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MATHEMATICS

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Topological Aspects of Random Polynomials

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ABSTRACT. **We introduce a class of random polynomials for which it is possible to estimate the mean value of topological degree of the gradient mapping. In particular, we estimate the rate of growth of mean gradient degree as the algebraic degree of random polynomial tends to infinity and in some cases find its exact value. Some applications and generalizations of these results are also discussed.**

Key words: Gaussian random variable, random polynomial, gradient mapping, Jacobian, topological degree, Euler characteristic.

1. Statistics of real roots of random real polynomials and systems of random polynomial equations was actively studied for more than fifty years [1 - 5]. Recent results about the statistical properties of real roots of random real polynomial systems [3 - 5] suggest some natural topological problems concerned with the fibers of random real polynomial mappings. Results of [3 - 5] were basically concerned with computing the expected number of real roots of random real polynomial systems with a certain Gaussian distribution of coefficients introduced in [3] but in those papers there were no attempts to deal with topological invariants.

Taking into account that some topological characteristics of real polynomials and polynomial endomorphisms (e.g., Euler characteristic of levels, mapping degree) can be defined by properly counting real roots of polynomial systems, we decided to consider some related problems of topological flavour for random polynomials. For example, given a Gaussian random polynomial system as in [3] one may wish to estimate the expected algebraic number of its real roots [6], in other words the mean topological (mapping) degree of the corresponding random polynomial endomorphism [7]. Natural as it seems, this problem appears to remain practically uninvestigated (only some elementary results for $n=2$ may be found in [8]) and our aim is to shed some light on this and related problems.

Another natural problem is to estimate the mean Euler characteristic of the fibers of a random real mapping with arbitrary nonnegative fiber dimension. This problem also remains largely uninvestigated despite interesting results on the mean.

Euler characteristic of Gaussian random hypersurfaces were obtained in [9]. Notice that the second problem in principle can be reduced to the first one, since the Euler characteristic of an algebraic set can be expressed through the local degree of the gradient of an auxiliary polynomial using the formulae presented in [7].

So we concentrate on the mean topological degree and describe some cases in which one can compute or estimate it in terms of the algebraic degrees of the given random polynomials. In particular, we find the asymptotic of the mean value of the absolute topological degree of gradient mapping of random real polynomial of algebraic degree m with rotation invariant Gaussian distribution of coefficients introduced in [3]. We also present some related results and formulate several open problems which seem to be tractable by the methods described in this paper.

2. We begin with presenting some definitions and auxiliary results concerned with polynomial mappings and random polynomials. If a real polynomial mapping F is defined

by a collection of t polynomials in s variables with real coefficients, we say that s is the source dimension while t is called the target dimension. We will only deal with situations when $s\leq t$ and in such cases the difference $s-t$ is called the fiber dimension of F. A mapping with vanishing fiber dimension ($s = t$) is called a polynomial endomorphism or a polynomial vector field [6,7]. One may somehow think of a vector field as a sort of drift and we use this word in the sequel.

The full preimage y^{-1} of a point y is called the fiber over y. A mapping F is called proper if the full preimage of any compact set is a compact set. For convenience and brevity, a proper real polynomial mapping F as above will be referred to as propomap. Correspondingly, a proper polynomial endomorphism will be called a propodrift. We only deal with real polynomials and mappings.

Obviously, regular fibers of a propomap F are compact smooth manifolds. By Sard's lemma [6], the set of singular values of F has measure zero so a "generic" fiber of F is a smooth compact manifold of dimension $s - t$.

In particular, if $s=t$ then each fiber is finite. In this case it appears reasonable to consider the algebraic number of preimages of a regular value, or the mapping degree F. Recall that for a propodrift F its topological (mapping) degree Deg F can be computed as

$$\text{tDeg } F=\Sigma\,\text{sign}\det JF(x)$$

where sign denotes the sign of a real number, JF is the Jacobi matrix of F, and y is an arbitrary regular value of F.

The sum in the right hand side does not depend on y and Deg F is invariant under proper homotopies of F. From the results of [7] it follows that the degree of a given propodrift can be computed in a purely algebraic way as the signature of a certain nondegenerate quadratic form which is explicitly constructible using the Taylor coefficients of components of F. This implies that the Euler characterisic of a given real algebraic set can also be computed in a purely algebraic way from the coefficients of defining equations [7].

We recall now some concepts and results concerned with Gaussian random polynomials. Speaking of such polynomial simply means that its coefficients are real random variables and have multivariate normal distribution [2,3]. The term "random polynomial" always refers to this situation and we freely use standard concepts and results from probability theory. Let us only mention that the word central indicates that the mean value of a random variable is equal to zero, in other words this is a distribution with zero mean. We only deal with central Gaussian distributions.

3. Let P be a central Gaussian random polynomial on R^s in the above sense. To specify this distribution it is sufficient to indicate the covariance matrix of its coefficients. For us especially important is the rotation invariant central Gaussian random polynomial introduced in [3] which we call a convenient random polynomial.

It can be described as follows. Denote by $H(m,n+1) = H_m(R^{n+1})$ the set of all real homogeneous polynomials of algebraic degree m (m-forms) in $n+1$ variable and consider a random homogeneous polynomial F from $H(m,n+1)$ whose coefficients are in dependent normal random variables with zero mean and variances

$$Ea_k^2 = \frac{m!}{\prod_{j=0}^{n} k_j!}$$

It was shown in [3] that such a random polynomial is invariant with respect to the natural action of the group $O(n+1)$ on $H(m,n+1)$. It is worthy of noting that such random polynomials appear in some problems of quantum physics [4].

Consider now a central Gaussian random polynomial on R^{n+1} of algebraic degree N such that each of its homogeneous components of degree $m \leq N$ has a rotation invariant central Gaussian distribution as above. In such situation we speak of a convenient Gaussian random polynomial of algebraic degree N. This is precisely the type of random polynomial we wish to consider. The main reason for such choice is that in this case there exist comprehensive results about the expected number of real roots [3,4] which can be used for estimating other invariants of such random polynomials.

4. We can now give rigorous description of some of our results. Suppose we are given a Gaussian random polynomial P in s variables. A whole series of nontrivial problems can be formulated in terms of computing the mean values of various topological invariants of the fibers of random polynomial mappings associated with convenient random polynomials. For example, by taking s of such polynomials we obtain a random drift in R^s.

It is easy to show that such a drift F is almost surely (a.s.) proper so its fibers are a.s. finite and its topological (mapping) degree Deg$F$ is a.s. well-defined so one may wish to compute the expectation $E(\mathrm{Deg}\ F)$ of this random variable and/or of its absolute value. We call these mean values the mean topological degree of F and mean absolute degree of F, respectively.

For a canonical random drift associated with convenient Gaussian polynomial as above, the expected number of real roots was computed in [3,4]. However it remains unclear how to find an explicit formula for its mean topological degree or mean absolute degree. Below we estimate these invariants in a different setting.

Namely, we consider a random propodrift $F=P'$ defined as the gradient mapping of a given random polynomial P, i.e., the components of P' are given by the partial derivatives of P. Notice that in this case the components of F are not independent. Such random drifts were earlier considered in [8]. We aim at finding the asymptotic of the mean gradient degree as m tends to infinity. To this end let us assume that P is a convenient Gaussian p-lolynomial as above.

Theorem 1. *The mean absolute gradient degree $E(|\mathrm{Deg}P'|)$ of convenient Gaussian random polynomial in $n+1$ variables of algebraic degree m, is asymptotically equivalent to $n^{-1}m^{n/2}$ as m tends to infinity.*

If the source is even-dimensional (notice that the source dimension in the above example is denoted by $n-1$) we can also find the mean gradient degree. So we suppose now that n is odd and P is again a convenient random polynomial of algebraic degree m. Introduce the number $M(n,m)$ by equality

$$M(n,m) = \frac{I_n(\sqrt{m})}{I_n(1)}, \text{ where } I_n(s) = \int_0^s (1-t^2)^{\frac{n-1}{2}}\, dt.$$

Theorem 2. *For an odd natural n, the average gradient degree $E(\mathrm{Deg}\ P')$ of convenient Gaussian random polynomial P in $n-1$ variables of algebraic degree m, is equal to $1-M(n,m)$.*

Proofs of these theorems rely on the results obtained in [3,4,9] and suggest that similar results can be obtained for convenient random drifts which are not necessarily gradient. For small odd n, it is easy to compute the average gradient degree by our formula. For example, for $n=1$ one gets $E(\mathrm{Deg}\ P)=1-\sqrt{m}$. which explicates the main result of [8] and confirms that the rate of growth found in [8] was correct.

Using the aforementioned explicit formulae for the Euler characteristic from [7], one can obtain some estimates for the rate of growth of the mean Euler characteristic of fibers of convenient random polynomial mapping. The problem of computing the average topological degree for an arbitrary Gaussian random drift is open and seems very difficult.

There exist many other problems of such kind which are quite interesting even in low imensions. For example, one may wish to find the expected number of cusps [6] of rotation invariant Gaussian random drift of the plane and/or the mean linking number of two Gaussian random curves in R^5 [7] and/or the expected number of umbilic points on Gaussian random surfaces appearing in optics [10]. In all these problems it appears crucial to estimate the mean topological degree of an auxiliary mapping, so our results give some estimates for those invariants as well. These and other problems of such kind will be considered in forthcoming publications of the author.

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REFERENCES

1. *M.Kac.* Bull. Amer. Math. Soc. 49, 1943, 314-320.
2. *A.Bharucha-Reid, M.Sambandham.* Random polynomials, Academic Press, New York, 1980.
3. *M.Shub, S.Smale.* Progr. Math. 109, 1993, 267 - 285.
4. *A.Edelman, E.Kostlan.* Bull. Amer. Math. Soc. 32(1995), 1 - 37.
5. *A.McLennan.* Amer. J. Math.124, 2002, 49 - 73.
6. *V.Arnold, A.Varchenko, S.Gusein-zade.* Singularities of differentiable mappings.1. Moscow, 1982 (Russian).
7. *G.Khimshiashvili.* *A.Razmadze* Math. Inst. Proc. 125, 2001, 1-121.
8. *G.Khimshiashvili, A.Ushveridze.* Bull. Georgian Acad. Sci. 159, 1999, 385-388.
9. *S.Podkorytov.* Algebra i Analiz. 11, 1999, 185-193.
10. *M.Berry, J.Hannay.* *J. Phys.* A: Math. Gen. 10, 1977, 1809-1821.

მათემატიკა

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