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# CYCLIC POLIGONS AS CRITICAL POINTS

Khimshiashvili G.

*I. Chavchavadze State University*
*32 I. Chavchavadze Av., 0179 Tbilisi, Georgia*
*A. Razmadze Mathematical Institute*
*1 M. Aleksidze Str., 0193 Tbilisi, Georgia*
*e.mail: gogikhim@yahoo.com*

**Abstract**. It is shown that cyclic polygons with the fixed lengths of the sides can be interpreted as the critical points of various functions on the moduli space of the corresponding polygonal linkage. The detailed results are given for the signed area and several generalizations to other functions on the moduli space are outlined. A few related conjectures and problems are also discussed.

**Keywords and phrases**: polygonal linkage, moduli space, cyclic polygon, signed area, electrostatic energy, critical point, Hessian matrix, Morse index.

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## Introduction

The aim of this paper is to describe a new point of view at cyclic polygons. As usual, under a cyclic polygon we understand a polygon which can be inscribed in a circle, i.e., such that there exists a point (center of circumscribed circle) equidistant from all vertices of the polygon (see, e.g., [4]). Study of cyclic polygons has a long history starting with elementary classical results such as Ptolemy theorem and Brahmagupta formula (see, e.g., [4]). Important results on existence and geometry of cyclic polygons were obtained by J. Steiner [4]. Cyclic polygons continue to attract considerable interest (see, e.g., [6], [16]), in particular, due to the results and conjectures of D. Robbins concerned with computation of the areas of cyclic polygons [14]. We suggest a new interpretation of cyclic polygons based on consideration of polygonal linkages. This approach enables us to reveal a seemingly new aspect of cyclic polygons by interpreting them as critical points of certain functions on moduli space of the corresponding polygonal linkage.

We will freely use results about polygonal linkages presented in [3] and [8]. Informally, linkages may be thought of as mechanisms build up from rigid bars (sticks) joined at flexible links (pin-joints). Linkages provide useful mathematical models of various mechanical and chemical systems and at the same time suggest some interesting mathematical problems. In particular, the concept of moduli (configuration) space of polygonal linkage appeared very fruitful and was actively studied in last few decades (see, e.g., [13], [15], [8]). In this context, it appeared possible to develop Morse theory of various functions on moduli spaces (see, e.g., [7], [8], [12]). Along these lines we consider several geometrically and physically meaningful functions on moduli space of polygonal linkage and show that their critical points are often given by cyclic configurations. The main attention will be given to the signed (oriented) area while a few possible generalizations will only be briefly mentioned.

It should be added that the interpretation of cyclic polygons as critical points of the signed area was suggested in [12]. It should be added that considerable information about cyclic configurations of any concrete polygon can be gained using the signature formulae for topological invariants [10]. Detailed results on cyclic configurations of planar quadrilaterals and pentagons were obtained in [5]. The aim of this paper is to extend the setting considered in [12], [5] and present a few new results in the same spirit.

## 1. Preliminaries on polygonal linkages

Polygonal linkages (or equivalently polygons with the fixed lengths of the sides [4]) were actively studied from various points of view for more than one century (cf., e. g., [9]). In particular, moduli (configuration) spaces of planar polygonal linkages were investigated in big detail [7], [8]. Those general results give a natural framework for our considerations and so we reproduce the necessary definitions in the form adjusted to our purposes.

Recall that an $n$-gonal linkage $L$ is defined by a $n$-tuple of nonnegative numbers $l_i$ (called sidelengths of $L$) each of which is not greater than the sum of all other ones [3]. We always assume that not all of sidelengths $l_i$ are equal to zero. The $N$-th *configuration space* $C_N(L)$ of such a linkage is defined as the collection of all $n$-tuples of points $v_i$ in $N$-dimensional Euclidean space $\mathbb{R}^{\mathbb{N}}$ such that the distance between $v_i$ and $v_{i+1}$ is equal to $l_i$, where $i = 1, \dots, n$ and $v_{n+1} = v_1$. Each such collection $V$ of points, as well as the corresponding polygon, is called a *configuration* of $L$. We assume that the corresponding $n$-gon is oriented by the given ordering of vertices. A configuration is called *cyclic* if all vertices lie on a certain circle and *aligned* if all vertices lie on the same straight line. Obviously, the latter type of configuration is a sort of limiting case of the former.

Factoring the configuration space $C_N(L)$ by the natural diagonal action of the group of orientation preserving isometries $Iso_+(N)$ of $\mathbb{R}^{\mathbb{N}}$ one obtains the $N$-th *moduli space* $M_N(L)$ [8]. Moduli spaces as well as configuration spaces are endowed with the natural topologies induced by Euclidean metric. For $N = 2$, the moduli space $M_2(L)$ is usually called the moduli space of planar polygonal linkage $L$, i.e., here one thinks of $L$ as a linkage lying in a fixed Euclidean plane $\mathbb{R}^2$.

In the sequel we basically consider the moduli space $M_2(L)$ and denote it simply by $M(L)$. It is easy to see that the moduli space $M(L)$

can be naturally identified with the subset of configurations such that $v_1 = (0,0), v_2 = (l_1, 0)$ and thus can be considered as embedded in $\mathbb{R}^{2n-4}$. It is also easy to realize that the moduli space is compact and can be represented as a level set of a certain quadratic mapping (see, e.g., [10]), which implies that, for generic values of $l_i$, the planar moduli space $M(L)$ has a natural structure of compact orientable manifold of dimension $n-3$. In fact, the condition of genericity needed in the last statement can be made quite precise. Let us say that linkage $L$ is *degenerate* if it has an aligned configuration. A minute thought shows that this happens if and only if there exists a $n$-tuple of "signs" $s_i = \pm 1$ such that $\sum s_i l_i = 0$. Now, it is possible to show that moduli space $M(L)$ is smooth (does not have singular points) if and only if linkage $L$ is nondegenerate (see, e. g., [8]).

One can now consider various geometrically meaningful functions on moduli space and study critical points of those functions. Notice that this makes sense even for a singular (non-smooth) moduli space because it has a natural structure of real algebraic variety and for such varieties one has a natural definition of critical point and many other concepts of differential topology (see, e.g., [2]). Taking into account the aforementioned embedding of $M(L)$ into $\mathbb{R}^{2n-4}$ we can consider restrictions to $M(L)$ of polynomial functions on $\mathbb{R}^{2n-4}$. If a moduli space $M(L)$ is smooth and function $f : M(L) \to \mathbb{R}$ arises as restriction of a certain smooth function $F$ on $\mathbb{R}^{2n-4}$ then the critical points of $f$ can be found by Lagrange method as the points $p \in M(L)$ such that $\mathrm{grad\,F}$ is orthogonal to the tangent space $T_p(M(L))$ [1]. For smooth moduli space, a natural idea is to investigate its topology using Morse theory of some natural smooth function on it, which requires a thorough investigation of its critical points. We proceed by applying this approach to *the signed (oriented) area* considered as a function on moduli space.

To this end recall that, for any configuration $V$ of $L$ with vertices $v_i = (x_i, y_i), i = 1, \ldots, n$, its signed area $A(V)$ is defined by

$$A(V) = (x_1y_2 - x_2y_1) + \ldots + (x_ny_1 - x_1y_n).$$

Obviously, this formula defines a smooth function on $\mathbb{R}^{2n}$. Now, to obtain a smooth function on moduli space $M(L)$ of any $n$-gonal linkage $L$ it is sufficient to make use of the chosen embedding of $M(L)$ into $\mathbb{R}^{2n-4}$ by putting $x_1 = y_1 = 0, x_2 = l_1, y_2 = 0$ in the above formula. If the moduli space is smooth, in this way we obtain a smooth function $A_L = A|M(L)$ on a compact manifold $M(L)$ and by said above we may find its critical points using the classical Lagrange method.

As will be shown below, $A$ is typically a Morse function on generic moduli space and so one can indeed use Morse theory to study the topology of moduli spaces if the amount and Morse indices of critical points are found. With this in mind, it was shown in [5] that, for $n = 4$ and $n = 5$, all critical points of $A_L$ in $M(L)$ are given by the cyclic configurations of a nondegenerate $n$-linkage $L$. In the next section we generalize these results by proving that, under certain additional assumptions of genericity, the same holds for arbitrary $n$.

## 2. Cyclic polygons as critical points of signed area

To begin with, let us prove the following result which was formulated in [12] as conjecture.

**Theorem 2.1.** *For any $n$ and any nondegenerate $n$-gonal linkage $L$, the signed area $A$ defines a Morse function on $M(L)$.*

**Proof.** Since $A$ is a homogeneous quadratic function its Hessian matrix with respect to standard coordinates on $\mathbb{R}^{2n}$ is a constant $(2n \times 2n)$-matrix. An obvious computation shows that its determinant does not vanish. Hence the second differential of $A$ is a nondegenerate quadratic form and its restriction on each linear subspace of $\mathbb{R}^{2n}$ also gives a nondegenerate quadratic form. It follows that the Hessian of restriction of $A$ to $M(L)$ is nondegenerate on each tangent space $T_V(M(L))$. Writing Lagrange equations for the critical points one sees that, for each nondegenerate $L$, the critical points of $A|M(L)$ are isolated. Hence we conclude that $A|M(L)$ is a Morse function, which completes the proof.

According to a classical result of J.Steiner, the signed area $A$ attains its maximum at the convex cyclic configuration [4]. Taking into account this fact and some heuristical evidence the present author conjectured in [12] that all critical points of $A$ on $M(L)$ are given by the cyclic configurations. This conjecture was proven in [5] for nondegenerate quadrilaterals and pentagons. We complement the results of [5] in two ways. First, we obtain a similar result for *arbitrary* quadrilaterals. Next, we show that, in the nondegenerate case, critical configurations are indeed given by cyclic configurations.

Since moduli spaces of degenerate linkages are singular, to formulate the first result one needs to have a rigorous definition of critical point of function on singular space. It turns out that the definition of critical point of a polynomial function on real algebraic variety with isolated singularities given in [2] is appropriate for our purposes. Thus in the formulation below the critical points are understood in the indicated sense. For further use notice that an aligned configuration is cyclic if and only if it has only two geometrically distinct vertices. Such a "highly degenerate" configuration exists for each regular $n$-linkage with even $n$ - just take the configuration where all vertices with odd indices coincide with $v_1$ and those with even indices coincide with $v_2$.

**Propositon 2.1.** *Let $L$ be an arbitrary quadrilateral with not all side-lengths equal to zero. A configuration $V \in M(L)$ is a critical point of $A|M(L)$ if and only if it is either a cyclic configuration or an aligned configuration.*

Since the case of nondegenerate $L$ was completely solved in [5], it remains to consider all possible degenerate quadrilaterals. The list of such quadrilaterals is given in [7]) and the structure of their moduli spaces is well-known (see, e.g., [11]). With this at hand, the proof can be obtained using case-by-case analysis. The details of the argument are quite standard and therefore we omit them. Instead we illustrate the essence of matter by considering the moduli space of rhomboid $R$ (all sidelengths are equal) which is in a sense the most "singular" case. As is easy to verify, $M(R)$ is homeomorphic to the union of three circles each pair of which has one common point which is a singular point of $M(R)$. The three singular points correspond to three aligned configurations, only one of which, the most degenerate one $V_0$ with $v_1 = v_3, v_2 = v_4$ is cyclic. The both components of $M(R)$ containing $V_0$ consist entirely of critical points of $A|M(R)$ and $A$ identically vanishes on those components. There are also one point of maximum ("upward square") and one of minimum ("downward square") both obviously cyclic. Thus it is pretty obvious in this case that all $A$-critical configurations are indeed either cyclic or aligned.

Notice that here the aligned critical configurations are singular points of $M(L)$. In fact, it can be proven that if the moduli space has no singular points then there can be no aligned configurations which are critical points of $A$. This observation leads us to the second main result. As was already mentioned, it was conjectured in [12] that, for any natural $n \geq 4$, a configuration $V$ of nondegenerate linkage $L$ is a critical point of $A|M(L)$ if and only if it is cyclic. Using Proposition 2.1 and the latter observation we can prove this conjecture in one direction. The reverse implication seems more delicate and remains open.

**Theorem 2.2.** *For a nondegenerate $n$-linkage $L$, each critical point of $A|M(L)$ is a cyclic configuration.*

**Proof.** Suppose $V = (v_1, ..., v_n)$ is a critical point of $A|M(L)$. For any $k = 1, \dots, n$, consider a quadruple of its consecutive vertices starting with $v_k$ and add the diagonal $v_k v_{k+3}$ to obtain two polygons: $P_k$ with the vertices $(v_1, ..., v_k, v_{k+3}, ..., v_n)$ and $Q_k$ with the vertices $(v_k, v_{k+1}, ..., v_{k+3})$.

Assume first that $Q_k$ is nondegenerate. Then taking only those deformations of $Q_k$ which leave unchanged all sides of $Q_k$ including the diagonal $v_k v_{k+3}$ we can interpret it as a linkage and consider its moduli space $M(Q^k)$. Then the latter moduli space is one-dimensional and we can identify it with a smooth curve in $M(L)$. Since the point $V \in M(L)$ is critical, the gradient grad A is orthogonal to $T_V(M(L))$ and a fortiori orthogonal to $T_{Q_k}(M(Q_k))$, which means that $Q_k$ is also a critical point of $A$ in the moduli space $M(Q_k)$. By the mentioned result of [5] this implies that $Q_k$ is cyclic, i.e. the four points $(v_k, v_{k+1}, ..., v_{k+3})$ lie on the same circle. Since this holds for each quadruple of consecutive vertices, the whole configuration $V$ is cyclic, as was claimed.

If $Q_k$ is degenerate, then by our Proposition 2.1 it is either cyclic or aligned. However, since by assumption the moduli space $M(L)$ is nonsingular, quadrilateral $Q_k$ should be cyclic and the argument can be finished in the same way as above. This completes the proof.

This theorem can be combined with the results of D.Robbins [14] and his followers [16], [6] to obtain some nontrivial information on the critical values of $A|M(L)$. For natural $m$, put

$$\triangle_m = \frac{1}{2}[(2m+1)C_{2m}^m - 2^{2m}],$$

where $C_{2m}^m$ denotes the corresponding binomial coefficient. Now define numbers $R_n$ as follows. For an odd $n = 2m+1$, put $R_n = \triangle_m$. For an even $n = 2m+2$, put $R_n = 2\triangle_m$.

**Proposition 2.2.** *The number of distinct critical values of the signed area function $A$ on the moduli space of nondegenerate $n$-gonal linkage does not exceed $R_n$. Moreover, all critical values of $A|M(L)$ can be found as the real roots of a certain polynomial coefficients of which are algebraically expressible through the sidelengths $l_i^2$.*

This follows directly from Theorem 2.2 and results of [14], [6]. The point of this result is that the critical values of $A|M(L)$ appear effectively computable, which was by no means obvious a priori.

Having these results one can try to develop Morse theory of area function and apply it to topological study of moduli spaces. The crucial step is of course to find a method of calculating Morse indices of cyclic configurations as critical points of signed area. This problem is largely open. In the first nontrivial case of nondegenerate pentagon $P$, by dimension reasons and the mentioned result of J.Steiner it follows that all critical points, except the global maximum and global minimum, are of index one, in other words, they are saddlepoints of $A|M(L)$. As was shown in [5], the signed area is not always a perfect (i.e., having the minimal possible amount of critical points) Morse function on $M(P)$. Thus we are led to another natural problem: find out for which pentagons $P$ the signed area is a perfect Morse function on $M(P)$.

It is clear that thorough investigation of these issues may yield an effective method of obtaining important information on the critical points of $A|M(L)$. In the rest of the paper we present some evidence that the same holds for several other natural functions on the moduli space of polygonal linkage. The main message is that the results and methods available for the signed area function may serve as a paradigm for further research of similar problems.

## 3. Critical points of energy functions on moduli spaces

A reasonable way to generalize the results and considerations presented above is to consider various geometrically or physically meaningful functions on the moduli space of polygonal linkage. Several natural candidates for such functions can be obtained as (potential) energy of a certain physical system which can be associated with a linkage. In fact, one can imagine a lot of such systems some of which may be considered as useful models of certain physical phenomena. For example, one can introduce certain forces acting between (the particles placed at) the vertices of linkage $L$ and consider the corresponding (potential) energy. If this energy function is translation- and rotation-invariant then we obviously obtain a function on the moduli space $M(L)$ and can embark on the study of its critical points along the lines described above.

Notice that the points of minima are especially interesting in this context since they correspond to the equilibria of the system considered. Moreover, this setting has some dynamical aspects concerned with the study of *relaxation processes* understood as the passage from a given configuration to the equilibrium. By general principles of mechanics the motion in process of relaxation should obey appropriate variational principles and can be possibly interpreted as the motion along a geodesic with respect to certain riemannian metric on the moduli space.

All these aspects seem reasonable and apparently deserve investigation. However, this is obviously too vast a program to be discussed in more depth in a short note like this one and so we will only present a few ideas and observations which emerged in the framework of this program without trying to reach maximal generality or rigour.

To begin with, consider functions $D_r$ defined as follows. For configuration $V \in M(L)$ put $D_r(V)$ equal to the product of the $r$th powers of all diagonals of polygon $V$, where $r$ is a fixed positive number. In other words, we put $D_r(V) = \tilde{\Pi} d_{ij}^r$, where $d_{ij}$ is the distance between vertices $v_i$ and $v_j$ of configuration $V$ and the product is taken over pairs $(i,j)$ with $|i-j| \geq 2$ (here as always we assume that $v_{n+1} = v_1$). Obviously, this formula defines a function on $M(L)$ which will be denoted by the same letter $D_r$ and we can consider its critical points in $M(L)$. Taking into account our results it seems reasonable to conjecture that, in some cases at least, the critical points could be given by cyclic configurations and the critical values can be effectively computed.

This is indeed the case but we cannot go into general formulations and describe just the simplest nontrivial situation of such kind. Namely, let us take $n = 4$ and $k = 1$. Then, in classical terms, we just wish to find the extrema of the product of two diagonals of quadrilateral with the fixed lengths of the sides. The answer perfectly fits our general paradigm.

**Proposition 3.1.** *For a nondegenerate quadrilateral linkage $Q$, the critical points of $D_1|M(Q)$ are given by the cyclic configurations of $Q$.*

The proof can be obtained by an easy analysis of Lagrange equations written in angular coordinates quite similar to the proof of Theorem 1 in [5]. For the point of maximum, this also follows from Ptolemy theorem and its converse (see, e.g., [4]). Indeed, these classical results state that, in our notations, $d_{13}d_{24} \leq l_1 l_3 + l_2 l_4$ and equality is achieved if and only if the configuration $V$ is cyclic (cf. [4]). Moreover, we see that the critical values are algebraically expressible in terms of sidelengths as was conjectured.

We do not have proof of similar statements in reasonable generality and at present there is no serious evidence to believe that this holds for arbitrary nondegenerate $n$-gon but there are many concrete examples where our conjectures were confirmed numerically. In any case this setting obviously suggests many interesting problems and deserves further investigation.

One can also consider the energy functions obtaining by summing the contributions coming from pairwise interactions. One class of such energy functions which seem relevant to physical models can be obtained as follows. We fix now a negative $r$ and consider the function $E_r$ given by the formula $E_r = \tilde{\sum} d_{ij}^r$, where as above $d_{ij}$ is the distance between vertices $v_i$ and $v_j$ of configuration $V$ and the sum is taken over pairs $(i, j)$ with $|i - j| \geq 2$. Notice that such functions have a certain physical flavour because one can think of identical particles placed at the vertices which pairwise interact through forces depending only on the distance between particles and then the potential energy of such a system is expressed by the formula of such type (modulo constant). For $r = -1$ we obviously obtain the electrostatic (Coulomb) energy $E = E_{-1}$ which in a sense is the most fundamental example. We believe that our approach may lead to new insights even in this classical case. In general, all such energy functions are naturally defined on the moduli spaces and we can again wonder if there critical points are related to critical configurations.

It should be said from the very beginning that, for such energy functions, the analysis of critical points is much more difficult and they are definitely NOT always given by cyclic configurations. This can already be observed for the electrostatic energy $E$. For example, if the linkage $L$ has several very short sides, the system behaves as if a big charge is placed instead of those short sides. If the number of short sides is sufficiently big, using numerical methods one can show that the equilibrium is not achieved at a cyclic configuration of such linkage. Thus we encounter a highly nontrivial problem of characterizing those linkages for which the stationary points of electrostatic energy are given by cyclic configurations. We were able to show that this is true for all nondegenerate quadrilateral linkages but already the case of pentagon linkages remains open.

Next, as soon as we know the stationary points of various functions on moduli space it is natural to investigate the dynamics of *relaxation process*, i.e., the process of passing from a given configuration to a stationary one. We present only a few brief remarks on this issue but it is obvious that a closer look into it could reveal many interesting aspects and problems.

First of all, according to the least action principle this should be a sort of movement along a geodesics in moduli space. Thus we are led to the necessity of exploring the geometry of moduli space in more depth. Geometry of moduli spaces of planar linkages is known in big detail (see, e.g., [8]) and so there is a good evidence that this may give some insights in the relaxation processes of planar linkages. In the case of signed area, it seems very likely that the motion always happens along a geodesic and terminates at a cyclic configuration. One may now wish to verify and/or explicate this conclusion by investigating the corresponding equations of motion.

The problem becomes more difficult for spatial linkages (i.e., for $N = 3$). On the one hand, for non-knotted initial configurations, the stationary configurations obtained as the results of relaxation process are often planar and hence cyclic. This can be proven for energy functions $D_r$ using the "diagonal bending flows" considered in [8]. On the other hand, in many cases the energy barrier prohibits self-intersections of linkage in process of relaxation and so if the initial configuration is knotted then it cannot relax to a cyclic configuration. Thus we are led to the problem of finding the limiting stationary configurations. Similar problems were considered in knot theory in several physically relevant contexts. It is our hope that such problems could be more easily studied for linkages, which might appear useful for studying relaxation processes for knots.

## 4. Concluding remarks

In conclusion we wish to indicate a few general problems suggested by our approach. First of all, one could try to characterize functions on moduli space having the property that all of their critical points are given by the cyclic configurations. Certain essential properties of such functions, like the invariance with respect to cyclic substitution of vertices, are more or less clear but the problem is far from obvious. One could at least try to indicate sufficiently wide classes of functions with such property.

We can further generalize this setting by introducing the general notion of *concritical* functions. Namely, given two smooth functions which are defined on moduli spaces of all $n$-gonal linkages we will say that these two functions are *concritical* if their critical sets coincide on each moduli spaces. In this terms, we can say that $A$, $D_1$ and $E_{-1}$ are concritical on moduli spaces of quadrilaterals. Having this concept one could try to characterize such pairs of functions axiomatically. It is also interesting to find out what could be the "dispersion" (maximal difference) of Morse indices of a given pair of concritical energies from the lists $D_r$, $E_r$ introduced above.

It is also natural to investigate if the results obtained in Section 2 hold for singular moduli spaces. As was mentioned in Section 1, all necessary concepts of differential topology are available for real algebraic varieties with isolated singularities [2] and so one can try to develop Morse theory for singular moduli spaces in the spirit of Goresky-McPherson.

Similar problems can be studied for "higher" moduli spaces $M_N(L)$ of which especially intriguing are moduli spaces of spatial linkages obtained for $N = 3$. Notice that in arbitrary dimension $N$ one has the oriented volume $\mathcal{V}_\mathcal{N}$ [4] which defines a smooth function on the $N$th configuration space of each $n$-gonal linkage. Up to our knowledge, no analogs of our Theorems 1 and 2 exist in this context so this seems to be a vast and promising research topic.

Finally, as was already mentioned, a plenty of practically unexplored problems is related to relaxation processes on moduli spaces of polygonal linkages. Especially interesting are the non-planar stationary configurations of knotted $3d$-linkages. There are examples showing that such stationary configurations appear in continual families and so the geometric aspects of this problem might appear quite involved. A natural way for getting more insight in problems of such kind is to use numerical methods and computer simulation. Thus there is good evidence that the problems and methods presented in this paper will lead to further results.

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