $\mathcal{L}$. *There is an isomorphism of simplicial Leibniz n-algebras*

$$\check{C}^{(m)}(\mathfrak{F})_* \cong E^{(m)}(\mathcal{M})_*,$$

where $\mathcal{M}$ *is the inclusion crossed m-cube of Leibniz n-algebras defined by the Leibniz n-algebra* $\mathfrak{F}(\emptyset)$ *and its n-sided ideals* $\mathcal{I}_i = \operatorname{Ker} \mathfrak{F}(\rho^{\emptyset}_{\{i\}})$, $i \in \langle m\rangle$ *(see Example 6).*

Proof. For $m = 1$ the required isomorphism

$$\lambda_* : E^{(1)}(\mathcal{I}_1 \hookrightarrow \mathfrak{F}(\emptyset))_* \xrightarrow{\approx} \check{C}(\mathfrak{F}(\emptyset) \to \mathcal{L})_*$$

is given by $\lambda_0 = id_{\mathfrak{F}(\emptyset)}$ and $\lambda_k(x_1, \ldots, x_k, f) = (x_1 + \cdots + x_k + f, x_2 + \cdots + x_k + f, \ldots, x_k + f, f)$ for all $k \geq 1$ and $(x_1, \ldots, x_k, f) \in \mathcal{I}_1 \rtimes \cdots \rtimes \mathcal{I}_1 \rtimes \mathfrak{F}(\emptyset)$. It is routine and we left to the reader to check that every λ_k is a homomorphism of Leibniz n-algebras and they commute with the face and degeneracy maps.

Then by repeated application of this isomorphism, we get an isomorphism of m-simplicial Leibniz n-algebras. Applying the diagonal we obtain the result for any m. □

Now we calculate the mth m-fold Čech derived functors of the abelianization functor $\mathfrak{Ab} : {}_{\mathbf{n}}\mathbf{Lb} \to \mathbf{Vect}$.

Theorem 15. *Let* $\mathcal{L}$ *be a Leibniz n-algebra and* $\mathfrak{F}$ *a free exact m-presentation of* $\mathcal{L}$. *Then there is an isomorphism*

$$L_m^{(m)}\mathfrak{Ab}(\mathcal{L}) \cong \frac{\bigcap\limits_{i \in \langle m\rangle} \mathcal{I}_i \cap [\mathcal{F}, \ldots, \mathcal{F}]}{\sum\limits_{A_1 \cup \cdots \cup A_n = \langle m\rangle} [\bigcap\limits_{i \in A_1} \mathcal{I}_i, \ldots, \bigcap\limits_{i \in A_n} \mathcal{I}_i]}, \quad m \geq 1,$$

where $\mathcal{F} = \mathfrak{F}(\emptyset)$ *and* $\mathcal{I}_i = \operatorname{Ker}(\mathfrak{F}(\emptyset) \to \mathfrak{F}(\{i\}))$ *for* $i \in \langle m\rangle$.

Proof. Using Lemma 14, we get $L_m^{(m)}\mathfrak{Ab}(\mathcal{L}) \cong \pi_m\big(\mathfrak{Ab}E^{(m)}(\mathcal{M})_*\big)$, where $\mathcal{M}$ is the inclusion crossed m-cube induced by the Leibniz n-algebra $\mathcal{F}$ and its n-sided ideals $\mathcal{I}_1, \ldots, \mathcal{I}_m$. Hence Proposition 9 implies an isomorphism $L_m^{(m)}\mathfrak{Ab}(\mathcal{L}) \cong \pi_m\big(E^{(m)}\mathfrak{Ab}^{(m)}(\mathcal{M})_*\big)$. Then, by [14, Proposition 13] (see also [6, Proposition 3.4]), there is an isomorphism

$$L_m^{(m)}\mathfrak{Ab}(\mathcal{L}) \cong \bigcap_{l\in\langle m\rangle} \operatorname{Ker}\left(\mathfrak{Ab}^{(m)}(\mathcal{M})_{\langle m\rangle} \xrightarrow{\widetilde{\mu}_l} \mathfrak{Ab}^{(m)}(\mathcal{M})_{\langle m\rangle\setminus\{l\}}\right). \tag{3}$$

By definition of the functor $\mathfrak{Ab}^{(m)}$, we have

$$\mathfrak{Ab}^{(m)}(\mathcal{M})_A = \frac{\bigcap\limits_{i\in A}\mathcal{I}_i}{\sum\limits_{A_1\cup\cdots\cup A_n=A}\big[\bigcap\limits_{i\in A_1}\mathcal{I}_i, \ldots, \bigcap\limits_{i\in A_n}\mathcal{I}_i\big]} \quad \text{for all } A\subseteq\langle m\rangle.$$

Now we set up the inductive hypothesis. Let $m = 1$, then

$$L_1^{(1)}\mathfrak{Ab}(\mathcal{L}) \cong \operatorname{Ker}\left(\frac{\mathcal{I}_1}{[\mathcal{I}_1, \mathcal{F}^{n-1}]} \longrightarrow \frac{\mathcal{F}}{[\mathcal{F}, \ldots, \mathcal{F}]}\right) = \frac{\mathcal{I}_1\cap[\mathcal{F}, \ldots, \mathcal{F}]}{[\mathcal{I}_1, \mathcal{F}^{n-1}]}.$$

Proceeding by induction, let $m \geq 2$ and suppose that the result is true for $m-1$ and we will prove it for m.

Let us consider $l \in \langle m\rangle$ and denote by $\mathfrak{F}^{\overline{\{l\}}}$ the restriction of the functor $\mathfrak{F} : \mathbf{C_m} \to {}_{\mathbf{n}}\mathbf{Lb}$ to the subcategory of $\mathbf{C_m}$ consisting of all subsets $A \subseteq \langle m\rangle$ with $l \notin A$. It is easy to check that $\mathfrak{F}^{\overline{\{l\}}}$ is a free exact $(m-1)$-presentation of the free Leibniz n-algebra $\mathfrak{F}(\langle m\rangle \setminus \{l\})$. Since $L_{m-1}^{(m-1)}\mathfrak{Ab}\big(\mathfrak{F}(\langle m\rangle\setminus\{l\})\big) = 0$, our inductive hypothesis implies that

$$\bigcap_{i\in\langle m\rangle\setminus\{l\}}\mathcal{I}_i \cap [\mathcal{F}, \ldots, \mathcal{F}] = \sum_{A_1\cup\cdots\cup A_n=\langle m\rangle\setminus\{l\}}\big[\bigcap_{i\in A_1}\mathcal{I}_i, \ldots, \bigcap_{i\in A_n}\mathcal{I}_i\big]. \tag{4}$$

Then from (3) and (4) we can easily deduce the required isomorphism. □

5.2. The main result

We finally give the main theorem of this paper, which expresses the homology of Leibniz n-algebras with trivial coefficients by Hopf type formulas.

We need the following lemma which is the Leibniz n-algebra analog of the well-known fact for simplicial groups (see for example [22]).

Lemma 16. *Let $\mathcal{L}_*$ be a simplicial Leibniz n-algebra and $A \subseteq \langle m\rangle$, $A \neq \langle m\rangle$. Then $d_m^m(\cap_{i\in A}\operatorname{Ker} d_{i-1}^m) = \cap_{i\in A}\operatorname{Ker} d_{i-1}^{m-1}$, $m \geq 2$.*

Theorem 17 (*Hopf Type Formula*). *Let $\mathcal{L}$ be a Leibniz n-algebra and $\mathfrak{F}$ a free exact m-presentation of $\mathcal{L}$. Then there is an isomorphism*

$${}_nHL_m(\mathcal{L}) \cong \frac{\bigcap\limits_{i\in\langle m\rangle}\mathcal{I}_i\cap[\mathcal{F}, \ldots, \mathcal{F}]}{\sum\limits_{A_1\cup\cdots\cup A_n=\langle m\rangle}\big[\bigcap\limits_{i\in A_1}\mathcal{I}_i, \ldots, \bigcap\limits_{i\in A_n}\mathcal{I}_i\big]}, \quad m \geq 1,$$

where $\mathcal{F} = \mathfrak{F}(\emptyset)$ and $\mathcal{I}_i = \operatorname{Ker}\big(\mathfrak{F}(\emptyset) \to \mathfrak{F}(\{i\})\big)$ for $i \in \langle m\rangle$.

Proof. Let $(F_*, d_0^0, \mathcal{L})$ be an aspherical augmented simplicial Leibniz n-algebra. Consider the short exact sequence of augmented simplicial Leibniz n-algebras

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & & [F_m,\ldots,F_m] & \overset{\widetilde{d_0^m}}{\underset{\widetilde{d_m^m}}{\vdots}}\cdots\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & [F_1,\ldots,F_1] & \overset{\widetilde{d_0^1}}{\underset{\widetilde{d_1^1}}{\rightrightarrows}} & [F_0,\ldots,F_0] & \xrightarrow{\widetilde{d_0^0}} & [\mathcal{L},\ldots,\mathcal{L}]\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & & F_m & \overset{d_0^m}{\underset{d_m^m}{\vdots}}\cdots\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & F_1 & \overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}} & F_0 & \xrightarrow{d_0^0} & \mathcal{L}\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & & \mathfrak{Ab}(F_m) & \vdots\cdots\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & \mathfrak{Ab}(F_1) & \rightrightarrows & \mathfrak{Ab}(F_0) & \longrightarrow & \mathfrak{Ab}(\mathcal{L})\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

By the induced long exact homotopy sequence and the fact that all homotopy groups of F_* are trivial in dimensions ≥ 1, we have the isomorphisms of vector spaces

$$\pi_m \mathfrak{Ab}(F_*) \cong \frac{\bigcap\limits_{i\in\langle m\rangle} \operatorname{Ker} \widetilde{d_{i-1}^{m-1}}}{\widetilde{d_m^m}(\bigcap\limits_{i\in\langle m\rangle} \operatorname{Ker} \widetilde{d_{i-1}^m})}, \quad m \geq 1. \tag{5}$$

Since $\widetilde{d_i^m}$ is the restriction of d_i^m to $[F_m, \dots, F_m]$, we have $\operatorname{Ker} \widetilde{d_i^m} = \operatorname{Ker} d_i^m \cap [F_m, \dots, F_m]$. Hence $\cap_{i\in\langle m\rangle} \operatorname{Ker} \widetilde{d_{i-1}^m} = \cap_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1}^m \cap [F_m, \dots, F_m]$ and $\cap_{i\in\langle m\rangle} \operatorname{Ker} \widetilde{d_{i-1}^{m-1}} = \cap_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1}^{m-1} \cap [F_{m-1}, \dots, F_{m-1}]$.

Since the shift of F_* is the contractible augmented simplicial object $(Dec(F_*), d_0^1, F_0)$ (see [23]), by Lemma 12 the m-cube of Leibniz n-algebras $\mathfrak{F}^{(m)}(Dec(F_*), d_0^1, F_0)$ is a free exact m-presentation of F_0. Hence, by Theorem 15 we have

$$L_m^{(m)}\mathfrak{Ab}(F_0) \cong \frac{\bigcap\limits_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1}^m \cap [F_m, \dots, F_m]}{\sum\limits_{A_1\cup\dots\cup A_n=\langle m\rangle} [\bigcap\limits_{i\in A_1} \operatorname{Ker} d_{i-1}^m, \dots, \bigcap\limits_{i\in A_n} \operatorname{Ker} d_{i-1}^m]} = 0, \quad m \geq 1,$$

implying, for $m \geq 1$, the following equality

$$\bigcap_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1}^m \cap [F_m, \dots, F_m] = \sum_{A_1\cup\dots\cup A_n=\langle m\rangle} [\bigcap_{i\in A_1} \operatorname{Ker} d_{i-1}^m, \dots, \bigcap_{i\in A_n} \operatorname{Ker} d_{i-1}^m]. \tag{6}$$

Since $(F_*, d_0^0, \mathcal{L})$ is an aspherical augmented simplicial Leibniz n-algebra, $d_m^m(\cap_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1}^m) = \cap_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1}^{m-1}$, $m \geq 1$. Using this fact, the equality (6) and Lemma 16, it is easy to see that

$$\begin{aligned}\widetilde{d_m^m}(\bigcap_{i\in\langle m\rangle} \operatorname{Ker} \widetilde{d_{i-1}^m}) &= d_m^m\Big(\sum_{A_1\cup\dots\cup A_n=\langle m\rangle} [\bigcap_{i\in A_1} \operatorname{Ker} d_{i-1}^m, \dots, \bigcap_{i\in A_n} \operatorname{Ker} d_{i-1}^m]\Big) \\ &= \sum_{A_1\cup\dots\cup A_n=\langle m\rangle} \Big[\bigcap_{i\in A_1} \operatorname{Ker} d_{i-1}^{m-1}, \dots, \bigcap_{i\in A_n} \operatorname{Ker} d_{i-1}^{m-1}\Big].\end{aligned}$$

Thus by (5) and Theorem 4 we have

$${}_nHL_m(\mathcal{L}) \cong \frac{(\bigcap\limits_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1}^{m-1}) \cap [F_{m-1}, \dots, F_{m-1}]}{\sum\limits_{A_1\cup\dots\cup A_n=\langle m\rangle} \Big[\bigcap\limits_{i\in A_1} \operatorname{Ker} d_{i-1}^{m-1}, \dots, \bigcap\limits_{i\in A} \operatorname{Ker} d_{i-1}^{m-1}\Big]}.$$

Using again Lemma 12 and Theorem 15 the proof is completed. □

This result extends the Hopf formula for the first homology of a Leibniz n-algebra [9] to higher homologies. Moreover, for $n = 2$, Theorem 17 describes the $(m+1)$th homology $HL_{m+1}(\mathcal{L})$ of a Leibniz algebra $\mathcal{L}$ via a Hopf type formula, for all $m \geq 1$.

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