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CYCLIC POLYGONS ARE CRITICAL POINTS OF AREA

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It is shown that typical critical points of the signed area function on the moduli space of a generic planar polygon are given by cyclic configurations, i.e., configurations that can be inscribed in a circle. Several related problems are briefly discussed in conclusion. Bibliography: 14 *titles.*

Introduction

As usual, by a cyclic polygon we understand a polygon that can be inscribed in a circle, i.e., there exists a point (the center of the circumscribed circle) equidistant from all vertices of the polygon (see, e.g., [4]). The study of cyclic polygons has a long history starting with elementary classical results such as Ptolemy's theorem and Brahmagupta's formula (see, e.g., [4]). Important results on the existence and geometry of cyclic polygons were obtained by J. Steiner [4]. This topic continues to attract considerable interest (see, e.g., [6, 14]), in particular, due to the results and conjectures of D. Robbins concerning the computation of the areas of cyclic polygons [12]. The aim of this note is to show that cyclic polygons can often be interpreted as critical points of the signed area function on the moduli space of the corresponding polygonal linkage.

Our considerations are performed in the context of polygonal linkages [3]. Informally, linkages may be thought of as mechanisms build up from rigid bars (sticks) joined at flexible links (pin-joints). Linkages provide useful mathematical models of various mechanical and chemical systems and suggest some interesting mathematical problems. Specifically, the moduli (configuration) spaces of polygonal linkages were actively studied in the last few decades (see, e.g., [7, 13, 8]). In particular, the Morse theory of various functions on moduli spaces was considered in [7, 8]. Along these lines, we consider the signed (oriented) area of a polygon [4] as a function on the moduli space of a generic planar polygonal linkage and show that, generically, its critical points are given by the cyclic configurations of the latter.

It should be added that the interpretation of cyclic polygons as critical points of the signed area function was suggested in [11]. As was shown in [5], this is indeed the case for nondegenerate planar quadrilaterals and pentagons. We extend these results by proving that the same holds for generic cyclic configurations of nondegenerate polygonal linkages with arbitrary number of vertices (pin-joints).

We tried to make the exposition (reasonably) self-contained. To this end, in the first section we give the necessary information about the configuration spaces of linkages and the signed area of planar polygons. The formulation and proof of the main result are presented in the second section. In the last section, we briefly discuss several related problems.

1. Preliminaries on polygonal linkages

Polygonal linkages (or, equivalently, polygons with fixed lengths of the sides [4]) were actively studied from various points of view for more than one century (cf., e.g., [9]). In particular, the moduli (configuration) spaces of planar polygonal linkages were investigated in big detail [7, 8]. Those general results give a natural framework for our considerations, and so we reproduce the necessary definitions in the form adjusted to our purposes.

Recall that an n-gonal linkage L is defined by an n-tuple of nonnegative numbers l_i (called the side lengths of L) each of which is not greater than the sum of all the other ones [3]. We also assume that not all of the side lengths l_i are equal to zero. The N-th *configuration space* $C_N(L)$ of such a linkage is defined as the collection of all n-tuples of points v_i in the N-dimensional Euclidean space $\mathbb{R}^N$ such that the distance between v_i and v_{i+1} is equal to l_i, where $i = 1, \dots, n$ and $v_{n+1} = v_1$. Each such collection V of points, as well as the corresponding polygon, is called a configuration of L. We assume that the corresponding n-gon is oriented by the given ordering of vertices. A configuration is called *cyclic* if all vertices lie on a certain circle and *aligned* if all vertices lie on the same straight line. Obviously, the latter type of configurations is a sort of limiting case of the former.

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Factoring the configuration space $C_N(L)$ by the natural diagonal action of the group $\mathrm{Iso}_+(N)$ of orientation-preserving isometries of $\mathbb{R}^N$, one obtains the N-th *moduli space* $M_N(L)$ [8]. Moduli spaces, as well as configuration spaces, are endowed with the natural topologies induced by the Euclidean metric. For $N = 2$, the moduli space $M_2(L)$ is usually called the moduli space of the planar polygonal linkage L, i.e., here one thinks of L as a linkage lying in a fixed Euclidean plane $\mathbb{R}^2$. In the sequel, we will only consider the moduli space $M_2(L)$ and denote it simply by $M(L)$. It is easy to see that the moduli space $M(L)$ can be naturally identified with the subset of configurations such that $v_1 = (0,0)$, $v_2 = (l_1, 0)$ and thus can be considered as embedded into $\mathbb{R}^{2n-4}$. It is also easy to realize that the moduli space is compact and can be represented as a level set of a certain quadratic mapping (see, e.g., [10]), which implies that, for generic values of l_i, the planar moduli space $M(L)$ has a natural structure of a compact orientable manifold of dimension $n-3$. In fact, the genericity condition needed in the last statement can be made quite precise. Let us say that a linkage L is *degenerate* if it has an aligned configuration. A minute's thought shows that this happens if and only if there exists an n-tuple of "signs" $s_i = \pm 1$ such that $\sum s_i l_i = 0$. Now, one can show that the moduli space $M(L)$ is smooth (does not have singular points) if and only if the linkage L is nondegenerate (see, e.g., [8]).

One can now consider various geometrically meaningful functions on the moduli space and study critical points of those functions. Note that this makes sense even for a singular (nonsmooth) moduli space, because it has a natural structure of a real algebraic variety, and for such varieties one has a natural definition of a critical point and many other notions of differential topology (see, e.g., [2]). Taking into account the aforementioned embedding of $M(L)$ into $\mathbb{R}^{2n-4}$, we can consider restrictions to $M(L)$ of polynomial functions on $\mathbb{R}^{2n-4}$. If the moduli space $M(L)$ is smooth and a function $f : M(L) \to \mathbb{R}$ arises as the restriction of a certain smooth function F on $\mathbb{R}^{2n-4}$, then the critical points of f can be found by the Lagrange method as the points $p \in M(L)$ such that $\operatorname{grad} F$ is orthogonal to the tangent space $T_p(M(L))$ [1]. For a smooth moduli space, a natural idea is to investigate its topology using the Morse theory of some natural smooth function on it, which requires a thorough investigation of critical points of this function. We apply this approach to the *signed (oriented) area* regarded as a function on the moduli space.

To this end, recall that for any configuration V of L with vertices $v_i = (x_i, y_i), i = 1, \ldots, n$, its signed area $A(V)$ is defined by

$$A(V) = (x_1 y_2 - x_2 y_1) + \ldots + (x_n y_1 - x_1 y_n).$$

Obviously, this formula defines a smooth function on $\mathbb{R}^{2n}$. Now, to obtain a smooth function on the moduli space $M(L)$ of any n-gonal linkage L, it is sufficient to make use of the chosen embedding of $M(L)$ into $\mathbb{R}^{2n-4}$ by putting $x_1 = y_1 = 0$, $x_2 = l_1$, $y_2 = 0$ in the above formula. If the moduli space is smooth, in this way we obtain a smooth function $A_L = A|M(L)$ on the compact manifold $M(L)$ and, as said above, we can find its critical points by the Lagrange method.

As was noticed in [11], from general principles of singularity theory it follows that A is a Morse function on a generic moduli space, and so one can indeed use Morse theory to study the topology of moduli spaces if the amount and indices of critical points are found. With this in mind, it was shown in [5] that, for $n = 4$ and $n = 5$, all critical points of A_L in $M(L)$ are given by the cyclic configurations of a nondegenerate n-linkage L. We generalize this result by proving that, under certain additional genericity assumptions, the same holds for arbitrary n. We conclude this section by presenting a few remarks on linkages and the signed area which will be used in the sequel.

Given an oriented configuration $V = (v_1, \ldots, v_n) \subset \mathbb{R}^2$ of a linkage L and a point $x \in \mathbb{R}^2$, we denote by $w_L(x)$ the winding number of L around the point x (cf. [12]). Assume now that two polygonal linkages L_1, L_2 have a common edge with opposite orientations. We define their sum $L = L_1 + L_2$ (which is again a polygonal linkage) as the homological sum of these two cycles. Further, assume that two configurations $V_1, V_2 \subset \mathbb{R}^2$ of L_1 and L_2 have a common edge with opposite orientations. Clearly, the homological sum of V_1 and V_2 is a configuration of $L_1 + L_2$. The following two properties of the signed area are well known and easy to prove directly using the above definitions and remarks.

Lemma 1. 1. *For the (signed) area of a configuration V one has*

$$A(V) = \int_{\mathbb{R}^2} w_L(x) d\lambda(x),$$

where λ denotes the Lebesgue measure in $\mathbb{R}^2$.

2. *If $V = V_1 + V_2$, then $A(V) = A(V_1) + A(V_2)$.*

For a configuration V of an n-linkage L and each $k \in [1, n]$, we denote by V^k the quadrilateral formed by the four consecutive vertices $v_k, v_{k+1}, v_{k+2}, v_{k+3}$ assuming that the diagonal $v_k v_{k+3}$ of V is added as the fourth side of V^k. Each such quadrilateral V^k will be called a *side quadrilateral* of V, and we denote by Q^k the corresponding quadrilateral linkage. We say that a configuration V is *strongly nondegenerate* if all of its side quadrilaterals are nondegenerate.

2. Cyclic configurations are critical points of the signed area

After these preparations, we are able to present the main result.

Theorem 1. *Let L be a nondegenerate n-gonal linkage. A strongly nondegenerate configuration V of L is a critical point of $A|M(L)$ if and only if V is cyclic.*

The proof is based on the similar result for $n = 4$ established in [5], which we reproduce as a lemma for the reader's convenience.

Lemma 2. *Let L be a nondegenerate quadrilateral linkage. Then a configuration V of L is a critical point of $A|M(L)$ if and only if V is cyclic.*

Note that all configurations of nondegenerate quadrilaterals are automatically strongly nondegenerate. It will be convenient for us to speak of *deformations* of a given configuration $V \in M(L)$, where the term "deformation" means any configuration V' of L sufficiently close to V. The heuristics behind this term is that, generically, one can in fact pass from V to V' by smoothly deforming the shape of V, or, which is the same, by changing the angles of V. With all these definitions and observations at hand, we can prove the main result.

Proof. ("Only if") Assume that $V = (v_1, \ldots, v_n)$ is a critical point of $A|M(L)$. Choose a natural number $k \in [1, n-3]$ and use the quadruple of consecutive vertices of V starting with v_k to decompose V into the sum of two polygons:

$$V = (v_1, \ldots, v_k, v_{k+3}, \ldots, v_n) + (v_k, v_{k+1}, \ldots, v_{k+3}) = \overline{V}^k + V^k.$$

In other words, we split the cycle along the diagonal $v_k v_{k+3}$. Let $M(L)$ (respectively, $M(Q^k)$) be the moduli space of L (respectively, of Q^k). Our assumptions obviously imply that $M(Q^k)$ is a compact smooth one-dimensional manifold and that in a neighborhood of the point V we have a natural smooth embedding $M(Q^k) \hookrightarrow M(L)$. Indeed, a deformation of the closed quadrilateral V^k yields a deformation of the whole L: we deform V^k and keep the rest (i.e., $\overline{V}^k$) fixed. (This is equivalent to saying that each configuration of Q^k sufficiently close to V^k gives a uniquely defined configuration of L.)

Since V is a critical point, the configuration V^k has to be a critical point of the area function on the moduli space $M(Q^k)$. By Lemma 2, the quadrilateral V^k is cyclic. Since this holds for any k, the whole L is cyclic as well.

("If") For a cyclic configuration $V = (v_1, \ldots, v_n)$ of a nondegenerate n-gonal linkage L, consider the tangent space $T_V(M(L))$ of the moduli space $M(L)$ at the point V. First, note that the nondegeneracy of L implies that, in a neighborhood of the point V, the moduli space $M(L)$ is smoothly parameterized by the angles $\alpha_1, \ldots, \alpha_{n-3}$ of the configuration at the vertices $v_1, \ldots, v_{n-3}$.

Next, each deformation of V can easily be represented as a composition of some deformations $(d_k)_{k=1}^{n-3}$ such that each of the deformations d_k keeps fixed the vertices $(v_1, \ldots, v_k, v_{k+3}, \ldots, v_n)$. In other words, only the quadrilateral $V^k = (v_k, v_{k+1}, v_{k+2}, v_{k+3})$ is deformed in course of the deformation d_k. Indeed, to decompose a deformation, we first choose a deformation d_1 that adjusts the angle α_1; next we choose a deformation d_2 that adjusts the angle α_2, and so on. This is obviously possible up to the angle α_{n-3}, and then the last three angles are determined uniquely.

Therefore the tangent vectors to the curves $M(Q^k)$, $k = 1, \ldots, n-3$, at the point V linearly generate the tangent space $T_V(M(L))$. Dimension reasons imply that these curves form a basis. Since each configuration V^k is cyclic, $\operatorname{grad} A$ regarded as a vector in $\mathbb{R}^{2n-4}$ is orthogonal to each of $M(Q^k)$, $k = 1, \ldots, n-3$, at the point V. Therefore $\operatorname{grad} A$ is orthogonal to $T_V(M(L))$ at the point V as well, which implies that dA vanishes on $T_V(M(L))$, i.e., V is a critical point of $A|M(L)$. This completes the proof.

3. Concluding remarks

A few remarks seem in place here. First of all, the condition of strong nondegeneracy is technical, since it is suggested by our method of proving the theorem. We believe that the same result should hold for an arbitrary critical configuration of a nondegenerate linkage, but there are subtleties caused by the possibility of the appearance of degenerate side quadrilaterals.

In fact, there may well exist cyclic configurations with degenerate side quadrilaterals. For example, the regular (all sides have the same length) pentagon linkage is nondegenerate but has an "isosceles-triangle-like" configuration with one "triple" side (cf. [14]), which is obviously cyclic but has a strongly degenerate side quadrilateral consisting of just one "thick" side obtained by piling four equal segments. Thus the direct application of our approach is impossible in this case, since we cannot refer to Lemma 2. In this respect, it might be interesting to look for conditions on the linkage L which guarantee that it does not have critical and cyclic configurations with degenerate side quadrilaterals.

Next, one could try to overcome these subtleties by obtaining an analog of Lemma 2 valid for all (not necessarily nondegenerate) quadrilaterals. In fact, there is good evidence that each A-critical configuration of a quadrilateral linkage is either cyclic or aligned if one properly defines the notion of a critical point on a singular moduli space. Making this idea precise seems reasonable and within reach using the machinery developed in [2], but we will not go into that here since it is not completely clear if this may eventually give the desired generalization. Let us illustrate possible complications by considering the moduli space $M(R)$ of a rhomboid R (all side lengths are equal). As is easy to verify, $M(R)$ is homeomorphic to the union of three circles each pair of which has one common point which is a singular point of $M(R)$. The three singular points correspond to three aligned configurations, only one of which, V_0 with $v_1 = v_3, v_2 = v_4$, is cyclic. Both components of $M(R)$ containing V_0 consist entirely of critical points of $A|M(R)$ at which A vanishes (there are also one point of maximum – "upward square," and one point of minimum – "downward square"). We see that, indeed, all A-critical configurations are either cyclic or aligned. However, the presence of continual components of critical points complicates the situation, and it is unclear if our argument can be applied in such situations. So here are several issues that require to be clarified.

Furthermore, a whole bunch of problems is related to calculating the Morse indices of cyclic configurations. Not much is known in this direction beyond the first nontrivial case of pentagon linkages (cf. [11]). As explained in [11], if one knows that the A-critical configurations of a linkage L coincide with the cyclic ones, then considerable information about the topology of $M(L)$ can be derived from the variety of results on the amount and geometry of cyclic configurations obtained in [12, 14, 6]. Thus stronger versions of our theorem may have concrete corollaries for linkages with fixed number of vertices.

All this shows that the relation between cyclic and critical configurations established in our theorem has a number of interesting and unexplored aspects. It is our belief that further research in this direction may appear rewarding.

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