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Mathematics

On Stable Quaternionic Polynomials

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ABSTRACT. We present several results on the location and structure of the zero-set of a quaternionic polynomial. Our main result provides an effectively verifiable criterion of stability of such polynomials. We also explain how one can find the number of components of the zero-set having negative real parts. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *unilateral quaternionic polynomial, stable polynomial, Euler characteristic, mapping degree, quadratic form, signature.*

1. We deal with the zero-sets of certain polynomials of one variable over the algebra of quaternions **H** [1]. Consider a so-called *unilateral quaternionic polynomial* of algebraic degree n having the form

$$P(q) = a_n q^n + a_{n-1} q^{n-1} + \ldots + a_1 q + a_0,\ a_0, \ldots , a_n \in \mathbf{H}.$$

Such polynomials naturally form a left **H**-module. It is also useful to consider products of such polynomials assuming that the variable q commutes with coefficients. If the coefficient by monomial of the highest degree is equal to one then P is called a monic (unilateral) quaternionic polynomial of degree n.

Obviously, while investigating the structure of the zero-set $Z(P)$ of P, without reall loss of generality one can assume that P is monic and we will do so in the sequel.

As was proved by S.Eilenberg and I .Niven [2], such a polynomial always has a root in **H** (see also [3], [4]). At the same time, it is well known that the zero-set of such a polynomial can be infinite. For example, the zero-set of polynomial $P(q) = q^2 + 1$ consists of all purely imaginary quaternions of modulus one which form a unit two-dimensional sphere in the hyperplane $\mathbf{H_0} = \{\mathrm{Re}\ q = 0\}$. It is also easy to produce examples like $q^3 - q^2 + q - 1 = (q-1)(q^2 + 1)$ where the zero-set contains isolated points as well as infinite components.

Recently, a comprehensive description of zero-sets of unilateral quaternionic polynomials was achieved in [3], [4], [5]. In particular, it was proved in [3], [4] that the zero-set $Z(P)$of such a polynomial P consists of points and two-dimensional metric spheres. Moreover, the results of [3] and [5] imply that the Euler characteristic of the structural sheaf of the zero-set is equal to the algebraic degree of polynomial. In other words, if one takes into account the multiplicities of components of the zero-set, then the number of points plus the doubled number of spheres is equal to n [5].

The latter result gives a natural analog of the "Fundamental Theorem of Algebra" for quaternionic polynomials. It should be added that the methods of [4] and [5] enable one to effectively calculate the number of spherical components of $Z(P)$ for any concrete polynomial P. Since the topological Euler characteristic of $Z(P)$ can be also calculated by the Bruce formula (cf., e.g., [5]), in this way one can find the number of isolated zeroes, which provides substantial information about the geometric structure of $Z(P)$.

Similar but less precise results have been obtained in a recent paper [6]. More precisely, it is shown in [6] that the number of isolated zeroes plus the doubled number of spherical zeroes does not exceed the algebraic degree of P, but there are no results about the sum of multiplicities of components analogous to the aforementioned quaternionic version of the Fundamental Theorem of Algebra obtained in [5]. Moreover, the problem of finding the number of spherical components of $Z(P)$ was not considered in [6].

2. Having the above results, one may investigate further problems concerned with the zero-sets of quaternionic polynomials. Several interesting problems of such kind were suggested in a recent paper [7] in relation with some problems of mathematical physics. According to [7], to establish the stability of solutions to certain quaternionic differential equations, it is important to have methods of checking that all roots of a given unilateral quaternionic polynomial have negative real parts.

This is a natural quaternionic analog of the classical Maxwell problem about the stable complex polynomials [8]. By analogy with the complex case let us say that a quaternionic polynomial is *stable* if all of its roots have negative real parts. In other words, a polynomial is stable if all of its roots lie in the left half-space defined by the hyperplane $\mathbf{H_0}$.

We aim at establishing an effective criterion of stability of quaternionic polynomial P which is in the spirit of the classical Stodola theorem [8]. In general case, one may wish to find the number of components of $Z(P)$ which lie on the left of $\mathbf{H_0}$. We shall show that this more general problem can also be solved effectively using the signature formulae for topological invariants presented in [5]. It should be added that our considerations and results make an essential use of the results obtained in [3-5].

3. We proceed with presenting the main result. Let P be a unilateral quaternionic polynomial as above. Introduce a new unilateral polynomial P^* which is obtained from P by changing each of its coefficients with its conjugate. In other words, we put $P^*(q) = \sum (a_s)^* q^s$, where the asterisk denotes quaternionic conjugation which acts by changing the sign of the imaginary part of the quaternion. Next, we put $N(P) = PP^*$. Obviously, $N(P)$ is monic and the algebraic degree of N(P) is 2n. The following fact, which was established in [3], can be verified by direct calculation.

Lemma 1. *All coefficients of polynomial $N(P)$ are real numbers.*

Notice now that, given a real polynomial R, one can consider the set $ZZ(R)$ of pairwise sums of roots of R where the number of appearances of each root of R *is equal to its multiplicity. Obviously, there exists a uniquely defined real monic polynomial $Q(R)$ such that its zero-set coincides with $ZZ(R)$.*

Lemma 2. *Coefficients of polynomial $Q(R)$ are algebraically expressible through coefficients of R.*

Indeed, the coefficients of $Q(R)$ are symmetric functions of the roots of R, so by the fundamental theorem on symmetric polynomials they can be algebraically expressed through the elementary symmetric functions of the roots which by the Viete theorem are expressible through the coefficients of R.

We are now in a position to formulate the main result.

Theorem 1. *A unilateral quaternionic polynomial P is stable if and only if all the coefficients of polynomials $N(P)$ and $Q(N(P))$ are positive*.

The proof relies on results of [4] and [8] and goes in two steps. First, one derives from the results of [4] that the set of the real parts of the roots of P is always finite and coincides with the set of the real parts of the roots of $N(P)$. Next, one uses the results on stable real polynomials to show that the stability of $N(P)$ is equivalent to the positivity of coefficients of $N(P)$ and $Q(N(P))$. Details will be presented elsewhere.

Notice that this criterion is indeed effective because the coefficients of $N(P)$ and $Q(N(P))$ can be algebraically computed from the coefficients of P. Moreover, it works even in the case if the zero-set of P is infinite. In this way we obtain a complete solution of the stability problem for unilateral quaternionic polynomials.

2. Let us now present a more general result concerned with the problem of finding the number of components of Z(P) having negative real parts. Notice that this problem is meaningful even in the case where $Z(P)$ is infinite because all roots lying in one spherical component have the same real parts.

In order to treat this problem consider P as a polynomial endomorphism of four-dimensional real vector space. Then, as was explained in [4], [5], one can use the Bruce formula [4] to algorithmically compute the geometric Euler characteristic $g(P)$ of $Z(P)$ which is equal to the number of geometrically distinct isolated roots plus the doubled number of the geometrically distinct spherical roots. Notice that, unlike the aforementioned sheaf-theoretic Euler characteristic of $Z(P)$, $g(P)$ need not be equal to n.

Analogously, using the results on the computation of the Euler characteristic of semi-algebraic set presented in [4] one can algorithmically compute the (geometric) Euler characteristic $g\_(P)$ of the set of roots lying in the left half-space $\mathbf{H}\_$. Taking into account that the final formula involves only the signatures of quadratic forms effectively constructible from the coefficients of P [4], we arrive at the second main result.

Theorem 2. *The Euler characteristic $g\_(P)$ of the set of the roots of P lying in the left half space* $\mathbf{H}\_$ *can be effectively calculated from the coefficients of P using a finite number of algebraic and logical operations.*

Obviously, if $g\_(P) = 0$, then polynomial P is stable. Thus Theorem 2 is indeed more general than Theorem 1.

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REFERENCES

1. *S. Lang* (1965), Algebra. N.Y.
2. *S. Eilenberg, I. Niven* (1942), Bull. Amer. Math. Soc. 28, 123-130.
3. *N. Topuridze* (2003), Georgian Math. Journal 10, 745-762.
4. *G. Khimshiashvili* (2001), Proc. A.Razmadze Math. Inst. 125, 1-121.
5. *G. Khimshiashvili* (2002), Bull. Georgian Acad. Sci. 165, No.3, 465-468.
6. *A. Pogorui, M. Shapiro* (2004), Complex Variables 49, No.6, 379-389.
7. *G. Ducati, S. Di Leo* (2001), J. Math Physics 42, 2036-2065.
8. *M. Postnikov*. (1982), Stable Polynomials. (Russian) Nauka, Moscow.
9. *P. Griffiths, J. Harris* (1982), Principles of Algebraic Geometry. N.Y.

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