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Mathematics

Holomorphic Structures in Seifert Fibrations

George Khimshiashvili*, Robert Wolak**

* *A. Razmadze Mathematical Institute, Tbilisi*
** *Yagiellonian University, Krakow, Poland*

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ABSTRACT. We show that each Seifert fibration of an orientable closed three-dimensional manifold has an intrinsic complex structure induced from the loop space of manifold. It is also shown that the natural mapping from the leaf space of Seifert fibration to the loop space of manifold is meromorphic. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *closed 3-dimensional manifold, Seifert fibration, loop space, complex structure, pseudoholomorphic curve, loop space, foliated solid torus, orbital invariants.*

1. The aim of this note is to establish an intrinsic relation between the geometry of three-dimensional manifolds (3-folds) and the theory of pseudoholomorphic curves in almost complex manifolds. Since the pioneering paper of M.Gromov [1], pseudo-holomorphic curves gradually became a standard tool of symplectic geometry and complex analysis (see, e.g, [2]). As is well known one can define such objects in any manifold endowed with an almost complex structure. Recently, some interesting examples of almost complex structures on loop spaces of 3-folds were constructed (see [3, 4]) which appeared useful in topology and differential geometry. This suggested that it might also be useful to consider pseudo-holomorphic curves in such loop spaces and a number of papers on this topic appeared quite recently [5-7].

In particular, it was shown in [7] that pseudoholomorphic curves in loop spaces of 3-folds locally look like solid tori foliated by closed curves and if a pseudoholomorphic curve is stable then it defines a Seifert fibration of the target manifold. This established a natural link between pseudoholomorphic curves in loop spaces and the theory of Seifert fibrations [8]. In the present note we elaborate upon some results of [7] by showing that each Seifert fibration generates a meromorphic curve in the loop space of manifold considered.

2. We recall first a few concepts of complex analysis. Let M be a smooth (infinitely differentiable) manifold with tangent bundle TM. An almost complex structure on M is defined by a linear endomorphism J on TM such that $J^2 = -I$, where I denotes the identity. Notice that this definition is also applicable in the case of an infinite dimensional manifold M modelled on a Banach or Frechet space [2]. If M is a complex manifold then its tangent planes T_pM are complex vector spaces so such an endomorphism J can be defined as multiplication by $\mathbf{i} = \sqrt{-1}$ in each tangent plane. If an almost complex structure arises in such way from a complex manifold, it is called integrable. A criterion of integrability is given by the famous Newlander-Nirenberg theorem [9].

Recall that a loop in a manifold M is defined as a continuous mapping g of the unit circle T into M. The totality of all loops in M is traditionally denoted ΛM. We will only work with smooth loops. Loop spaces we shall be dealing with, have earlier been considered by J.-L.Brylinski [3] and L.Lempert [4]. Recall that the immersed loop space BX of Riemannian 3-fold X has a natural almost complex structure J introduced by J.-L.Brylinski [3]. The main objects of our study are holomorphic maps of the unit disc D into the almost complex manifold (BX,J) which is called the Brylinski loop space of X [4]. We briefly recall the basic construction from [3].

Let X be a three-dimensional oriented smooth manifold endowed with Riemannian structure. Denote by M the totality of smooth immersed knots in X [3, 4]. As is explained in [3], M can be endowed with a natural almost complex structure which is non-integrable [4]. In order to make our exposition self-contained we briefly recall the basic definition from [3] which states that M consists of equivalence classes of smooth immersions of the circle S^1 into X. Two immersions are called equivalent if one can be obtained from another by composing with an orientation preserving diffeomorphism of S^1. Equivalence class of an immersion f is denoted by [f]. Elements of M are called immersed loops.

The standard construction of topology on mapping spaces applied to M turns it into an infinite dimensional manifold modelled on Frechet space $C^\infty(S^1, \mathbf{R}^2)$ [3]. The tangent plane to M at point [f] is naturally identified with the space of smooth sections of the normal bundle N to $C = f(S^1)$ in X. A natural almost complex structure on X arises as follows. One defines an endomorphism J on T_CM by describing its action on a normal vector field v along C. Namely, Jv = w if at each point $p\in C$ vectors v(p) and w(p) are orthogonal, have the same length, and the triple (t(p),v(p),w(p)) is positively oriented, where t(p) is the unit vector in the direction of tangent vector to C (this direction is well-defined since we only used orientation preserving diffeomorphisms in the definition of equivalence classes). It is geometrically obvious that $J^2 = -I$ and it is easy to see that this is a smooth endomorphism of the tangent bundle.

So we obtain an almost complex structure on M. M with this almost complex strucutre is called the *Brylinski loop space* of X and denoted BX [4]. Thus it becomes possible to speak of pseudoholomorpic curves in M. For brevity and convenience, we call them *loopy holomorphic curves* (LHC*)*. As usual, holomorphicity of a differentiable map between two almost complex manifolds means that its differential intertwines the operators of almost complex structures considered [1]. We often omit the prefix “pseudo” and speak simply of holomorphic maps and curves when this cannot lead to a misunderstanding.

Basic existence theorems for loopy holomorphic curves were proved in [4, 7]. A closer look at the geometry and topology of their images reveals that they typically look like solid tori foliated by closed curves, which exhibits obvious analogy with Seifert fibrations [7]. We are basically interested in studying the images of (the germs of) LHC and introduce some relevant concepts.

Definition 1. A holomorphic doughnut in M is defined as an image of a (pseudo)holomorphic map $F: D \to BM$, where D is a smooth simply connected domain in **C**. For a given immersed loop [f] in BM, we say that F is a holomorphic doughnut through [f] if $0 \in D$ and [F(0)] = [f]. Loop [f] is called the core of F.

Intuitively, one can think of a holomorphic doughnut as a family of immersed loops holomorphically depending on a complex parameter. Generically such a family constitutes a foliated solid torus in X which is a standard pattern in Seifert fibrations theory [8]. In general, solid tori play a big role in three-dimensional geometry and topology. In particular, many 3-folds can be obtained by gluing solid tori along their boundaries [9]. For example, it is well known that a 3-sphere S^3 is diffeomorphic to the union of two solid tori $\mathbf{T}_1$, $\mathbf{T}_2$ glued along their boundaries by a diffeomorphism which identifies meridians of $\mathbf{T}_1$ with parallels of $\mathbf{T}_2$. As was shown in [4, 5], the inverse to Hopf fibration $H: S^3 \to S^2$ is an LHC in BS^3. In other words, the arising natural map $H^{-1}: \mathbf{P} \to BS^3$ is pseudoholomorphic and each of $\mathbf{T}_1$, $\mathbf{T}_2$ is a loopy analytic disc. Here **P** is the Riemann sphere with the canonical complex structure.

3. Our nearest aim is to show that a LHC locally looks like a foliated solid torus. The following result shows that all solid tori foliated by closed curves appear in this way up to leafwise diffeomorphism. Recall that derivative $F'(0)$ of a loopy holomorphic disc is a section of the pull-back on the normal vector bundle N_MK, K = [F(0)], and it can be thought of as a normal vector field along [F(0)]. If such a vector field is nowhere vanishing then its winding number *indv* along [F(0)] is well-defined [5].

Theorem 1. *Let $f: S^1 \to X$ be a real-analytic embedded knot in a 3-fold X and v be a nowhere vanishing real analytic section of the pull-back bundle $f^*(N_M(f(S^1)))$. Then there exists a loopy holomorphic disc $F: D \to BX$ such that $F(0) = f(S^1)$, $F'(0) = v$, the loops [F(s)], $s \in D$, do not intersect, and their union $[F] = \cup \{[F(s)], s \in D\}$ is diffeomorphic to a solid torus. Moreover, the images [F(s)], $s \in D$ define an analytic foliation of [F], and the core multiplicity of [F] is equal to indv.*

The result follows using an explicit geometric construction of a special tubular neighbourhood of $K = f(S^1)$ suggested in [7].

In fact, this theorem is also applicable to LHC which arise from multiple loops like $z \to z^p$. The resulting loopy holomorphic curve is a p times covered solid torus foliated by toric (p,q)-curves, where q is the winding number of v along the core. Strictly speaking, multiple loops are not immersed and do not belong to the Brylinski loop space as defined in [3]. However it is not difficult to extend the setting so that becomes applicable for multiple loops as well.

4. Suppose now we are given a Seifert fibration S of a 3-fold X as above and denote by L the leaf space of S [8]. It is well known that L is a real two-dimensional or bifold and locally looks like a factor of unit disc with respect to cyclic group. Moreover, the set E of exceptional (singular) fibers is finite. Thus $L^* = L - E$ is an open two-dimensional surface. It is well known that such a surface can be endowed with a complex structure and all those complex structures are equivalent. Notice that we also have a natural map of L into BX.

Theorem 2. *With the above assumptions the natural map $F:L \to BX$ is meromorphic and all of its singular points belong to L^*.*

This means that the collection of regular fibres of Seifert fibration possesses an intrinsic complex structure. The singularities at points of E are indeed essential and so one cannot extend F to the whole of L in a holomorphic way. An interesting problem is to characterize those Seifert fibrations for which the mapping constrcuted above is holomorphic. Taking into account that Seifert fibrations are stable in the sense of foliation theory [8], one might hope to establish certain stability properties of the corresponding loopy holomorphic curves using the standard methods of singularity theory. Actually, in the next section we establish a result which may be considered as a sort of converse.

5. The general definition of stable mapping used in singularity theory [9], in our setting sounds as follows. First of all, the set of all LHC in a given 3-fold has a natural topology making it a Frechet manifold [3]. A holomorphic curve $F: Y \to BX$ is called stable if, for any other LHC G sufficiently close to F in Frechet topology, there exist diffeomorphisms S and T of Y and X, respectively, such that F = TFS, in other words, G is right-left equivalent to F (cf. [9]).

Theorem 3. *Let X be an orientable Riemannian 3-fold, Y a compact Riemann surface, and $F: Y \to BX$ a stable loopy holomorphic curve. Then the velocity vector fields $F'(x)$ are nowhere vanishing for all $x \in X$, images of loops $[F(y)]$, $y \in Y$ are nonintersecting, and the collection of loops $[F(y)]$, $y \in Y$, defines a Seifert fibration of X.*

The proof of Theorem 3 is rather lengthy but the crucial points are quite natural and easy to describe. First, one shows that intersecting loops cannot appear in a stable situation because any intersection can be eliminated by a small holomorphic perturbation. Next, one proves that if a velocity vector field vanishes at the certain point, then this situation can be also eliminated by a small holomorphic perturbation. It remains to show that the loops [F(y)] behave like leaves of a foliation.

In conclusion we mention that most of the concepts and problems discussed above are meaningful for Seifert fibrations of orientable manifolds of an arbitrary odd dimension. An intriguing problem is to obtain analogs of our results for holomorphic mappings of higher-dimensional complex manifolds into loop spaces.

Summing up, the relation between Seifert fibrations and holomorphic curves in loop spaces established above suggests a number of problems which are apparently worthy of further investigation.

მათემატიკა

ჰოლომორფული სტრუქტურები ზაიფერტის ფიბრაციებში

გ. ხიმშიაშვილი[*], რ. ვოლაკი[**]

* *ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი*
** *იაგელონიანის უნივერსიტეტი, კრაკოვი, პოლონეთი*

(წარმოდგენილია აკადემიკოს გ. ხარატიშვილის მიერ)

ნაჩვენებია, რომ ყოველ ზაიფერტის ფიბრაციას ჩაკეტილ ორიენტირებულ სამგანზომილებიან მრავალნაირობაზე გააჩნია კომპლექსური სტრუქტურა, რომელიც ინდუცირებულია მრავალნაირობის მარყუჟთა სივრციდან. ნაჩვენებია აგრეთვე, რომ ბუნებრივი ასახვა ფიბრაციის ფენათა სივრციდან მრავალნაირობის მარყუჟთა სივრცეში მერომორფულია.

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