Journal of Mathematical Sciences, Vol. 160, No. 6, 2009

ON STOCHASTICALLY INDEPENDENT CONTINUOUS FUNCTIONS

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Abstract. We discuss continuous functions that are stochastically independent as random variables. It is shown that such functions are closely related to Peano curves which fill parallelotopes in Euclidean spaces. An explicit construction of independent functions is presented which leads to a sufficiently complete description of collections of stochastically independent continuous functions on an interval. Several related results are also presented and a few open problems are formulated.

CONTENTS

Introduction

The notion of independent random variables is of fundamental importance in probability theory [1, 8]. Some curious and nontraditional aspects of this concept arise in connection with the following definition which was considered in several papers (see, e.g., [3, 4, 7, 9]). We develop these aspects further here.

Let us say that two functions on a compact probability space are stochastically independent if they are independent as random variables. It seems that topology was not considered relevant to the study of stochastically independent functions, perhaps because those were implicitly assumed to have bad regularity properties. In fact, the typical examples of independent random variables, (e.g., Rademacher functions and their various modifications [8]) involve discontinuous functions. There are even written statements that independent continuous functions do not exist (see, e.g., the editor's foreword to the Russian edition of M. Kac's famous book on stochastic independence [8]).

However, nonconstant continuous independent (NCI) functions do exist (see, e.g., [3, 7]) and they are naturally connected with interesting topological constructions like the famous Peano curve [11]. Moreover, one can ask interesting and quite nontrivial questions about stochastically independent functions even in the context of smooth and real-analytic functions (e.g., the so-called *Eidlin problem* [4]).

In this paper, we address some basic geometric and topological aspects of independent continuous and smooth functions. Our discussion consists of two parts. In the first (and shorter) part (Secs. 1 and 2) we present several simple results about *smooth* and (*real-*)*analytic* NCI functions on certain manifolds, in particular, on two-dimensional closed surfaces. To this end we use a few basic notions of differential topology. As usual, here and below the word "smooth" means "infinitely differentiable."

In the second part, we deal with NCI *continuous* functions on intervals and parallelotopes. In particular, we establish an intrinsic connection between independent continuous functions and Peano

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 59, Algebra and Geometry, 2008.

1072–3374/09/1606–0717

curves and give a comprehensive description of NCI functions on an interval (Theorem 3.1). The discussion in this part is based on a few classical constructions presented in an excellent textbook of N. Lusin [11] and deep results about homeomorphisms of measures obtained by J. von Neumann, W. Oxtoby, and S. Ulam (see, e.g., [5, 12, 13]).

To give a more precise idea of the problems and results discussed below, let us first add a few words about the setting accepted in the first part. By a closed surface, as usual, we mean a compact oriented connected two-dimensional smooth manifold S without boundary. Recall that, for such a surface S, one can consider its genus $g(S)$ (or, equivalently, the Euler characteristic $\chi(S) = 2 - 2g(S)$), which is a complete topological invariant of closed surfaces [6]. Each such surface can be embedded in three-dimensional Euclidean space $\mathbb{R}^3$ as an algebraic surface, hence it also has a natural structure of real-analytic manifold [6]. Assuming that such an embedding is fixed, we can endow S with measure P which is induced by the Lebesgue measure in ambient space and normalize it by requiring that $P(S) = 1$. This turns S into a probability space and enables one to speak of (stochastically) independent functions on S. Hence one wonders how many independent continuous functions of certain regularity class (e.g., Hölder, smooth, real-analytic) exist on S. We present some simple results in this direction, which may hopefully serve as a sample for further investigation of independent smooth functions on manifolds.

As to the second part, it is based on a well-known fact that independent continuous functions on an interval can be constructed using Peano curves which fill parallelotopes in Euclidean spaces [3, 7]. We complement this observation by showing that this description is complete in a certain natural sense. Namely, our Theorem 3.1 shows that constructing independent continuous functions on an interval is essentially *equivalent* to constructing Peano curves. More precisely, each Peano curve can be turned into one with independent components by applying an appropriate homeomorphism of its image. Combined with other results presented in Sec. 3, this gives a seemingly curious description of independent continuous functions in terms of Peano curves. The proof is based on some classical results from [13]. We also outline connections of independent continuous functions with another type of space-filling curves provided by the so-called *thread theorem* (see [5]).

Our joint research on this topic was started during a long-term visit of the first author to the Institute of Computational Mathematics of the Georgian Academy of Sciences. The authors wish to thank N. Vakhania and T. Shervashidze for stimulating discussions and valuable comments.

1. Preliminary Remarks

We need to use several standard concepts and paradigms of probability theory for which we refer to [1, 15] and differential topology for which we refer to [6]. Let $(X, \mathcal{B}, P)$ be a probability space [15]. Recall that any measurable function $f : X \to \mathbb{R}$ can be naturally considered as a random variable.

Definition 1.1. Two measurable real functions $f, g : X \to \mathbb{R}$ considered as random variables are called (stochastically) independent if they are independent as random variables, i.e., for any $a, b \in \mathbb{R}$, one has $P\{f < a, g < b\} = P\{f < a\}P\{g < b\}$. In this case we say that g is an independent companion for f (and vice versa) or that (f, g) is a pair of (stochastically) independent functions.

For brevity we omit the word "stochastically" and write simply "independent functions." A constant function is obviously independent of any other (measurable) function and we wish to exclude this trivial case from the very beginning. So let us say that a continuous function f on X is (*stochastically*) *tolerant* if it is nonconstant and admits a smooth independent nonconstant companion. One of the problems we are interested in is to describe peculiar properties of tolerant functions.

Remark 1.1. The notion of a stochastically independent collection of functions $(f_1, \dots, f_k)$ with $k > 2$ is defined similarly. One should keep in mind that in higher dimensions this is a much stronger condition than their pairwise independence [15].

The results presented delow are based on two observations which we formulate as lemmas. Assume that X is endowed with topology which turns it into a compact connected topological space. In such case we say that X is a compact probability space.

Lemma 1.1. *If $f_1, \dots, f_k$ are independent continuous functions on a compact probability space X, then the image of mapping $F = (f, g) : X \to \mathbb{R}^k$ fills up a k-dimensional parallelotope (direct product of k segments).*

Proof. First, assume that $k = 2$. The space X is compact and connected and the images $f(X)$ and $g(X)$ are compact and connected subsets of the real line. Hence they both are closed segments and we can write $f(X) = [a, b]$ and $g(X) = [c, d]$. We set $B = [a, b] \times [c, d]$. Then it is easy to show that $F(X) = B$. Indeed, if we assume that there exists an interior point $(x, y) \in B$ outside $F(X)$, then by compactness of $F(X)$ there exists a small quadrangle $Q = [x-r, x+r] \times [y-r, y+r]$ in B lying outside $F(X)$. Since f and g are continuous, the numbers $P\{x - r < f < x + r\}$ and $P\{y - r < g < y + r\}$ are both strictly positive. Thus by the independence condition the number $P\{F \in Q\}$ is also positive, which contradicts the assumption and proves the statement. It is pretty clear now that the same argument with obvious changes works for arbitrary k. □

If a probability space X has in addition a structure of compact smooth manifold, then we say that X is a *smooth probability space.* In such case we can speak of smooth functions, differentials, gradients, regular and singular values, and other notions of differential topology [6]. Obviously, Definition 1.1 applied to smooth functions gives the notion of independent smooth functions.

Lemma 1.2. *Let $f_1, \dots, f_k$ be independent smooth functions on a smooth probability space. Then a point $c = (c_1, \dots, c_k)$ is a regular value of $F = (f_1, \dots, f_k)$ if and only if each c_j is a regular value of f_j.*

Proof. It is known and easy to verify that, for smooth independent smooth random variables, the probability density of their joint distribution F is equal to the product of probability densities of the components f_j. In our situation, if we introduce a Riemannian metric on X, the densities become nonzero multiples of the lengths of gradients $\operatorname{grad} f_j$. It follows that $J(p)$ is nonzero if and only if all gradients at this point are nonzero, which is equivalent to the statement of the lemma. □

Corollary 1.1. *On an n-dimensional compact connected manifold X, there cannot exist collections of $k > n$ independent smooth functions.*

Indeed, it is well known that a smooth mapping cannot increase the dimension [6] and so the image of an n-dimensional manifold cannot fill up a parallelotope of dimension bigger than n. Taking this into account, it remains to refer to Lemma 1.1.

We emphasize that the condition of smoothness is essential. In fact, a similar statement is not correct for continuous functions. As will be explained below, for any natural k, one can construct a collection of k independent smooth functions on the unit interval $I = [0, 1]$ by taking the coordinate functions of an appropriate *Peano curve* which regularly fills up the cube I_k (see [2]).

Corollary 1.2. *On a closed surface S, there can only exist pairs of independent smooth functions.*

In the next section we will take a closer look at independent smooth functions on two-dimensional surfaces.

2. Independent Functions on Two-Dimensional Surfaces

It is obvious that the standard angular coordinates on the torus $T^2 = S^1 \times S^1$ provide a pair of smooth (in fact, real analytic) functions on T^2. Moreover, pairs of independent smooth functions on a closed surface S of arbitrary genus $g > 1$ can be constructed from the polar coordinates on the unit disc using its well-known representation as the universal covering space of S (see, e.g., [9]).

Proposition 2.1. *On each orientable closed surface there exist pairs of independent nonconstant smooth functions.*

We omit the proof because more general statements will be established in the sequel. It is now natural to wonder if there exist essentially different pairs of smooth independent functions. Notice that, given such a pair (f, g), one can obtain new independent pair $(f\circ\Phi, g\circ\Phi)$ by composing it (on the right) with any measure preserving diffeomorphism $\Phi : S \to S$. This provides a natural equivalence relation on the set of pairs of independent smooth functions on S and it would be interesting to describe the set of equivalence classes. For independent functions on the unit interval I, some results in this direction are presented in the next section. We conclude this section by giving some "no-go" results in the real-analytic case.

For a smooth function f, denote by $E(f)$ the set of all points where f has a local extremum (minimum or maximum). The next geometric property of independent functions follows from the above remarks (cf. also [4]).

Lemma 2.1. *Let f, g be independent smooth functions on S and let $p \in E(f)$ be such that g is nonconstant in an arbitrary small neighborhood U of p. Then $g(U \setminus E(f))$ is a set with nonempty interior in $\mathbb{R}$ (i.e., it contains an open interval).*

We apply this to a pair of real analytic functions. Since a nonconstant analytic function g does not vanish on any open subset and cannot increase dimension, we see that the connected components of $E(f)$ are one-dimensional.

Corollary 2.1. *A tolerant analytic function on a closed surface attains its maxima and minima on one-dimensional subsets.*

Remark 2.1. Note an apparent similarity with the so-called round functions whose critical sets consist of several smooth one-dimensional components [10]. This connection may appear useful because round functions on closed surfaces are sufficiently well understood [10].

From the above it follows that the sets of maxima and minima of an analytic tolerant function f on a closed surface S define one-dimensional analytic closed chains [4]. Thus they define certain elements in the first homology group $H_1(S, \mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$ and one can show that at least one of these elements should be nontrivial. Since the homology groups and the fundamental groups of closed surfaces are well known, we can derive some concrete conclusions.

Proposition 2.2. *There are no nonconstant tolerant analytic functions on a two-dimensional sphere S^2.*

This follows from the preceding remarks because $H_1(S^2) = 0$. In other words, on S^2 do not exist independent nonconstant analytic functions (notice that S^2 admits no round functions as well [10]). We feel that the same conclusion should be correct for any closed surface except T^2, but the above argument with homology groups is insufficient. Similar results are available for nonorientable compact surfaces. For example, no independent analytic functions exist on the projective plane $\mathbb{RP}^2$.

Of course, the same problems are meaningful and nontrivial in any dimension. However, at present we are not able to add much about the higher-dimensional cases and so we postpone a more detailed discussion of independent smooth functions for future publications. In the rest of this paper we concentrate on investigating independent continuous functions on intervals and parallelotopes.

3. Independent Functions and Space Filling Curves

Apparently, the most direct and simple way of constructing independent continuous functions is based on the use of certain space filling curves. Let us begin with the two-dimensional case. Given two independent continuous functions u and v on the segment $I = [0, 1]$, consider the function $f = u + \imath v : I \to \mathbb{C}$. Let $u(I) = [a_1, a_2]$ and $v(I) = [b_1, b_2]$. Then by Lemma 1.1 we conclude that $f(I) = [a_1, a_2], [b_1, b_2]$.

Since u and v are continuous we obtain a continuous curve (path) which fills up a rectangle. Such curves are well known and are often called *Peano curves* (see [2, 11]). A rich class of such curves including the original Peano curve have indeed the property that the components are independent. We reproduce here an explicit construction of such curves given in [3]. Let

$$X = \{f : I \to I^2\}, \quad X_0 = \{f \in X \cap C(I) : f(0) = 0,\ f(1) = 1\}.$$

It is well known that X and X_0 endowed with the uniform norm are complete metric spaces. Consider the operator $T : X \to X$ defined by the formulas

$$Tf(t) = \imath\overline{f}(4t)/2 \text{ if } t \in [0, 1/4); \quad Tf(t) = [f(4t-1) + \imath]/2 \text{ if } t \in [1/4, 1/2);$$

$$Tf(t) = [1 + \imath + f(4t-2)]/2 \text{ if } t \in [1/2, 3/4); \quad Tf(t) = [1 + \imath f(4-4t)]/2 \text{ if } t \in [3/4, 1].$$

It is easy to verify that T is a contraction and that

$$\|Tf - Tg\| \le \frac{1}{2}\|f - g\|.$$

According to Banach's fixed point theorem, there exists exactly one function $F \in X$ such that $TF = F$; moreover, $T^n f \to F$ for every $f \in X$. It turns out that F is a Peano function which fills up the square. Set $F = u + \imath v$ and denote by λ the Lebesque measure. The main properties of the function F are collected in the following proposition.

Proposition 3.1 (see [3]). (i) $T(X_0) \subset X_0$;
(ii) $F \in X_0$ *and* $F(I) = I^2$;
(iii) $\lambda \circ F^{-1} = \lambda \otimes \lambda$, *i.e., u and v are independent and uniformly distributed.*

From this proposition one derives the following corollary which shows that there are sufficiently many pairs of NCI functions on an interval.

Corollary 3.1. *Let $u, v : I \to \mathbb{R}$ be two continuous function. Then there exist two independent continuous functions U and V such that $\lambda \circ u^{-1} = \lambda \circ U^{-1}$ and $\lambda \circ v^{-1} = \lambda \circ V^{-1}$.*

Indeed, one can just set $U = u(\operatorname{Re} F)$ and $V = v(\operatorname{Im} F)$, where F is the Peano function constructed above.

As was shown in [3], the above construction of Peano curves can be generalized by using a more general definition of operator T. In this way, one obtains further examples of independent continuous functions on I. Furthermore, using the same construction and reasoning one can construct continuous curves which fill up parallelotopes of arbitrary fixed dimension and such that their components are independent. In this way one obtains collections of independent continuous functions of arbitrary finite cardinality. Moreover, one can strengthen the latter conclusion by constructing continuous curves which fill up the Hilbert cube and such that their components are independent. This enables one to show that there exist even countable collections of independent continuous functions on the unit interval [7].

Thus each collection of independent continuous functions provides a Peano curve filling up a parallelotope. At the same time, it is easy to construct a Peano curve such that its components are not independent. In fact, this can be done by an arbitrarily small perturbation of the Peano curve constructed above. This can be used to show that, in a certain explicit sense, "generic" Peano curves do not come from independent continuous functions. However, it turns out that each Peano curve can be transformed into a curve with independent components by applying an appropriate homeomorphism of its image. This is the content of Theorem 3.1 which is our main result. We present it here for convenience of the reader but the proof will be given in the next section after recalling necessary concepts and results concerned with transformations of measures.

Theorem 3.1. *Let $F : I \to I^n$ be a continuous surjective mapping (Peano curve). There exists a homeomorphism $\Phi : I^n \to I^n$ such that the components of the curve $\Phi \circ F$ are independent continuous functions.*

This gives a sufficiently complete description of independent continuous functions on an interval. It is now quite easy to construct NCI functions on parallelotopes. First of all, coordinate functions are obviously independent on each parallelotope, so one has at least d independent continuous (actually, real-analytic functions) on a d-dimensional parallelotope. Collections of independent continuous functions of cardinality bigger than the dimension of parallelotope considered can be constructed from obvious analogs of Peano curves. More precisely, there always exist surjective continuous mappings $I^d \to I^n$ and they can be constructed in such a way that their components are independent. Thus their exist arbitrarily long collections of independent continuous functions on each parallelotope and even countable collections of independent continuous functions. One can also prove a natural analog of Theorem 3.1 for parallelotopes of arbitrary dimension. We do not go into details which are rather obvious. Thus with regard to independent functions the case of I^d is completely analogous to the case of I. The reason why this is so will become clear from the discussion in Sec. 4.

It is now natural to wonder what are the possible regularity classes of independent functions on I. It can be shown that there do not exist independent functions of finite variation [3]; in particular, there are no differentiable independent functions (the latter fact was already mentioned above). However, as was shown in [7], one can construct independent functions which are Hölder continuous with an exponent not exceeding one-half. We will not go into these aspects and conclude this section by formulating a number of further problems.

Recall that, given a map $F : X \to Y$, the multiplicity $m_F(y)$ of a point $y \in Y$ with respect to F is defined as the cardinality of $F^{-1}(y)$ (if this set is infinite we put $m_F(x) = \infty$). If $y \notin F(X)$ then we put $m_F(y) = 0$. The multiplicity of map is defined as the maximum of point multiplcities. One may wonder what is the multiplicity of Peano curves constructed above and in [3]. For the curve constructed above, from the discussion in [11] it follows that the multiplicity is equal to 4. As was shown by Hilbert, there exist Peano curves filling the square with multiplicities not exceeding 3 (see [11]). Since composing with a homeomorphism does not change the multiplicity, from the mentioned result of Hilbert and our Theorem 3.1 it follows that their exist pairs of NCI functions of multiplicity 3, which apparently would be not so easy to prove directly.

For n-dimensional parallelotopes, it can be shown that the multiplicities of Peano curves provided by the construction presented above are equal to 2^n. Thus we arrive at a natural problem: what is the minimum of multiplicities of continuous curves filling n-dimensional parallelotopes and such that their components are independent functions? Our conjecture is that there exist continuous curves filling an n-dimensional parallelotope with multiplicities not exceeding $n + 1$ but for proving this one needs a thorough analysis of multidimensional analogs of Hilbert's construction.

Thus we have achieved a reasonable understanding of the structure and storage of independent continuous functions on an interval and it becomes natural to address the same questions for more complicated probability spaces, but we cannot go into that in this paper. In the next section we present some general results of measure theory which are crucial for proving Theorem 3.1.

4. Independent Functions and Measure Preserving Automorphisms

One can also approach the construction of independent continuous and smooth functions using general results on measure preserving automorphisms of compact spaces obtained by J. von Neumann, W. Oxtoby, and S. Ulam [12, 13]. To describe this approach we introduce the necessary definitions and terminology.

A finite outer measure on a set (space) E is a function m^* defined for all subsets of E and satisfying three conditions:

(1) $0 \leq m^*A \leq m^*e, 0 < m^*E \geq \infty, m^*(\emptyset) = 0$;
(2) $m^*A \leq m^*B$ if $A \subset B$;
(3) $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

A set A is measurable with respect to m^* if $m^*W = m^*(W \cap A) + m^*(W \cap (E\backslash))$ for every subset $W \subset E$. The function m^* is completely additive with respect to its class of measurable sets, and the measure function thus derived from m^* by restricting its domains is denoted by m.

If E is a metric space, a Caratheodory outer measure is defined as an outer measure which also satisfies the condition

(4) $m^*(A + B) = m^*A + m^*B$ if the distance between A and B is positive.

The measurable sets then include all Borel sets. By a Lebesgue–Stieltjes (LS) outer measure on a polyhedron, we shall understand a finite Caratheodory outer measure which satisfies the further condition

(5) $m^*A = \inf m^*G$ for all open $G \supset A$.

An LS measure on an r-dimensional polyhedron E, $r \geq 1$, will be called r-dimensional if it is zero for points, zero for the set of singular points (lower-dimensional faces), and positive for neighborhoods of regular points. The set of all automorphisms of a polyhedron E (or of any compact metric space) is made into a metric space $H[E]$ by the definition

$$d(g, h) = \max[\max d(g(x), h(x)), \max d(g^{-1}(x), h^{-1}(x))], \quad x \in E.$$

The closed subgroup of $H[E]$ consisting of automorphisms that leave all boundary points fixed will be denoted by $H^{[E]}$. The subspace of $H[E]$ consisting of all measure-preserving automorphisms (isomerisms) with respect to a given LS measure m will be denoted by $M[E, m]$. We shall write simply $M[E]$ if m is the ordinary r-dimensional Lebesgue measure on $E \subset \mathbb{R}^n$, $r < n$. The closed subgroup of $M[E]$ consisting of automorphisms that leave all boundary points fixed will be denoted by $M^{[E]}$.

The following two results obtained by J. von Neumann, W. Oxtoby, and S. Ulam [12, 13] give a comprehensive description of automorphisms of measures (measure preserving homeomorphisms). The second of them is of crucial importance for our approach while the first one is presented for completeness and so its proof is omitted.

Theorem 4.1 (see [13]). *Let E be any regularly connected polyhedron of dimension $r \geq 2$, and let m be any r-dimensional LS measure on E. Then the set of all metrically transitive automorphisms is a residual G^δ-set in the space $M[E, m]$ of measure-preserving automorphisms of E.*

Theorem 4.2 (see [13]). *In order that a given outer measure m^* defined for all subsets of polyhedron $E \subset \mathbb{R}^n$ be automorphic to the Lebesgue outer measure L^* it is necessary and sufficient that it satisfies conditions* (1)–(5)*, and also*

(6) *$m^*G > 0$ if G is a nonempty open subset*;

(7) *$m^*p = 0$ for every point p*;

(8) *$m^*(\mathrm{bd}, E) = 0$ and $m^*E = L(E)$.*

If m^* satisfies these conditions there exists an automorphism h of E such that $m^*A = L^*h(A)$ for every $A \subset E$, and such that h leaves the boundary fixed.

In the case where $r = 1$, Theorem 4.2 is trivial. Indeed, suppose E is the unit interval $[0, 1]$ and put $h(x) = m([0, x])$. Then it is easy to verify that h satisfies all conditions listed in Theorem 4.2. For arbitrary r, the proof is based on a sequence of lemmas whose motivation lies in the idea of securing first that $m(A) = L(h(A))$ for all sets of a division automorphic to a dyadic subdivision. Then h is modified within each of these sets so as to secure equality for the sets of a finer subdivision. Finally, a convergent sequence of such modifications is obtained and the limiting automorphism affects the desired transformation for all sets. Before presenting the lemmas let us show that Theorem 4.2 indeed implies Theorem 3.1.

Proof of Theorem 3.1. Consider a Peano curve $F : I \to I^n$ filling in I^n. Introduce a measure on $E = I^n$ by setting $m(X) = \lambda_1 F^{-1}(X)$ for any Borel set $X \subset E$, where λ_1 denotes the one-dimensional Lebesgue measure on I. Then it is easy to verify that measure m satisfies all the conditions of

Theorem 4.2. Hence by the latter theorem, there exists an automorphism h of E such that $\lambda_n h(X) = m(X) = \lambda_1 F^{-1}(X)$. Thus for $G = h \circ F$ we get that $\lambda_n(X) = \lambda_1 G^{-1}(X)$ for any Borel subset $X \subset E$, which by definition means that the components of the curve $h \circ F$ are independent. Thus, Theorem 4.2 indeed implies Theorem 3.1. □

We now formulate the lemmas needed for proving Theorem 4.2 and present outlines of their proofs.

Lemma 4.1. *Let m be any LS measure on E that is zero for points and for the boundary and let a be any number in the interval $(0, m(E))$. There exists an open set G contained in the interior of E such that $m(G) = a$.*

Lemma 4.2. *Let m be any LS measure on E that is zero for points and for the boundary. Let e_1 and e_2 be the two cells obtained by bisecting E perpendicularly to one of its edges. Let a_1 and a_2 be any two positive numbers such that $a_1 + a_2 = m(E)$. There exist an automorphism $h \in H^{[E]}$ such that $m(h(e_1)) = a_1, m(h(e_2)) = a_2$.*

Lemma 4.3. *Let m be any LS measure in E that is zero for points and for the boundary. Let $s_1, \ldots, s_N$ be the cells of any dyadic subdivision of E, and let $a_1, \ldots, a_N$ be associated positive numbers whose sum is equal to mR. There exists an automorphism $h \in H^{[E]}$ such that $mh(s_i) = a_i$, $i = 1, \ldots, N$.*

Lemma 4.4. *Let m be any LS measure on E that is zero for points and for the boundary. There exists an automorphism $h \in H^{[E]}$ such that for every dyadic cell s we have $m(h(\operatorname{bd} s) = 0$.*

Lemma 4.5. *Let m, v be two r-dimensional LS measures on E such that $mE = vE$, and let $d > 0$ be given. There exists automorphisms $g, h \in H^{[E]}$ such that, for each cell s of a certain dyadic subdivision of E, we have $m(g(s)) = v(h(s))$; $m(g(\operatorname{bd} s)) = v(h(\operatorname{bd} s)) = 0$; $\operatorname{diam} g(s) < d$, $\operatorname{diam} h(s) < d$, $\operatorname{diam} s < d$.*

Lemma 4.6. *Any two r-dimensional LS measures m, v on E such that $m(E) = v(E)$ are automorphic to each other under an automorphism that leaves the boundary fixed.*

Proof of Lemma 4.1. Since $m(E) < \infty$, there can be at most countably many planes parallel to the faces of E that intersect E in sets of positive m-measure, hence we can divide E into a finite number of rectangular r-cells $s_1, \ldots, s_N$ of diam $< 1/2$ whose boundaries all have m-measure zero. Let i be the least integer such that $m(s_1 + \cdots + s_i) \geq a$ and let G_1 be the union of interiors of $s_1, \ldots, s_{i-1}$. Then $m(G_1) < a$, but $m(G_1) + m(s_1) \geq a$. □

Now consider the cell s_i, denote it by R_1 and divide it into rectangular cells $s_{i,1}, \ldots, s_{i,N_i}$ of diam $<$ $1/4$. Again, we find an open set $G_2 \subset R_1$ such that $m(G_1 + G_2) < a$ and $m(G_1 + G_2) + m(R_2) = a$, where R_2 has an analogous meaning. Proceeding in this manner, we find disjoint open sets $G_1, G_2, \ldots$ contained in the interior of E and sequence of rectangular r-cells $R_1, R_2, \ldots$ such that $\operatorname{diam} R_n \to 0$ and $a - m(R_n) \leq m(G_1 + \ldots + G_n) < a$, $n \geq 1$. The cells R_n intersect in a point p and we have $\lim m(R_n) = m(p) = 0$ (by hypothesis), hence $m(G) = a$.

The next lemma is crucial. Its complete proof is rather tedious and so we only present an outline of the argument. The details can be found in [13].

Sketch of proof of Lemma 4.2. Let H_1 be the set of automorphisms $h \in H_0^{[R]}$ such that $m(h(R_1)) \geq a_1$ and $m(h(R_2)) \geq a_2$. This is a closed set. Let $\{h_n\}$ be any sequence of automorphisms in H_1 tending uniformly to h; then any neighborhood of $h(R_i)$, $i = 1, 2, \ldots$, contains $h_n(R_i)$ for all sufficiently large n, and therefore has m-measure at least a_i. It can be proved that H_1 is a nonempty closed subset of a complete space, hence it is also complete. For each natural n, put $E_n = \{h \in H_1 : m(h(R_1)) \geq a_1 + 1/n\}$. For each n, it is a set of first category in the sense of Baire, hence the set $H_1 \setminus \bigcup_{n=1}^{\infty} E_n$ is residual. The same is done for R_2, i.e., it is proved that the set of automorphisms h such that

$m(h(R_2)) = a_2$ is also residual in H_1. Thus we get two sets of second category in H_1, hence their intersection is dense in H_1. It remains to note that each element of this intersection satisfies the desired property, which completes the proof of Lemma 4.2. □

Lemma 4.3 is a direct generalization of Lemma 4.2 to the case of N cells. The proof is purely technical and is based on repeatedly applying Lemma 4.2 to a dyadic sequence of subdivisions in which each subdivision is a refinement of the preceding one. Details can be found in [13].

Sketch of proof of Lemma 4.4. The proof is again based on considerations involving the Baire category. Let R_1 be the $(r-1)$-dimensional cell in which E is intersected by any one of the planes used in forming dyadic subdivisions. Let A_n be the set of automorphisms $h \in H_0^{[R]}$ such that $m(h(R_1)) \geq 1/n$. Then the set A_n is nowhere dense, and $H_0 - \bigcup_{n=1}^{\infty} E_n$ is residual. The intersections of the residual sets corresponding to each of the countably many planes used in forming dyadic subdivisions is therefore also residual, and the automorphisms belonging to this set have the required property. □

In order to prove Lemma 4.5 it is sufficient to apply the preceding lemmas to a pair (*measure, automorphism*) for each of the two measures separately. This immediately gives the existence of automorphisms with the required properties.

Sketch of the proof of Lemma 4.6. The proof is conceptually simple but technically rather involved. First, one constructs a sequence of partitions of E into cells $s_1, \ldots, s_n$ (elements of each partition are numbered by upper indices) and two sequences of automorphisms $g_n, h_n \in H^{[E]}$ satisfying four properties, three of which are the same as in Lemma 4.5 but for $e_n - (1/2)^n \operatorname{diam} E$, and the fourth one reads: $g_n = g_{n-1}$ on s_i^{n-1}, $i = 1, \ldots, N_{n-1}$ and similarly for h_n and h_{n-1}. It is now possible to show that sequences g_n and h_n are convergent in $H^{[E]}$, say, to g and h, respectively. Then for measures $m'(A) = m(g(A))$ and $v'(A) = v(h(A))$, one can prove that

$$m'(G) = \sum_{k=1}^{\infty} m'(s_k) = \sum_{k=1}^{\infty} v'(s_k) = v'(G),$$

where $G = \bigcup_{k=1}^{\infty} s_k$ is an open set. Hence m'^* and v'^* agree for open sets and therefore for all sets. Thus $m^*(A) = v^*(h(g^{-1}(A)))$ for every set, and Lemma 4.6 is proved. □

Now Theorem 4.2 follows at once from Lemma 4.6 by taking $v = L$ the Lebesgue measure, which, as was shown above, gives also the proof of Theorem 3.1.

In order to prove the existence of independent smooth functions on closed manifolds we need a refinement of Theorem 4.2. Let us say that measure is of class C^k, $k = 1, 2, \ldots, \infty$, if it possesses density of the same regularity class, in other words, if there exists a C^k-function p such that $m(A) = \int_A p$.

Theorem 4.3. *Let m^* be an outer LS-measure of class C^k defined for all subsets of polyhedron $E \subset \mathbb{R}^n$ which satisfies all conditions of Theorem 4.2. Then there exists a C^k-diffeomorphism h of E such that $m^*A = L^*h(A)$ for every $A \subset E$ and h leaves all points of the boundary fixed.*

The proof reduces to checking that, with the assumptions of the latter theorem, the homeomorphism h constructed in the proof of Theorem 4.2 appears smooth of the same regularity class. The details are fairly standard and therefore we omit them. Having this theorem one may repeat the argument used for deriving Theorem 3.1 from Theorem 4.2. Recall that there always exists a smooth surjective map f of a closed manifold on a cube E of the same dimension such that the set of singularities has measure zero [6]. Having such a map we obtain a measure m on E by putting $m(A) = \mu(f^{-1}(A))$ for each measurable set A, where μ is an a priori fixed measure on the manifold. It is easy to see that m satisfies all conditions of Theorem 4.3. Hence there exists a diffeomorphism h transforming measure m in the Lebesque measure on E. Obviously, components of the composition $h \circ f$ provide a collection of independent smooth functions on the given manifold. In this way we obtain a generalization of Proposition 2.1.

Theorem 4.4. *Each closed n-dimensional C^k-manifold possesses collections of n nonconstant independent C^k functions.*

Thus all the results formulated in this paper are proved. We postpone the discussion of its generalizations and applications for future publications.

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