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MATHEMATICS

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Counting Roots of Quaternionic Polynomials

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ABSTRACT. **We present several results on the structure of zero-set of a quaternionic polynomial. In particular, it is shown that the Euler characteristic of the zero-set is equal to its algebraic degree, which may be considered as a generalization of the Fundamental Theorem of Algebra. We also indicate an effectively verifiable sufficient condition which guarantees that the zero-set is finite.**

Key words: quaternionic polynomial in standard form, Jacobian matrix, Euler characteristic, mapping degree, signature of a quadratic form.

1. We consider the zero-sets of polynomials of one variable over the algebra of quaternions **H** [1]. More precisely, we deal with the so-called monic polynomial of algebraic degree n in standard form

$$P(q) = \Sigma a_s q^s, \quad a_0, \dots, a_n = 1 \in \mathbf{H}.$$

The highest coefficient a_n is always set to be equal to one and P is referred to as a standard quaternionic polynomial of degree n.

As was proved by S.Eilenberg and I.Niven [2] such a polynomial always has a root in **H** (see also [3]). At the same time, it is well known that the set of roots of such a polynomial can be infinite. For example, the zero-set of polynomial $P(q) = q^2 + 1$ consists of all purely imaginary quaternions of modulus one which form the unit two-dimensional sphere in the hyperplane $\{\mathrm{Re}\, q = 0\}$. It is also easy to produce examples where the zero-set contains isolated points as well as infinite components. One may wonder, how to detect such cases and what is a proper way of counting roots of such polynomials. In this note we explain how one can do that using some concepts from topology and algebraic geometry. Some results in this direction were also obtained in [3].

2. Let us first give an effectively verifiable sufficient condition of the finiteness of zero-set $Z=Z(P)$. Consider P as a polynomial endomorphism of a four-dimensional Eulcidean space and denote by J its Jacobian matrix. As is well known, its determinant j (Jacobian of P) is a nonnegative polynomial [2,3]. By inverse function theorem, if q_0 is a root of P and $j(q_0) \neq 0$, then P is a local diffeomorphism at q_0 and in particular q_0 is an isolated root of P. Thus if j does not vanish on the zero-set of P we can be sure that the zero-set consists of isolated points.

Notice that the endomorphism defined by P is proper, i.e., full-preimage of any compact set is compact [3]. Thus the set of isolated zeroes cannot be infinite because otherwise they should accumulate at infinity which would contradict properness of P. Actually, the amount of isolated roots cannot exceed n. Indeed, as was proved in [2] and [3] the (global) topological degree of P is equal to n. Recalling the definition of mapping degree in terms of Jacobian [4] one gets that, if all roots are simple then each of them is counted with plus sign (since j is nonnegative everywhere) so the amount of such zeroes exactly coincides with the value of degree equal to n. If some of isolated roots are multiple, this means that some of simple roots have collided at these points so the total amount of isolated roots cannot exceed n.

Thus if we have somehow established that j does not vanish on Z, we may be sure that Z is finite. Notice that this is actually a question about existence of common real roots of five polynomials on R^4, four of them P_1, P_2, P_3, P_4 being components of P and the fifth is j. Recall now that there exists the so-called signature technique of counting real roots [4] which enables one to effectively solve such problems. Under an effective solution as usual it is understood that the answer is obtained using a finite number of algebraic and logical operations over the coefficients of given polynomials (in our case, over coefficients of P) [4]. Actually, one can give an explicit formula for the number of real solutions of such a system.

To this end introduce five auxiliary polynomials of five variables

$$h_i(x_0,\dots,x_4) = (x_0)^{n+1} P_i(x_1/x_0,\dots,x_4/x_0),\ i=1,2,3,4,\quad H_P = \Sigma\,(h_i)^2 - \Sigma\,(x_i)^{2n+4}.$$

As was proved in [4], polynomial H_P has an isolated critical point at the origin of $\mathbf{R}^5$ so that the local topological degree of its gradient at the origin, $\deg_0 \operatorname{grad} H_P$, is well-defined.

Theorem 1. *If $\deg_0 \operatorname{grad} H_P = 1$ then the zero-set of P is finite. This condition may be effectively verified using a finite number of algebraic and logical operations over coefficients of P.*

This follows from the explicit formula for the cardinality M of the common real roots in $\mathbf{R}^n$ given in Theorem 8.2 of [4] which, for n=4, reads

$$2M = 1 - \deg_0 \operatorname{grad} H_P.$$

The compactness condition needed in that theorem is fulfilled due to the properness of P. We would like to add that nowadays there already exist computer algorithms for calculating the local topological degree [5] so this condition can be easily verified using a computer.

3. Let us now investigate the structure of zero-set when it is infinite. Some properties of such zero-sets were established in [3], in particular it was shown that such phenomenon takes place for polynomials with real coefficients which possess non-real roots. In that case each nonreal root gave rise to a two-dimensional sphere of roots due to the process of „cloaning" roots by means of conjugation by purely imaginary quaternions of unit length [3]. It is straightforward to check that each pair of complex-conjugate roots produces a whole two-dimensional sphere of quaternionic roots of P.

More precisely, it is easy to see that every complex number $z = a + ib$ is „quaternion conjugate" to its complex conjugate $a - ib$. Thus in such case the zero-set consists of isolated points and several smooth two-dimensional spheres. In other words, the zero-set consists of several smooth submanifolds and this can be proved in general by a direct argument using properties of jacobian matrix J.

Indeed, by a direct inspection of such matrices it is possible to prove that their ranks at the roots of P can only take three values: 0, 2, 4. The first one is excluded for a non-zero polynomial P, while the third one corresponds to an isolated root. In remaining case when $rk\,J = 2$, by implicit function theorem one concludes that the zero-set near such a point is a smooth two-dimensional surface.

Theorem 2. *Each component of the zero-set of a standard quaternionic polynomial is a compact smooth submanifold of dimension zero or two.*

Actually, one can show that the zero-set is always a union of a finite set of points and several smooth copies of two-dimensional sphere S^2. This suggests that in this case one should find a way of properly counting components of the zero-set and this would lead to a generalization of the Fundamental Theorem of Algebra. It turns out that this can be done using an appropriate version of the concept of Euler characteristic.

More precisely, we use the Euler characteristic of the structural sheaf of a complex algebraic subset which plays considerable role in many topics of algebraic geometry [6]. This becomes possible since in our case the zero-set Z has a natural structure of a complex algebraic set. In order to show that, notice that by interpreting **H** as a two-dimensional complex space, P defines a polynomial endomorphism of C^2 called complexification of P. Thus the zero-set Z can be represented as the zero-set of two complex polynomials appearing as the components of the complexification of P.

Notice that in the case of a smooth subset, the sheaf-theoretical Euler characteristic reduces to the usual Euler characteristic of this subset multiplied by the multiplicity of its generic point. In our situation this is the usual multiplicity of an isolated root or the multiplicity of a two-sphere appearing as a component of Z.

For example, if all isolated roots and two-dimensional components are simple (of multiplicity one) then the value of this Euler characteristic is equal to the number of isolated points plus two times the number of two-spheres. In general case each summand should be multiplied by the corresponding multiplicity. If all roots are simple and isolated this apparently coincides with the algebraic degree n.

Notice now that from the properness of standard polynomials it follows that its complexification is a flat holomorphic mapping in the sense of algebraic geometry [6]. By a fundamental theorem of Grothendieck this implies that, for every $w \in \mathbf{H}$, the Euler characteristic of the structural sheaf of pre-image $P^{-1}(w)$ remains unchanged [6]. Taking into account that in our case this is the usual Euler characteristic counted with multiplicity and that it is equal to n for every regular value w of P, we arrive to the main result.

Theorem 3. *The Euler characteristic of the zero-set of a standard quaternionic polynomial is equal to its algebraic degree.*

Generically the roots have multiplicity one, and this results implies that the number of isolated roots plus two times the number of spheres is equal to the algebraic degree. Moreover, it is possible to show that, if all coefficients of P lie in the same two-dimensional subalgebra of **H** then all roots are isolated, lie in the same subalgebra, and their amount is equal to n, which is apparently a reformulation of the Fundamental Theorem of Algebra [1].

It is instructive to have a closer look at this result in the case when all coefficients are real numbers. If such a P has n real roots then there are no other roots and the result is trivial. If among the roots appear some complex conjugate pairs then, as was explained above, each complex conjugate pair generates a two-dimensional sphere of roots via „cloaning" of roots by conjugations (see [3]). Apparently the amount of two-spheres will be exactly such as predicted by **Theorem 3**.

4. We would like to conclude by explaining how one can obtain a direct proof of Theorem 3 without referring to general results of algebraic geometry. To this end one can use an alternative approach to criterion of finiteness developed by N.Topuridze [3]. Recall that the centralizer (commutant) $C(P)$ of polynomial P is defined as the intersection of centralizers $C(a_s)$ of all of its coefficients. As is easy to see, $C(P)$ can have dimension 1, 2, or 4 [3].

The latter case just means that all coefficients of P are real and one can successfully study this case using the signature technique for counting real roots [5]. It follows that the set of roots is finite if and only if all roots are real and this can be effectively checked using signature formulae for the topological degree [5]. If there are non-real roots, one can apply the aforementioned analysis and see that Theorem 3 holds in this case.

If dim $C(P) = 2$, then it is easy to see that all coefficients a_s belong to the same two-dimensional subalgebra isomorphic to C. By the Fundamental Theorem of Algebra, there exist n roots (counted with multiplicites) of P belonging to the same subalgebra. Using the fact that Jacobian of P is nonnegative one can show that there can be no isolated roots outside this subalgebra (otherwise the value of deg P would exceed n). Some pairs of roots can again generate two-dimensional spheres by „cloaning" but it is not difficult to verify that the amount of those spheres is exactly such as stated in Theorem 3. So the theorem remains true also in this case.

The most troublesome is the case when dim $C(P) = 1$. As was explained in [3] one can use homotopies preserving dim $C(P)$ to reduce the whole matter to the case of trinomials. So it remains to verify **Theorem 3** for an arbitrary trinomial, which can be done by a direct application of signature formulae [5]. Details of the argument will be published elsewhere.

These results may be generalized in several directions of which we mention only two. First, one can analogously treat standard polynomials over an arbitrary finite-dimensional associative algebra. Effective criteria for the finiteness of the zero-set are again available using signature technique. It would be interesting to find some geometric interpretations in the case of a Clifford algebra [1].

Another natural possibility is to consider arbitrary noncommutative polynomials which are by definition the finite sums of finite „words" of the form *aqbqcq*..... It is known that such polynomials need not always have a root in H, which may be observed already for linear equations of the form $aq + bq = c$. At the same time, if such a polynomial contains only one word of the maximal length, then it has a quaternionic root [1]. In such case one can again apply the result of Grothendieck and obtain the constancy of the Euler characteristic of the structural sheaf of zero-set.

However, this result is less illuminating because it is not a priori known what can be the components of zero-set in this case so it is not clear if one can obtain a visual statement like Theorem 3 in this case. Correspondingly, it would be very interesting to obtain some general conclusions about the possible structure of zero-set of noncommutative quaternionic polynomials.

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მათემატი

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კვატერნიონულ პოლინომთა ფესვების დათვლა

რეზიუმე. აღწერილია კვატერნიონული პოლინომის ფესვთა სიმრავლის ტოპ
ლოგიური სტრუქტურა. კერძოდ, დამტკიცებულია, რომ ფესვთა სიმრავლის ეილ
რის მახასიათებელი უდრის პოლინომის ალგებრულ ხარისხს, რაც გვაძლევს აღ
გებრის ძირითადი თეორემის განზოგადებას.

Z. Tediash

Uniqueness for Two-Dimensional D
of the Potentia

Presented by Corr. Member of the Acad

ABSTRACT. **Inverse problems of the poten sional domains. The uniqueness theorems are boundaries of the domains and densities of pote**

Key words: inverse problems, potential theo

In the inverse problems of the potential th density of a body (or one of them) if the outer p most principal question in the study of these ill lem.

Let Ω be some bounded domain in $\mathbf{R}^2$ $B(x,\varepsilon)=\{y\in\mathbf{R}^2:|x-y|<\varepsilon\}$ is an open ball cente

$K(x,y) := \frac{1}{2\pi}\ln|x-y|$ stands for the fundame

$V_{\partial\Omega}(\mu)(x)= \int_{\partial\Omega} K(x,y)\mu(y)dS_y$ is a single layer p

If Ω_1 and Ω_2 are two bounded domains in

connected component of $\mathbf{R}^2 \setminus \overline{\Omega_1 \cup \Omega_2}$ and Ω

Formulation of the problem. Let Ω_1, Ω_2 be t function defined on $\partial\Omega_1\cup\partial\Omega_1$. Moreover, let

$$V_{\partial\Omega_1}(\mu)(x) = V_{\partial\Omega_2}(\mu)(x) \text{ for al}$$

By this condition we have to define the loc respect to each other.

To prove the uniqueness theorems we essenti

Lemma 1. *Let Ω_1 and Ω_2 be two bounded function on $\partial\Omega_1\cup\partial\Omega_2$ such that $V_{\partial\Omega_1}(\mu)(x) = V_{\partial\Omega}$*

Then

$$\int_{\partial\Omega_1} u(y)\mu(y)dS_y = \int$$