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MATHEMATICS

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On the Complex Points of Planar Endomorphisms

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ABSTRACT. **We present an explicit formula for the number of complex points on the graph of a polynomial endomorphism of the plane. We also show that one can compute the number of elliptic complex points and explain relation of our results to Bishop's problem.**

Key words: complex point of a surface, elliptic complex point, Gaussian curvature.

1. We deal with certain aspects of the so-called Bishop's problem concerned with the existence of analytic discs attached to the graph of a complex valued function in the plane [1]. As was shown by Bishop, in many cases a positive solution to this problem follows from the existence of points where the tangent plane to graph is a complex line in $\mathbf{C}^2$. Such points are called complex points of a given function (and its graph) and they can be of the two essentially different types according to the local geometric properties of the graph near the point in question. Thus it is very desirable to obtain detailed information about the existence and amount of the complex points of both types and this is exactly the problem which we are going to attack. In this note we give an effective solution of the latter problem in the case when the real and imaginary parts of the given function are (real) polynomials of the real and imaginary parts of the argument. Obviously the same assumption may be expressed by saying that we are given a polynomial endomorphism of the plane which will be called a planar endomorphism (plend).

Following the general strategy of singularity theory [2] it is reasonable to consider first „generic" plends which satisfy certain jet transversality conditions. For our purposes it is appropriate to require that a plend is proper and the Gauss mapping of its graph is transversal to the subset of complex lines G_C in the real Grassmanian Gr[2,4] (cf. [3], [4]). We call them „perfect plends" (by analogy with „excellent maps" of H.Whitney [5]).

It is well known that such plends are indeed generic in the standard sense [1], i.e., they form an open dense subset in the space of all plends [2], [3]. In particular any plend can be approximated by arbitrarily close perfect ones (so-called perfect perturbations). From our transversality condition by dimension reasons it follows that a perfect plend can only have a finite number of complex points. A general „deformation paradigm" of singularity theory suggests then that important information on an arbitrary plend can be obtained by counting complex points of its sufficiently small perfect perturbations.

2. In line with that we concentrate on counting complex points of a perfect plend. Let $F = (f,g)\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a perfect endomorphism defined by real polynomials $f(x,y)$, $g(x,y)$. Denote by G_F its graph in $\mathbf{R}^2\times\mathbf{R}^2$ which is identified with C^2 in the usual way. Recall that a point $p\in\mathbf{R}^2$ is called a complex point of F (and G_F) if the tangent plane T_pG_F is a complex line in C^2.

Denote by $\partial_* = 1/2\,(\partial/\partial x + i\,\partial/\partial y)$ the usual ∂-bar operator in the plane and put $F_* = \partial_* F$. Complex points admit a simple analytic characterization.

Lemma 1. *A point p is a complex point of plend F if and only if* $\partial_* F(p) = 0$.

In other words, the set of complex points $C(F)$ coincides with the zero set of plend F_*, which suggests that one can calculate the number of complex points $c(F)$ as the cardinality of algebraic subset $\{F_* = 0\}$, which can be done in an algorithmic way using results of [6]. In order to realize that, it is necessary to verify that F_* satisfies conditions of Theorem 8.2 of [6].

Lemma 2. *If F is a perfect plend then F_* is a proper planar endomorphism.*

Indeed, the homogeneous forms of highest degree of the components of F_* are easily seen to have no non-trivial common zeroes, which implies that $|F_*|$ is growing at infinity, hence F_* is proper. We can now use Theorem 8.2 of [6], which leads to an explicit algorithm for computing $c(F)$. Suppose that the maximum of algebraic degrees of f and g is equal to d. Introduce auxiliary polynomials

$$P(x,y,z) = z^{d+1} f_*(x/z, y/z),\quad Q(x,y,z) = z^{d+1} g_*(x/z, y/z),\quad H(x,y,z) = P^2 + Q^2 - (x^{2d+4} + y^{2d+4} + z^{2d+4}).$$

Recall that the local topological degree $\deg_p G$ of a plend G is defined at any point p which is isolated in the full preimage of its image [6]. Our first main result is an explicit formula for $c(F)$.

Theorem 1. *Polynomial H has an isolated critical point at the origin of* R^3 *and*

$$c(F) = 1/2\,(1 - \deg_0 \operatorname{grad} H),$$

where grad H denotes the gradient mapping of polynomial H.

Corollary 1. *The number of complex points of a perfect planar endomorphism can be effectively computed using a finite number of algebraic and logical operations over its coefficients.*

Both these results follow from the discussion in Chapter 8 of [6] so we omit the details. It may be added that there already exist computer algorithms for calculation of the local topological degree [8] so our theorem enables one to compute the number of complex points in many concrete cases.

3. Our next task is to obtain more detailed information about the structure of complex points of a given perfect plend F. By a fundamental result of E.Bishop, near a generic complex point there exits a germ of biholomorphism of C^2 such that, up to infinitesimals of the third order, the graph G_F of F lies in the three-dimensional subspace $\{\operatorname{Im} z_2 = 0\}$ and coincides with the set $\{\operatorname{Re} z_2 = h(x,y)\}$ [1]. In such a representation, there can happen two essentially different situations - either the graph of F lies on one side of the tangent plane (elliptic case) or the tangent plane intersects any arbitrarily small part of the graph near this point (hyperbolic case). In the first case the point is called an elliptic complex point while in the second case it is called a hyperbolic complex point.

As was shown by E.Bishop in the same paper [1], near each elliptic complex point of F there exists a family of analytic discs attached to the graph of F. Thus the Bishop's problem is automatically solved if one is able to guarantee existence of elliptic complex points (such strategy was applied in [3,4]). So our next goal becomes to compute the number $e(F)$ of elliptic complex points of F.

To this end we use results of [6] on counting points of finite semi-algebraic subsets. This becomes possible because the set of elliptic complex points $C_e(F)$ of a perfect plend appears to be a semi-algebraic subset of the plane. The simplest way to show that, is to

appeal to the notion of Gaussian curvature
Gaussian curvature at point p of the graph

Lemma 3. *A complex point p is an*
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Now we can apply results of Chapter 9
the effective computability of $e(F)$.

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Actually, this number can be expressed
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4. In conclusion we show that some in
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appeal to the notion of Gaussian curvature of a two-surface [7]. We denote by $K_F(p)$ the Gaussian curvature at point p of the graph of a perfect plend F.

Lemma 3. *A complex point p is an elliptic complex point if and only if the Gaussian curvature* $K_F(p)$ *is positive.*

This follows from the aforementioned representation of the graph G_F as a hypersurface in three-dimensional Euclidean space, because in that representation it becomes obvious that $K_F(p)$ is positive exactly in the case when the graph lies on one side of the tangent plane at p.

Proposition 2. *The set of elliptic complex points is a semi-algebraic subset of the plane.*

From preceding remarks it is clear that $C_e(F) = \{F_* = 0, K_F > 0\}$ and the result follows if we show that the second condition can be expressed as a polynomial inequality. In order to show that, we refer to some well-known results about the Gaussian curvature of a parameterized two-dimensional surface [7]. First of all, from the general formula for the Gaussian curvature in terms of Riemann tensor [7], it follows that in our case the sign of K_F coincides with the sign of the component R_{1212} of the Riemann tensor R of G_F. The component R_{1212} can be computed by simple explicit formulae which show that, for a two-dimensional surface given by a polynomial parametrization, it is a polynomial function of parameters. In our case this means that R_{1212} is a polynomial in x,y so the condition of positivity of K_F is indeed a polynomial inequality, which shows that $C_e(F)$ is indeed a semi-algebraic subset.

Now we can apply results of Chapter 9 of [7] and obtain the desired conclusion about the effective computability of $e(F)$.

Theorem 2. *The number of elliptic complex points of a perfect plend can be effectively computed using a finite number of algebraic and logical operations on its coefficients.*

Actually, this number can be expressed through signatures of explicitly constructible quadratic forms on the factor-algebra of polynomial algebra over the ideal generated by components of F_* (cf. [6]). In concrete cases necessary computations can be done using the program from [8].

4. In conclusion we show that some information about the complex points of small perfect deformations of an arbitrary proper plend may be obtained in a purely algebraic way from its coefficients. To this end we use a version of the Maslov index which was introduced in [3, 4]. In our setting it is convenient to define the Maslov index of a proper plend F in geometric way in the spirit of [3, 4]. For a proper plend, the number of complex points is always finite so its graph is a totally real surface „at infinity", i.e., outside sufficiently big discs. Take a circle S of sufficiently big radius and consider its image $G(S)$ in $Gr(2,4)$ under the Gauss map G of the graph G_F. Obviously $G(S)$ is an oriented one-dimensional submanifold which does not intersect the oriented two-dimensional submanifold of complex lines G_c and the sum of dimensions of these two oriented submanifolds is by one less than the dimension of the ambient manifold $Gr(2,4)$ so one can define the linking number of these two submanifolds.

Actually, in order that complex points are counted properly it is necessary to consider both connected components G_+, and G_- of G_c consisting of complex lines with their natural orientation,and with the opposite orientation respectively. Then the Maslov index of G_F is defined as the linking number $L(G(S), G_+ \cup -G_-)$, where $-G_-$ denotes the component taken with the opposite orientation (cf. [3, 4]). It is easy to verify that it does not depend on a (sufficiently big) circle S and is invariant under homotopies of proper plends. If the complex points are generic then the Maslov index can be computed by properly counting complex points of various types, in particular formulae from [3] and [4] can be easily extended to this situation. Since Maslov index is homotopy invariant this means that it gives some information about complex points of small perfect perturbations. Thus our next result can be considered as a first step towards counting complex points of perfect perturbations.

Theorem 3. *The Maslov index of a sufficiently small perfect perturbations of a proper plend F can be computed as the local topological degree of an explicitly constructible planar endomorphism.*

Corollary 2. *The Maslov index is equal to the signature of the so-called Gorenstein quadratic form on the local algebra of an explicitly constructible polynomial endomorphism.*

Both these statements follow from the method of computing Maslov index in terms of the local topological degree developed in [3, 4] and from the well-known algebraic formula for the local topological degree (see, e.g., [6], Chapter 5).

Using estimates for the topological degree of homogeneous polynomial endomrophisms it is now easy to give an exact estimate for the possible values of Maslov index of a planar endomorphism with fixed algebraic degrees of its components (cf. [6], Chapter 6). These estimates and similar estimates for the number of complex points and elliptic complex points will be presented elsewhere.

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მათემატიკა

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სიბრტყის ენდომორფიზმთა კომპლექსური წერტილების შესახებ

რეზიუმე: დადგენილია ზუსტი ფორმულა სიბრტყის ენდომორფიზმის კომპლექ-სური წერტილების რიცხვისათვის. გამოთვლილია აგრეთვე ელიფსური წერტილე-ბის რაოდენობა.

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Member of the Acad

On Marcinkiewicz Type

ABSTRACT. **In the paper the best existence and integrability of Marcin erators are established.**

Key words: multidimensional

1. We use some notations from elements of n ($n \geq 2$)-dimensional

2. We consider the functions $f \in L(T^n)$. We denote

3. In the paper the best possible function