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MATHEMATICS

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On the Fibers of Proper Polynomial Mappings

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ABSTRACT. **We establish that each orientable compact smooth manifold is diffeomorphic to a component of fiber of a certain proper quadratic mapping. In the opposite direction, it is shown that one can obtain considerable topological information about the smooth fibers of a given proper polynomial mapping using signature formulae for the mapping degree and Euler characteristic. We also discuss some related problems about fibers of proper polynomial mappings.**

Key words: smooth manifold, proper polynomial mapping, bifurcation diagram, isolated singular point, mapping degree, Euler characteristic

1. Polynomial mappings of real vector spaces constitute a natural and important class of mappings which can be studied from many points of view. In particular, it is known that the structure of their fibers is quite complicated and exhibits a lot of interesting phenomena studied in real algebraic geometry and singularity theory [1,2].

From the classical results of J.Nash and R.Tognoli it follows that each orientable smooth compact manifold is diffeomorphic to a regular (smooth) fiber of a certain polynomial mapping [3]. This establishes a fundamental intrinsic relation between differential topology and real algebraic geometry and shows that in some sense the topological study of general smooth manifolds can be reduced to the investigation of fibers of polynomial mappings. In this note we develop some aspects of this relation suggested by recent results of W.Thurston concerned with moduli spaces of planar linkages [4,5].

It turns out that those results of W.Thurston imply that in the aforementioned result of J.Nash and R.Tognoli it is sufficient to consider only the fibers of proper real quadratic mappings. Thus the study of real quadratic mappings becomes of special importance and suggests some natural problems which are discussed in the sequel. In particular, in this context it becomes especially desirable to have some effective methods for the topological study of their fibers.

The topology of quadrics and their intersections for a long time was ([6], Ch.13) and remains [7] an object of intensive study. In particular, A.Agrachev and R.Gamkrelidze developed an effective approach to the computation of Euler characteristics of the fibers of quadratic mappings [7]. The topology of fibers can be also studied using the signature formulae for the mapping Euler characteristic (see, e.g., [8]). In the sequel we describe some situations in which these approaches lead to quite effective results.

2. Recall that a polynomial mapping $F: R^s \to R^t$ is defined by a collection of t polynomials in s variables with real coefficients. We say that s is the source dimension while t is called the target dimension. The difference s-t is called the descent of F. The mapping of zero descent ($s = t$) is called a polynomial endomorphism [8] or a polynomial vector field [2]. For any $y \in R^s$, the set $F^{-1}(y)$ is called a fiber of F over the point y. Obviously, each fiber of a polynomial mapping is a (real) algebraic variety. It is well known that each regular fiber (i.e., the one which does not contain singular points of F) is a smooth manifold of dimension equal to the descent s-t.

As usual, a mapping F is called proper if preimage $F^{-1}(X)$ of any compact set X is a compact set. For convenience and brevity, a proper real polynomial mapping F as above will be referred to as a propomap. The fibers of a propomap are compact algebraic varieties. Correspondingly, regular fibers of a propomap F are orientable compact smooth manifolds. By Sard's lemma [3], the set of singular values of F is of measure zero so a "generic" fiber of F is a smooth manifold of dimension s-t. A natural question if each compact closed (i.e., without boundary) smooth manifold can be represented as a regular fiber of a propomap was given a positive answer in the works of J.Nash and R.Tognoli [3]. Their result can be formulated in terms of polynomial mappings as follows.

Theorem 1. *For each compact closed smooth manifold M, there exists a polynomial mapping F such that M is diffeomorphic to a connected component of some fiber of F.*

If all components of F are polynomials (not necessarily homogeneous) of algebraic degree two then F is called a quadratic map. Quadratic maps are the simplest nonlinear propomaps which appear in many problems of analysis and geometry so it is natural to ask which smooth manifolds can be represented as the regular fibers of a quadratic maps and this is exactly the question to be addressed to in the sequel.

3. To this end we need to recall some constructions from the theory of linkages [4,5]. A linkage (L,l) is defined as a graph L with a positive real number $l(e)$ assigned to each edge e. We assume that there is chosen a distinguished oriented edge e^* in L and in such case we speak of a based linkage. The planar moduli space $M(L)$ of a based linkage L is defined as the set of all maps from the vertex set V of L into the Euclidean plane such that the image of edge e^* coincides with the segment $[(0,0), (l(e^*),0)]$ and the distance between the images of each pair of vertices joined by an edge e is equal to $l(e)$ [5].

It is easy to see that $M(L)$ comes with a natural topology inherited from the Euclidean plane. As is well known, for a generic linkage L the (planar) moduli space $M(L)$ is an orientable compact smooth manifold of the dimension $2n - m - 3$, where n is the number of vertices of L and m is the number of its edges [5]. Moduli spaces of linkages were studied in various contexts. In particular, W.Thurston proved that for any orientable smooth compact manifold M there exists a linkage L such that its planar moduli space is diffeomorphic to a disjoint union of a number of copies of M. In other words, M is diffeomorphic to some (actually any) connected component of $M(L)$. In many cases L can be chosen so that $M(L)$ is connected and is itself diffeomorphic to M.

We want to use this result to show that each compact smooth manifold can be represented as a regular fiber of a propomap. To this end let us take into account that each planar moduli space $M(L)$ has actually a natural structure of a real algebraic variety. Indeed, if one writes down the condition that the square length between the images of ends of each edge e is equal to the square of $l(e)$ then in terms of Euclidean coordinates in the plane then it turns out that the moduli space is exactly the fiber over the origin of a certain propomap $F : R^{2n-4} \to R^{m-1}$. Moreover, all components of this map are easily seen to be quadratic polynomials. This means that we can actually represent $M(L)$ as a fiber of a proper quadratic map Q which is called the map associated with L. Taking into account the cited result of W.Thurston we come to the following improvement of Theorem 1.

Theorem 2. *For each compact closed smooth manifold M, there exists a proper quadratic mapping Q such that M is diffeomorphic to a connected component of some fiber of Q.*

Actually, by applying the argument which was used by A.Tognoli [9] to elaborate the original result of J.Nash [3], one can show that there exists a quadratic mapping Q such that M is diffeomorphic to a (whole) fiber of Q. In such situation we say that Q exhibits M. If moreover quadratic map Q is associated with a planar linkage L in the way described above, we say that Q exhibits M. Notice that there is no natural way of constructing a quadratic map or linkage which exhibit a given manifold so a number of natural problems can be formulated in this setting.

For example, it is pretty clear that there exist many quadratic maps exhibiting a given manifold M and one may wonder what is the simplest possible choice. In other words, what can be the minimal values of the source and target dimension of such quadratic map Q.

Some easy observations are immediate. For example, the cases when $t=1$ are very rare since the possible topological types of a real quadric are well known for all source dimensions. So in most case we have really to deal with intersections of several quadrics. Thus a natural approach to the above problem is to classify possible toplogical types of quadratic mappings with fixed values of s and t. Many results are known for $t=2$ ([6], Ch. 13, [7]). In general case, Euler characteristics of the fibers can be estimated using the signature formulae from [8].

For example a torus T^2 cannot be diffeomorphic to a quadric but one can easily verify that it can be represented as an intersection of two quadrics in R^4 so it is exhibited by the corresponding quadratic mapping into R^2. Obviously, two is the minimal possible source dimension in this case. One can add that the sought map Q can be chosen homogeneous.

When we have already found a proper quadratic map Q exhibiting M it is natural to have a look at other regular (smooth) fibers of Q. The corresponding smooth manifolds can be called quadratically adjacent to M. The same notion makes sense for homeomorphy (diffeomorphy) types of those fibers. One can now wonder what are the smooth manifolds quadratically adjacent to a given manifold M or what is the maximal possible number of pairwise nonhomeomorphic fibers exhibited by quadratic maps with fixed values of source and target dimension. Obviously this is closely related to the possible topological changes happening in the fibers of a given quadratic mapping.

A natural approach is to describe the possible types of bifurcation diagrams [2] of quadratic mappings with the fixed source and target dimension. Another interesting aspect is related with the minimal possible number of monomials entering in the components of quadratic maps exhibiting a given manifold.

All these questions can be answered for quadratic mappings in low dimensions but we will not dwell upon them in short note like this one. We only point out that these problems become especially visual and attractive in the context of planar linkages exhibiting a given manifold and we wish to present some brief remarks on the latter issue.

4. An arbitrary quadratic mapping need not be associated with a planar linkage so it also makes sense to ask what is the "simplest" linkage exhibiting a given M. As a natural measure of "simplicity" one can take the number of vertices of a linkage exhibiting M This problem can be easily solved for one-dimensional M but already the case of a two-dimensional compact closed surface M appears nontrivial. We suggest a universal approach to this problem based on signature formulae for topological invariants [8].

For any nonnegative integer M let $M(g)$ denote the orientable compact closed two-dimensional surface of genus g ("two-sphere with g handles") and let $n(g)$ denote the minimal value of integer n such that there exists a planar graph with n vertices and configuration space homeomorphic (hence diffeomorphic) to $M(g)$. In other words, $n(g)$ is the minimal cardinality of vertices among the linkages exhibiting $M(g)$. In view of said above, the number $n(g)$ is well-defined and it is desirable to compute it as a function of g or at least find some estimates which give its rate of growth as g tends to infinity.

By a bifurcation diagram argument in the spirit of [2] it is easy to show that, for each fixed n, there exist a finite set of values of g such that $M(g)$ s homeomorphic to the configuration space of a planar linkage with n vertices. Hence $n(g)$ certainly cannot remain bounded as g tends to infinity. In fact, $n(g)$ exhibits rather nonregular behaviour as function of g, as will become clear from our next result. However one can consecutively find it as follows.

For a number $n>2$ and consider all connected linkages with n vertices and $2n$-5 edges having a fixed combinatorial type. Obviously the number of such combinatorial types is finite for each fixed n one can find the associated quadratic mapping and its bifurcation diagram for all of them and get a finite list of possible homeomorphy types according to components of the complements to bifurcation diagrams. Then one determines the topological type of regular fiber in each case by computing its Euler characteristic using formulae from [8] and ends up with the finite list of two-dimensional surfaces exhibited by linkages with n vertices. In virtue of Thurston's theorem, each $M(g)$ will appear in this list at certain step and the number of this step is exactly $n(g)$. Notice that as a by product we can also get the tables of quadratic adjacency for small values of g.

For example, for $n=4$ one can exhibit two-torus T^2 by taking a three-arm (open chain with three sides) so $n(1)=4$. We also managed to study the case $n=5$.

Theorem 3. *One has: $n(g)=5$ for $n=2,3,4$.*

Applying the same strategy, we can prove for example that $n(5)=6$ and $n(6)$ does not exceed 7.

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მათემატიკა

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